PHILOSOPHICAL SOCIETY OF WASHINGTON
BULLETIN, Vol. XIV, pp. 247-276

ATMOSPHERIC FRICTION

WITH

SPECIAL REFERENCE TO AERONAUTICS

BY

A. F. ZAHM

READ BEFORE THE PHILOSOPHICAL SOCIETY OF WASHINGTON
FEBRUARY 27, 1904

WASHINGTON
PUBLISHED BY THE SOCIETY
JUNE, 1904

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[Read before the Society February 27, 1904.]

MEASUREMENTS.

The experiments here described were made to determine the magnitude of the friction of air flowing over even surfaces, both smooth and rough ones, and the law of its variation with the speed of flow, the length, and quality of surface. The primary purpose of the investigation was to establish a basis for calculations in engineering, and particularly in aerial navigation; but it is hoped that the measurements are sufficiently accurate to be of value also to the general dynamics of fluid motion.

It has long been known to marine science that in a well-formed vessel one of the chief elements of resistance is the skin-friction of the water on its sides; and, by analogy, it was surmised that a fair-shaped body in the air might be retarded in a similar way by the tangential drag of that fluid. But the measurements of several prominent experimenters led them to affirm that the skin-friction of the air is negligible, even for bodies of fair outline. The present research, however, seems to prove that the frictional resistance is at least as great for air as water, in proportion to their densities. In other words, it amounts to a decided obstacle in high-speed transportation. In aeronautics it is one of the chief elements of resistance, both to hull-shaped bodies and to aero-surfaces gliding at efficient angles of flight. It seems important, therefore, that the main laws of this resistance should be carefully determined.

To measure the tangential force of the air on even surfaces, various skin-friction planes were suspended inside a wind-tunnel by means of two fine steel wires attached to the top of the laboratory, as shown in figure 1. The tunnel itself, standing on the floor of the laboratory, measuring 40 feet long by 6 feet square, has a 5-foot electric suction fan at one end, and a cheese-cloth screen, or two, at the other, to straighten the current of inflowing air. A boy with a rheostat and tachometer holds the fan at any desired speed, accurately to a fraction of 1 per cent., thus giving an even

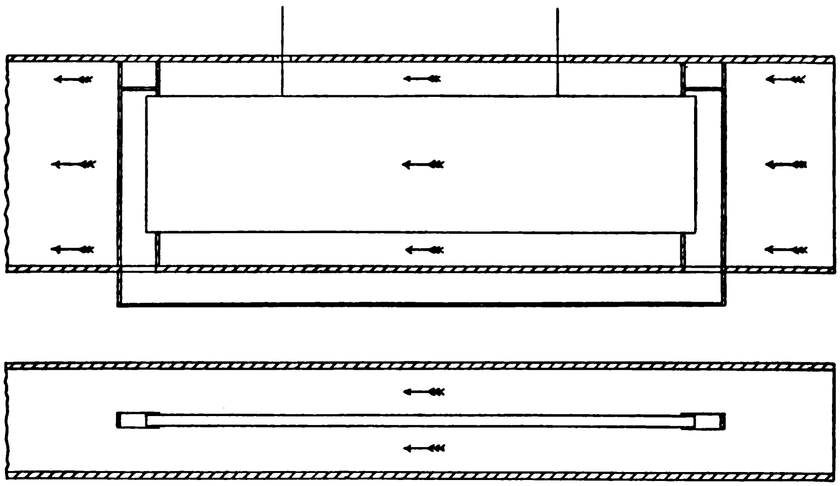

FIG. 1.—*Section of Wind-tunnel with Suspended Skin-friction Plane.*

flow of air of like constancy. As the wind-friction moves the plane endwise the displacement is determined by the motion of a sharp pointer attached to one suspension wire and traveling over a fine scale lying on top of the tunnel. The swing of the plane can be measured accurately to five-thousandths of an inch, and the force on the plane is exactly proportional to the scale readings. The wind-speed is usually measured by a pressure-tube anemometer, though other kinds have been employed for comparison.

In the first attempt to determine the coefficient of skin-friction a thick plane was used, having wind-shields fore and aft, as shown in figure 1, to protect it from end-thrust. The plane is 16 feet long, 4 feet wide, and 4 inches thick. The shields are made of sheet zinc, their cross-section measuring 4½ by 12 inches inside, and each shield closely envelops one end of the plane, yet has ample space farther within to allow the air to flow very freely from one shield to the other, through a large connecting pipe underneath the floor of the tunnel, thus equalizing the pressure. This pipe, or flue, measures 1 square foot in cross-section.

The static pressures in the two wind-shields deserve careful attention. If they are equal, the resultant end-thrust is nothing, and the only deflecting force on the plane is the friction of the air along its sides. But in practice there is a difference of static pressure, which is measured by connecting the shields, by means of rubber hose, to a differential pressure-gauge graduated to millionths of an atmosphere, and usually read to one-ten-millionth.* Computing the end-thrust from the differential pressure, and adding or subtracting the result, gives the total skin-friction on the plane. The correction thus introduced is about 5 per cent. of the whole deflecting force.

Considerable care was taken in the design of the plane to make it light and keep it perfectly straight. A frame was made of organ tubes and covered with paper in such a way as to be adjustable for warpage. As shown in figure 2, the paper is glued, not directly to the organ-tube frame, but to ⅛-inch boards which slide over the four outer faces of the frame. As the paper was fastened on wet, it now remains very taut on all but the dampest days, and of course holds the sliding pieces firmly to the frame. The process of adjustment is as follows: The two end sliding pieces are set vertical by means of plumb-lines, thus bringing the four corners accurately into a mathematical plane. The four corners are then joined by tight threads and the other sliding pieces tapped into line with a mallet. The operation re-

* See "Measurement of Air Velocity and Pressure," Physical Review, December, 1903.

quires less than half an hour, and the plane can easily be made true to less than $\frac{1}{32}$ of an inch. The warpage of the 16-foot plane sometimes amounts to one-eighth of an inch in twenty-four hours, and may be more than a quarter of an inch in several days; but in practice the plane is kept straight by timely adjustment.

During each experiment one assistant controlled the fan speed by means of a rheostat, and noted the revolutions per minute with a Schaeffer and Budenberg tachometer; another assistant read the deflection of the plane, while a third observed the differential pressure in the wind-shields by means of a manometer, and the wind velocity as given by a pressure-tube anemometer or a Robinson cup anemometer. The duration of an experiment was usually about an hour and comprised ten different wind velocities.

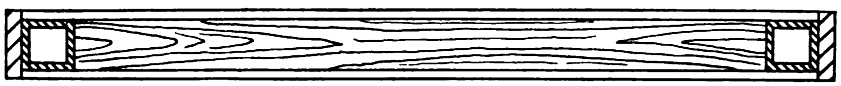

FIG. 2.—*Cross-section of 16-foot Plane, Showing Paper Glued to Sliding Pieces.*

The following page from the laboratory note-book for January 30, 1903, gives the results obtained after some skill had been acquired in using the various instruments. Similar observations had been taken in July, 1902, and this much of the present paper was communicated to the American Association for the Advancement of Science in December, 1902.

A few essential data may be prefaced: surface of plane between wind shields, 138.08 square feet; cross-section of plane, 202.1 square inches; weight, 58 pounds; 1-inch swing of plane, =0.296 pound deflecting force; 1 milligram per square centimeter differential pressure in the wind shields equals 0.00287 pound end-thrust on the plane; mean temperature of experiment, 4°.5 C.; barometric pressure, 29″.74; time, 3.30 to 4.30; weather, dry; mouth of tunnel not screened.

TABLE I.

Skin-friction on Plane Measuring 16′ × 4′ × 4″.

Speed of fan.	Swing of plane.	Force causing swing.	Differential pressure in shields.	End-thrust on plane.	Pressure-tube anemometer.	Wind speed.	Friction per square foot.
Rev. min.	*In.*	*Lbs.*	*Mg. sq. cm.*	*Lbs.*	*Mg. sq. cm.*	*Ft. sec.*	*Lbs.*
150	0.27	0.080	0.0	0.0	70	11.11	.000579
200	0.41	0.121	0.0	0.0	105	13.63	.000875
250	0.54	0.160	0.0	0.0	155	16.16	.001156
300	0.76	0.225	1.0	0.003	225	19.46	.00165
350	0.95	0.277	1.7	0.005	295	22.30	.00203
400	1.19	0.352	3.9	0.011	375	25.14	.00262
450	1.45	0.428	7.1	0.019	465	28.0	.00324
500	1.74	0.515	9.5	0.026	570	31.0	.00392
550	2.04	0.603	13.7	0.037	670	33.6	.00463
600	2.39	0.701	16.5	0.045	815	37.0	.00539

The force in the third column is computed from the observed swing of the plane. Adding the end-thrust, since the differential pressure opposed the deflection of the plane, there results the actual skin-friction on the exposed surface. Dividing by the area of the surface gives the values recorded in the last column. The wind speed is computed from the pressure-tube readings by a theoretical formula, which has been carefully verified by a special series of experiments which were published in the *Physical Review*, December, 1903.

The values of the wind velocity and skin-friction have been plotted on logarithmic cross-section paper, as shown in figure 3. Their relation in this, as in subsequent experiments, is invariably expressed by a straight line—that is, by the relation,

$$F = av^n \quad . \; . \; . \quad (a)$$

in which F is the total friction, v the wind speed, a, n, numerical constants. The concrete relation obtained from the numerical values of table I is, for a plane 16 feet long,

$$f = 0.00000671\, v^{1.85} \;.\;.\;.\;.\;.\;.\;. \; (v = \text{ft. sec.}),$$
$$f = 0.00001363\, v^{1.85} \;.\;.\;.\;.\;.\;.\;. \; (v = \text{mi. hr.}),$$

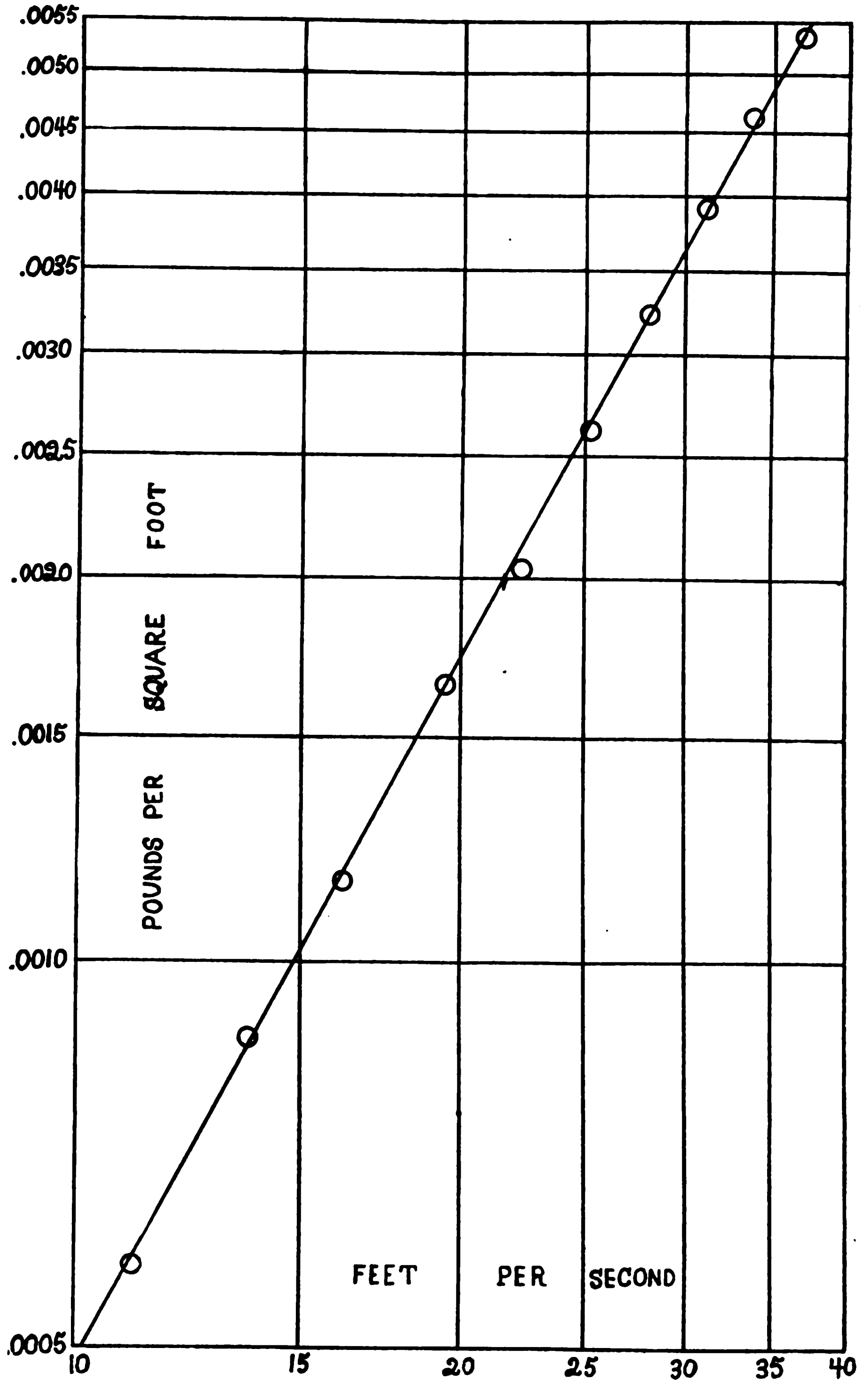

FIG. 3.—*Relation between Velocity and Unit Friction for 16-foot Plane.*

in which f is the average friction in pounds per square foot of surface, and v is the wind velocity in the units indicated within the parentheses. This relation was corroborated by later experiments in which no wind-shields were used.

Having fairly established the law of variation of the skin friction with the air velocity, an effort was made to discover its variation with the length of surface. A simpler method was then adopted which had been considered, but was discarded in the beginning as appearing hardly delicate enough to measure such extremely small forces as the friction was at first conceived to be.

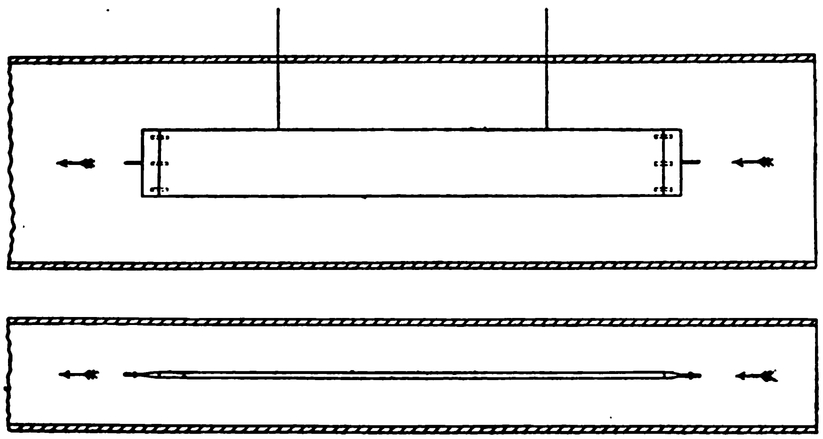

FIG. 4.—*Skin-friction Plane with Sharp Ends, Suspended in Wind-tunnel.*

Planes were now constructed similar to those commonly used to determine the skin-friction of water. The first was a pine board 4 feet long, 25.5 inches wide, and 1 inch thick, carefully trued and varnished, and suspended in the wind tunnel, as usual, by steel wires 0.025 of an inch in diameter. It was provided with a 7-inch pine prow and stern, both of ogival form. These were held on by dowel pins, as shown in figure 4, and each terminated in a sharp edge, from the center of which a steel pin protruded along stream between guides to steady the plain against wabbling. As the doweling was carefully executed, straight planes of any length could be made by adding extra boards, the lengths most employed being 2, 4, 8, 12, and 16 feet.

The method of using the planes to determine the surface friction was as follows: The total force was measured, at various velocities, using the prow and stern first on the 16-foot board, then successively on the 12, 8, 4, and 2 foot boards, and finally with nothing between them. Subtracting this last force from each of the others gave the friction on those five lengths. It may not be absolutely true that the end resistance was the same for each of those lengths, but the error of this assumption is regarded as very slight for several reasons: (1) the end resistance is but a small part of the total; (2) the stream lines are so slightly disturbed that the flow about the ends must be practically the same in all cases; (3) the results harmonize very well with those obtained by other methods.

Table II, taken from the laboratory note-book, exhibits the observed and computed values for the two-foot friction-board. The mouth of the tunnel was screened with cheese-cloth to steady the flow of the air, in order to obviate wabbling in so small a board. The velocity was thus reduced, it is true, but sufficient values are given to make a reliable diagram.

The following data may be prefaced: surface of the two-foot plane without prow and stern, 8.83 square feet; weight of plane with end pieces, 17 pounds; 1 inch swing of plane = 0.0862 pounds wind force; barometric pressure, 29.80 inches; mean temperature of experiment, 24°.2 C.

TABLE II.

Surface Friction by 24″ × 25.5″ Pine Board with Prow and Stern.

Speed of fan.	Swing of plane.	Force causing swing.	Force on prows.	Net friction.	Pressure tube anem.	Wind speed.	Friction per sq. ft.
Rev. min.	*In.*	*Lbs.*	*Lbs.*	*Lbs.*	*Mg. sq. cm.*	*Ft. sec.*	*Lbs.*
200	0.050	0.00431	0.00233	0.00198	22.8	6.38	0.000224
250	0.080	0.00690	0.00365	0.00325	37.0	8.12	0.000368
300	0.120	0.01034	0.00553	0.00481	58.0	10.18	0.000545
350	0.158	0.01362	0.00734	0.00628	78.5	11.80	0.000710
400	0.205	0.01767	0.00949	0.00818	103.5	13.52	0.000925
450	0.260	0.02240	0.01188	0.01052	132.0	15.34	0.001188
500	0.310	0.02586	0.01379	0.01207	155.0	16.61	0.001366

The third column gives the whole force on the friction-board with its prow, stern, and suspension wires. The fourth column gives the force on the latter alone, which, deducted from the whole force, gives the friction on the sides of the two-foot length. Dividing this net friction by 8.83, the area of the true friction surface, gives the values in the last column.

Similar tables were obtained for the other friction-boards, of lengths 4, 8, 12, and 16 feet respectively. When the values from the five tables are plotted on logarithmic cross-section paper they give five separate straight lines, all having the same inclination as the one shown in figure 3, in which the slope tangent is 1.85. This means that, for all the velocities and lengths of surface employed in this research, the skin-friction is expressed by an equation of the form

$$F = av^{1.85} \ldots\ldots (\beta),$$

a being a numerical constant, and v the wind speed. Hence if the net friction on each board is known for any velocity, it can readily be computed for any other velocity.

In practice the computations illustrated in tables I and II were obviated, for all the tables, by a simple expedient. The observed anemometer readings and swing of the plane were plotted while the measurements were in progress, giving five straight lines, all of the same slope. Then a point was selected on each line representing a wind speed of ten feet per second, and the corresponding friction per square foot of surface noted. From these values the numerical equations between F and v can at once be written. The observed values are given in the subjoined table.

TABLE III.

Skin Friction at 10 Feet per Second for Various Lengths of Surface.

Length of friction board.	2	4	8	12	16
Average friction, pounds per square foot.......	0.000524	0.000500	0.000475	0.000467	0.000457

Knowing, then, the friction at the same speed on five different boards, there remained to determine the law of its variation with length of surface. To that end, the values in

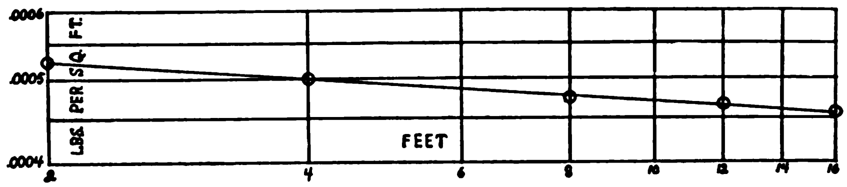

FIG. 5.—*Relation between Length and Unit Friction at 10 Feet per Second.*

table III were plotted on logarithmic cross-section paper, as shown in figure 5. The result is a straight line whose equation is of the form,

$$f = al,^{-0.07}$$

whence $$F = fl = al^{.93} \ldots \quad (\gamma),$$

in which f is the average friction in pounds per square foot and l is the length of surface in feet. At one foot per second the coefficient is 0.00000778; hence at any speed, v feet a second, the average friction per square foot is

$$f = 0.00000778\, l^{-0.07}\, v^{1.85} \ldots \quad (v = \text{ft. sec.}),$$
$$f = 0.0000158\, l^{-0.07}\, v^{1.85} \ldots \quad (v = \text{mi. hr.}).$$

Assuming the two laws thus far derived to be true for the planes and wind speeds employed, we may readily express the total friction on a plane of any length from 2 to 16 feet, moving at any speed from 5 to 40 feet a second. Thus, by the last equation, the total friction F on a surface 1 foot wide and 1 foot long is

$$F = fl = 0.00000778\, l^{.93}\, v^{1.85} \ldots \quad (v = \text{ft. sec.}).$$
$$F = 0.0000158\, l^{.93}\, v^{1.85} \ldots \quad (v = \text{mi. hr.}).$$

Of course this value of F must be doubled for a material plane of length l and width one foot, since a material plane has two sides.

In order to facilitate the computation of skin-friction in practice, the following table has been derived from the equation $f = 0.0000158\ l^{-0.07}\ v^{1.85}$. The friction for any intermediate velocity, or length of surface, may be found by interpolation. If the surface is of variable length it may be divided into longitudinal strips, the force on each strip being the product of the area of the strip multiplied by the average friction for its particular length. Only the values in heavy type lie within the range of the experiments above described.

TABLE IV.

Friction per Square Foot for Various Speeds and Lengths of Surface.

Wind speed.	Average friction in pounds per square foot.					
	1′ plane.	2′ plane.	4′ plane.	8′ plane.	16′ plane.	32′ plane.
mi. hr.						
5	0.000303	**0.000289**	**0.000275**	**0.000262**	**0.000250**	0.000238
10	0.00112	**0.00105**	**0 00101**	**0.000967**	**0 000922**	0.000878
15	0.00237	**0.00226**	**0 00215**	**0 00205**	**0.00195**	0.00186
20	0.00402	**0.00384**	**0.00365**	**0.00349**	**0.00332**	0.00317
25	0.00606	**0.00579**	**0.00551**	**0 00527**	**0.00501**	0.00478
30	0.00850	0 00810	0.00772	0.00736	0.00701	0.00668
35	0.01130	0.0108	0.0103	0.0098	0.00932	0.00888
40	0.0145	0.0138	0.0132	0.0125	0.0125	0.0114
50	0.0219	0.0209	0.0199	0.0190	0.0181	0.0172
60	0.0307	0.0293	0.0279	0.0265	0.0253	0.0242
70	0.0407	0.0390	0.0370	0.0353	0.0337	0.0321
80	0.0522	0.0500	0.0474	0.0452	0.0431	0.0411
90	0.0650	0.0621	0.0590	0.0563	0.0536	0.0511
100	0.0792	0.0755	0.0719	0.0685	0.0652	0.0622

It may now be inquired what other circumstances alter the surface friction. Perhaps the chief of these are the atmospheric changes of density and the unevenness of surface.

No effort was made to determine the relation between the density and skin-friction of the air, partly for want of time, partly because, with the apparatus in hand, too great changes of density would be needed to reveal such relation accurately. Doubtless the friction increases with the density, since it is due to the inertia of the fluid near the friction surface. Of course, in steady motion at low velocity, such as studied by Maxwell, the conditions are different. He found that when one plane moved edgewise near and parallel to another plane, at a constant speed below one-twelfth of an inch per second, the friction is independent of the pressure and proportional to the absolute temperature for such atmospheric conditions as prevail near the earth's surface.

Some measurements were made with the four-foot friction-board covered with various materials to observe the effect of quality of surface upon the tangential resistance. Practically the same friction was observed, whether the board was covered with dry varnish, or wet, sticky varnish, or sprinkled with water, or covered with calendered or uncalendered paper, or glazed cambric, or sheet zinc, or old English draughting paper, which feels rough to the touch. But when the plane was covered with coarse buckram, having sixteen meshes to the inch, the friction at ten feet a second was 10 to 15 per cent. greater than for the uncovered surface, and the friction increased as the velocity raised to the power 2.05, or approximately as the square of the speed.

The fact that such a variety of materials exhibit practically the same friction seems to indicate that this is a shearing force between the swiftly gliding air and the comparatively stationary film adhering to the surface, or embedded in its pores. If, as seems to be true, there is much slipping, this means that the internal resistance of the air is less at the surface than at a sensible distance away. As the shearing strength of a gas is due to the interlacing of its molecules, owing to their rapid motion normal to the shearing plane, it may be that the diminution of shear near a boundary surface is due to the dampening, within the film, of the component of molecular translation normal to the surface.

To summarize the results attained thus far, it may be said that, within the ascribed limits of size and wind speed

(1.) The total resistance of all bodies of fixed size, shape and aspect is expressed by an equation of the form

$$R = av^n \; . \; . \; . \; . \; . \; (a),$$

R being the resistance, v the wind speed, a, n, numerical constants.

(2.) For smooth planes of constant length and variable speed the tangential resistance may be written

$$R = av^{1.85} \quad \dots\dots (\beta).$$

(3.) For smooth planes of variable length, l, and constant width and speed the friction is

$$R = al^{.93} \quad \dots\dots (\gamma).$$

(4.) All even surfaces have approximately the same coefficient of skin-friction.

(5.) Uneven surfaces have a greater coefficient of skin-friction, and the resistance increases approximately as the square of the velocity.

The equation $R = av^n$ was found to express very accurately the resistance of all the shapes tested at speeds from five to forty feet a second. For normal planes, spheres, cylinders, and blunt bodies generally, except very small ones, n equals 2, very approximately; for thin, tapering bodies n may have any value from 2 to 1.85; but in every case, if the form and aspect of the model remain fixed, a and n are found to remain practically invariable for all the speeds employed. This was manifested by plotting the speed and resistance on logarithmic cross-section paper and observing that the diagram was invariably a straight line for all the models tested. The statement cannot be true for a great range of speeds.

Such were the results obtained in a wind of uniform velocity and direction. When, however, the current is turbulent a and n are found to vary considerably; but since the flow of a turbulent wind cannot be specified, the measurements obtained in one such current cannot well be applied to determine the resistance in a different one. For that reason it seemed better to make the measurements in a uniform wind, where, moreover, the instruments give steadier readings.

On comparing the above results with those obtained by Dr. Froude for water, it is found that the equations are very similar. The exponents are nearly the same, and the coefficients are to each other roughly as the densities of air and water. Using varnished friction-boards, Froude finds $n = 1.85$ for a surface 8 to 20 feet long, and $n = 2.00$ for a plane 2 feet long; I find $n = 1.85$ for all lengths from 2 to 16 feet. By Froude's measurements the friction varies as the power 0.83 of the length for varnished planes 2 to 20 feet long; I find it to vary as the power 0.93. With a varnished board 2 feet long, moving 10 feet a second, the ratio of our coefficients of friction for air and water is 1.08 times the ratio of the densities of those media under the conditions of the experiment.

But in some respects Froude's results are quite unlike mine. For several surfaces he finds the skin-friction to vary as the square of the velocity, or nearly so, which is the relation I observed in a turbulent current and when the friction-board was vibrating slightly. He finds the friction of calico about twice that of a varnished surface; I find that glazed cambric has about the same friction as a varnished surface; but if the cambric is roughened, so as to expose a fine down, the friction is very much increased.

The fact that for some surfaces the coefficients of friction for air and water are roughly as their densities is of considerable interest. It is well known that the head resistances of the two fluids are directly as their densities, and if their friction coefficients also bear that ratio, the total resistances must be approximately as the densities. Hence the data obtained from water-resistance measurements on such surfaces may be fairly well applied to estimate the air resistance on various shaped models.

It is not, however, self-evident that the surface friction of any two fluids is proportional to their densities, and should not be taken for granted. It happens to be roughly true for varnished wooden surfaces in air and water, but appears to be wholly untrue for calico surfaces. In default, therefore, of an adequate physical theory of surface friction the magnitude in any given case can be determined only by direct experiment.

To complete the theory of the skin-friction board, two steps further remain to be taken. First, the equations of motion for a viscous fluid must be integrated to find the velocity at all points in the disturbed region about a thin material plane. Then the speed of flow must be measured at all points next the plane and at some distance away. The writer expects soon to map the stream-lines and measure the velocity. If, then, the equations can be integrated so as to give the speed as a function of the space coördinates, the computed and observed values can be directly compared. It is hoped that some one may obtain sufficiently general solutions of the equations to be of practical value, particularly for the simpler case in which the plane is indefinitely wide and in which the edge conditions are negligible. The integrals, if sufficiently general, will be of great importance to the science of surface friction, and may at once be applied to the mass of accurate data that for a generation has been accumulating in the laboratories of marine engineers.

APPLICATIONS.

The laws of skin-friction have both theoretical and practical application. They serve theory by explaining some apparently anomalous phenomena and by leading to more complete formulæ for total resistance. They are of practical use because in many of the forms employed in transportation the skin-friction is a large part of the total resistance.

It has been the common practice of writers on air resistance to employ the Newtonian formula,

$$R = av^2,$$

in which a is regarded as constant for a surface of fixed form and aspect; but this equation is far from true (1) for blunt bodies moving at high speeds, and (2) for bodies of easy shape moving at moderate speeds.

For blunt bodies at speeds below 1,400 feet a second the resistance is expressed more accurately by the equation

$$R = av^2 + bv^3,$$

in which a and b are constants. This has been shown analytically by Duchemin,* and has been proved experimentally by the writer † for speeds below 1,000 feet a second. It was also corroborated by Duchemin by citations from the experiments of others.

For bodies of easy shape and moderate speed the coefficient a in the Newtonian formula gradually diminishes with the velocity. This was observed by Langley and Canovetti, and now one reason seems apparent. The resistance cannot vary as the square of the velocity because a large part of it is friction, which varies as a lower power.

A good general formula may be obtained by writing the total resistance as the sum of two terms, one giving the head resistance proper, the other the skin-friction. Thus for ordinary transportation speeds we have

$$R = av^2 + bv^{1.85},$$

in which the body constants, a and b, are each a function of the dimensions and aspect of the given figure. A like formula may be used for a family of figures.

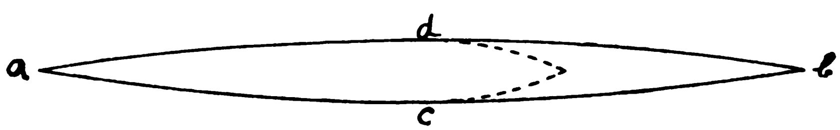

FIG. 6.—*Symmetrical Ogival Wedge of Minimum Resistance.*

As an example of the influence of the friction term, let it be required to find the resistance per unit length of a post

* "Les Lois de la Resistance de l'air."

† "Resistance of the air at speeds below one thousand feet a second," Philosophical Magazine, May, 1901.

having the form of cross-section shown in figure 6. The head resistance proper may be written equal to that of the major section, taken normal to the velocity multiplied by the sign of half the angle of the edge of the post. Thus

$$R_1 = c \sin a,$$

in which c is the resistance of the major section and a is the angle *abd*. Again, the skin-friction resolved parallel to the velocity is

$$R_2 = 2\int f_s\, ds \cdot \frac{dx}{ds} = 2\int f_s\, dx,$$

in which f_s is the coefficient of friction for the element of surface, and dx is an element of the width *ab*. Hence the total resistance may be written

$$R = c \sin a + 2fx,$$

in which f is the average friction per unit surface.

A glance at the above equation reveals its chief features. For x equal to zero, the second term vanishes, and the first becomes

$$R = c,$$

which is the normal resistance of the major section. For x very large the first term is negligible, and there remains

$$R = 2\,fx,$$

which is the formula for a simple plane moving edgewise. Thus the total resistance is comparatively large when x equals zero; then becomes smaller and smaller till a minimum is reached, and finally continuously larger as x goes on increasing. The width giving a minimum resistance is, of course, obtained by placing the derivative of R equal to zero and solving for x.

What has been said of this particular shape is true of all the figures of a family in which the major cross-section is kept constant while the length varies. There is some length for which the resistance is a minimum, and beyond that the resistance increases with the length up to infinity. To illustrate these features, let the equation for the total resistance be applied to the data of an experiment.

For practical engineering purposes, which need not be detailed here, it was found desirable to measure the total resistance of a number of wedge forms such as shown in figure 6. The models are all 1 inch thick and of the widths given in the second column of table V. The size of the models is given in the first column as so many calibers, their outlines being circular arcs whose radii are an even number of times the thickness of the wedge. The actual measured values of the resistance per unit length of post at 10 feet a second are given in the last column of the accompanying table and shown diagrammatically in figure 7 by the little circles.

TABLE V.

Computed and Observed Resistances of Duangular Cylinders One Inch Thick, One Foot Long, and of Various Widths.

Caliber of model.	Width of model.	Computed resistance.			Observed resistance.
		Head.	Frictional.	Total.	
1	1.76	0.00687	0.000212	0.00708	0.00702
5	4.41	0.00307	0.000511	0.00358	0.00375
10	6.20	0.00221	0.000687	0.00290	0.00298
20	8.88	0.00155	0.000960	0.00251	0.00267
30	11.05	0.00125	0.001178	0.00243	0.00250
40	12.77	0.00108	0.001348	0.00243	0.00238
50	14.31	0.000968	0.001500	0.00247	0.00235
60	16.00	0.000870	0.001664	0.00253	0.00253
70	16.87	0.000822	0.001746	0.00257	0.00253
80	18.25	0.000772	0.001884	0.00266	0.00261
100	20.12	0.000690	0.002061	0.00275	0.00285
150	24.87	0.000557	0.002505	0.00306	0.00299

Now let us apply to these data the equation

$$R = c \sin a + 2\, fx.$$

Taking $c = 0.0139$ pounds, the normal resistance of the major section at 10 feet a second, as computed by Langley's coefficient; also $\sin a = 1 / \sqrt{1 + x^2}$, x being inches; and $2f = 0.0001263\, x^{-0.07}$, the numerical equation becomes

$$R = \frac{0.0139}{\sqrt{1 + x^2}} + 0.0001263\, x^{.93}.$$

The values obtained by substituting for x the various widths of the models are given in the table and shown by means of the curves in figure 7.

The diagram portrays the main features of the equation very clearly. The total resistance falls rapidly at first, becomes a minimum when the wedge is about one foot wide, then increases indefinitely with the width. The true head

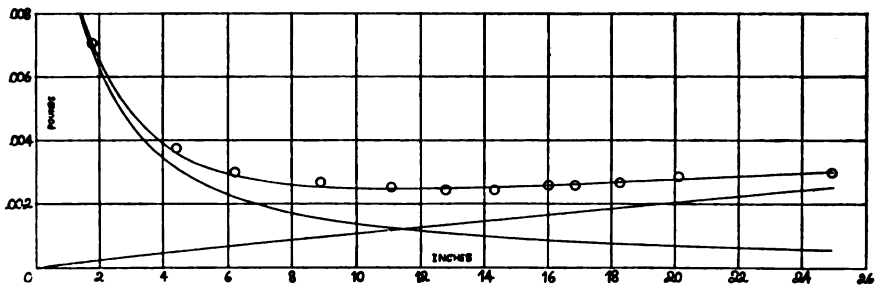

FIG. 7.—*Computed and Observed Resistances of Symmetrical Wedges.*

resistance and the skin-friction, as shown by the lower curves, approach each other, becoming equal when the width of the wedge is a little below one foot, then diverge indefinitely, the friction being four times the true head resistance when the width of the wedge becomes two feet.

We have thus found a formula which accords very well with the data of experiment; but its first term expresses only approximately the true head resistance and is here employed merely tentatively. In fact, the coefficient f had to be somewhat increased to make the computed and observed values agree. Thus the term $0.000126x$ makes the skin-friction equal to 0.00127 of a pound, when x equals one foot, whereas by table IV it should be 0.00113. So probably the term $c \sin a$ gives values for the head resistance which are somewhat too small. Possibly, also, the values of f given in table IV for short planes should be slightly increased.

It should be remarked that the minimum resistance given above is such only for the symmetrical shapes in question, but not necessarily a minimum for all possible shapes having the same major section. In fact, when a five-caliber bow, shown by the dotted line in figure 7, was combined with a fifty-caliber stern, the resistance was much diminished, and it was found incidentally that the ratio of the resistance of a good model to that of its major section can be made less than one part in eight. What the ratio may be for the shape of least possible resistance has not been ascertained.

Similar experiments were made with spindles having the outline shown in figure 8, and with like results. These are

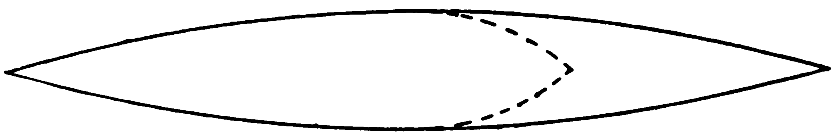

FIG. 8.—*Symmetrical Ogival Spindle of Minimum Resistance.*

still unfinished; but it may be mentioned, in passing, that the frictional effect is very manifest. The total resistance of a symmetrical spindle having such outline is again half friction, and has its minimum value in a model of about twelve calibers, for which the length is nearly seven times the major diameter—a relation given by Rankine for well-formed ships. A still less resistance is found when a two-caliber bow, shown dotted in figure 8, is combined with a twelve-caliber stern, in which case the length is about five times the major diameter. The ratio of the resistances of the spindle and its major section has been reduced to about one part in eight. What the smallest possible ratio may be for a given velocity has still to be ascertained and may well form the object of a special research.

The foregoing examples suffice to indicate the importance of the friction term in the general equations of aerodynamics. We may now notice its bearing on problems of transportation, and particularly the cost of propulsion in aeronautics. Let us consider the soaring plane, first assuming it smooth, then frictional.

Let A be the area of the plane, W its weight, v its velocity, a its angle of flight, R its resistance, H the propulsive power, and ς the density of the fluid in which it is moving. Then, if the plane is frictionless and steadily soaring on a horizontal course in still air,

$$R = W \tan a, \quad \dots \dots \dots \quad (a)$$
$$H = R v, \quad \dots \dots \dots \quad (b)$$
$$W = \frac{2 k \varsigma A v^2 \sin a \cos a}{1 + \sin^2 a}, \quad \dots \quad (c)$$

the last expression being the lift as given by Duchemin's formula, in which k is a constant of figure.

The relations of these seven variables contain much that is of interest in the theory of the aeroplane. For example, let us find the mileage cost and the propulsive power when the plane is just soaring.

The mileage cost is proportional to the resistance divided by the load, and hence, as shown by equation (a), it is directly proportional to the tangent of the angle of flight. It may therefore have any value, from zero to infinity, according to the inclination of the plane, and if this be kept constant the mileage cost is the same for all velocities, for whatever extent of surface, and for all densities of the medium, from mountain air to sea water.

In a similar way the mileage cost may be studied as a function of any of the other variables. Thus from equation (c) we obtain

$$\tan a = \frac{W (1 + \sin^2 a)}{2 k \varsigma A v^2 (1 - \sin^2 a)}, \quad \dots \quad (d)$$

in which the ratio of the parenthetical factors is practically unity for small values of a. Hence, writing

$$\tan a = \frac{W}{2 k \varsigma A v^2}, \quad \dots \quad (d')$$

it is at once evident that the mileage cost is directly proportional to the load, and inversely proportional to the density of the medium, the area of the plane, and the square of its velocity.

The propulsive power may be obtained directly from the last equation. Thus,

$$H = Wv \tan \alpha = \frac{W^2}{2 k \varsigma A v}.$$

This shows that the power varies directly as the square of the load, and inversely as the density of the medium, the area and speed of the plane.

This last relation, viz., that if W, ς, and A remain constant, H varies inversely as v, has been more emphasized than the other relations by the various writers on aeronautics. It was first proved, though in a different manner, by A. Du Roy de Bruignac,* and formally enunciated by him in 1875, as follows: "Providing the angle of a heavy plane, moving in the air, be maintained at the minimum necessary to sustain its weight, the work of translation diminishes as the velocity increases." Mr. Curtis† gives a different analytical proof, and Lord Rayleigh, in his interesting memoir on "The Mechanical Principles of Flight," demonstrates analytically that "if frictional forces can be neglected, a high speed is all that is required in order to glide without energy. Mr. Chanute‡ has shown, by numerical computation, that De Bruignac's statement may be applied to birds and flying machines moving at limited speeds, say thirty to forty miles an hour; and Professor Langley has concluded from his experiments that the propulsive power of a material soaring plane diminishes with the speed up to at least 66 feet a second, if the edge resistance be left out of the account.

Nearly identical with the expression for power is the equa-

* "Recherches sur la Navigation Aerienne."

† "Experiments in Aerodynamics," Langley.

‡ "Aerial Navigation."

tion for the speed of fall of a horizontal plane having lateral motion. If v_x be its edgewise speed, v_y the speed of fall, then its true speed, v, equals $\sqrt{v_x^2+v_y^2}$, and the angle a, between v and the plane, is determined by the equation $\tan a = v_y / v_x$. Substituting this value of a in equation (d), we have

$$\frac{v_y}{v_x} = \frac{W(1+\sin^2 a)}{2\,k\varsigma\,Av^2(1-\sin^2 a)},$$

which, for high speeds and moderate loading, becomes,

$$v_y = \frac{W(1+\sin^2 a)}{2\,k\varsigma\,Av},$$

since a is small, and v_x is nearly equal to v. Under these conditions the speed of fall varies inversely as the speed of flight, which means that the rate of descent and the power expended may be made indefinitely small by sufficiently increasing the speed. Of course, if the air has an upward trend equal to or greater than v_y the plane will soar continuously on a horizontal or ascending course.

Suppose the gliding plane to dip a degrees below the horizon, and to have a forward resistance. The angle of impact of the air is $\delta = \vartheta - a$, in which $\tan\,\vartheta = v_y / v_x$, as before; and, when steady motion is established, the horizontal component of the air pressure, $\frac{2\,k\varsigma\,Av^2 \sin\delta \sin a}{1+\sin^2\delta}$, just equals the horizontal resistance. Accordingly the plane will glide continuously with the constant component velocities, v_x forward and v_y downward. If, however, the air has an upward trend equal to v_y, or greater, the plane will glide continuously on a horizontal, or ascending course. This is the principle of one kind of soaring practiced by the birds.*

* The Wright brothers report that they can glide continuously down a seven-degree slope at a speed of 18 miles an hour in still air. This means that if the air has an upward trend of $18 \times \sin 7° = 2\frac{1}{4}$ miles an hour, they can glide on a horizontal course indefinitely at a speed of $18 \cos 7° = 17.06$ miles an hour. Hence in a soaring pavilion having a forced

It can be proved, by a slight extension of this argument, that soaring is possible even in a wind that alternately rises and falls.

Many other relations between these variables might be pointed out, but it would be foreign to the purpose of this paper. In passing it may be observed that, for a plane of given size, weight, and speed, it is more than eight hundred times easier to glide through water than through air, since the power varies inversely as the density of the medium. An interesting hydroplane has in fact been constructed by Professor Williams, of Cornell University, and made to "soar" through the water of Lake Cayuga.

In the foregoing discussions it has been assumed that Duchemin's formula is a true expression for the resistance of a smooth plane. This is not true for all planes at all angles, though at small angles it is doubtless true; for at these the formula makes the normal resistance on the oblique plane proportional to the sine of the angle of flight, which is unquestionably true.

So much for a smooth mathematical plane. Let us now consider the effect of surface friction. If the friction per square foot is $f' v^{1.85}$, and the angle of flight is small, equation (a) may be written

$$R = W \tan \alpha + 2 f' A v^{1.85},$$

the other equations remaining practically the same. Substituting in this the value of $\tan \alpha$ in (d') we have

$$R = \frac{W^2}{2 k \ A v^2} + 2 f' A v,^{1.85}$$

$$H = \frac{W^2}{2 k s A v^2} + 2 f' A v^{2.85}.$$

upward draft of, say, 3 miles an hour, a group of machines could glide all day without motive power, rising and falling at pleasure. The power of such a draught is about $\frac{1}{10}$ of a foot pound per second over each square foot of floor surface. Hence two horse-power can maintain such a draught continuously over 5,500 square feet of surface, working at an efficiency of 50 per cent.

These equations show that for high speeds both R and H, that is, both the mileage cost and propulsive power, increase with the velocity. In the limit the mileage cost varies as $v^{1.85}$, while the power varies as $v^{2.85}$. By giving concrete, practical values to W, ς, and A, it is easy to show that both the resistance and power of a soaring plane have minimum values at some small angle, say between one and ten degrees. An example will illustrate this.

Let it be required to find the power necessary to propel a soaring plane one foot square weighing one pound. The soaring angle, α, is given in terms of the velocity by the equation (c) by making $k\varsigma = 0.004$, A being one square foot, W one pound, and v being miles an hour. The resistance may then be computed from the formula

$$R = \tan\alpha + 2f,$$

f being the coefficient of friction, as given by table IV. The power and pounds carried per horse-power are obtained by obvious means. The computations for such a plane are given in table VI.

TABLE VI.

Computed Power and Speed for a Soaring Plane; Area, One Square Foot; Weight, One Pound.

Soaring speed.	Soaring angle.	Computed resistance.			Tow-line power.	Tow-line horse load.
		Drift.	Friction.	Total.		
Mi. hr.	*Deg.*	*Lb.*	*Lb.*	*Lb.*	*Ft. lb. sec.*	*Lbs.*
30	8.25	0.145	0.0170	0.162	7.13	77.1
35	5.94	0.104	0.0226	0.1266	6.51	84.3
40	4.52	0.790	0.0289	0.1079	6.32	86.7
45	3.55	0.0621	0.0360	0.0981	6.39	86.1
50	2.88	0.0500	0.0439	0.0939	6.89	80.2
60	2.03	0.0354	0.0614	0.0962	8.50	64.7
70	1.47	0.0257	0.0814	0.1071	11.00	50.0
80	1.12	0.0195	0.1045	0.1240	14.56	35.8
90	0.88	0.0154	0.1300	0.1454	19.17	28.7
100	0.71	0.0124	0.1584	0.1708	25.00	22.0

The effect of friction is very manifest. Owing to it, the power reaches a minimum at about forty miles an hour. The mileage cost attains its least value at about fifty miles an hour and at an angle of less than three degrees. This latter relation is more clearly shown in figure 9, where the soaring angle and resistance are coördinated. The drift curve is nearly a straight line for the small range of angles plotted, but later turns rapidly upward, becoming infinity and vertical at an angle of ninety degrees. The friction curve begins at infinity, falls rapidly, and becomes zero at a soaring angle

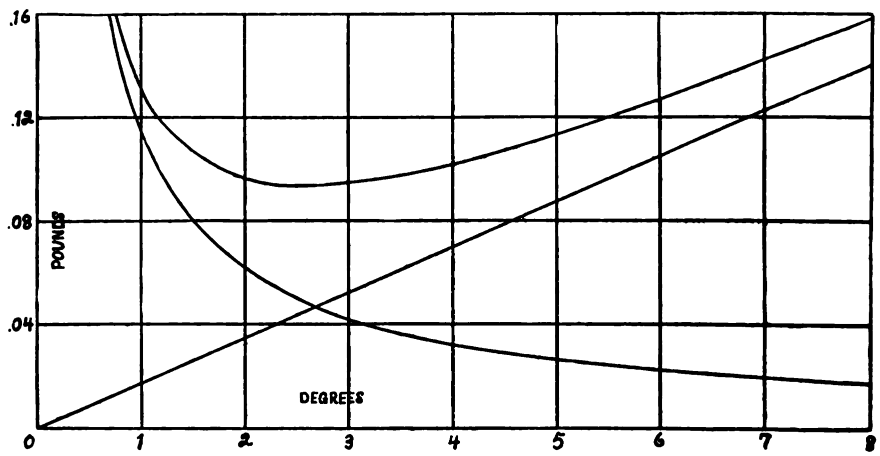

FIG. 9.—*Soaring Angle and Computed Resistances for a Foot Square Plane Weighing One Pound.*

of ninety degrees. The total resistance is asymptotic to the others, and has its minimum at about two and a half degrees. This angle and the corresponding speed are, therefore, the most economical for a thin foot-square soaring plane weighing one pound.

It will be observed in the last column that the maximum weight carried per tow-line horse-power is scarcely ninety pounds. This is very small, but may be increased in several ways: by lightening the load and letting the plane soar at a lower speed; by arching the surface like a vulture's wing; by changing the foot-square plane to a rectangle and towing it long side foremost. The latter device has been tested experimentally by Mr. Langley. His results are presented in table VII, together with corrections for skin-friction made by the present writer.

TABLE VII.

Data for Soaring of 30 × 4.8 Inch Plane; Weight, 500 Grammes.

Soaring angle.	Soaring speed.	Horizontal resistance.	Corrected for friction.	Horse-load.	Corrected for friction.
Deg.	*Ft. sec.*	*Gms.*	*Gms.*	*Lbs.*	*Lbs.*
10	40.7	88	95.04	77	71.3
5	49.8	45	55.34	122	99.2
2	65.6	20	37.69	209	110.9

The last column shows, after correction for friction, that the plane in question may carry about 111 pounds per tow-line horse-power at an angle of two degrees if the edge resistance be neglected. This ratio of weight to power is still not very large, but it may be augmented by arching the plane and by lessening the load. This latter device is being pushed to an extraordinary degree by Dr. A. G. Bell, and it will be very interesting to learn the horse-load of his most efficient kites.

So much for soaring planes. But these are of less substantial interest than arched surfaces, which, besides other advantages, carry a larger burden per horse-power. This fact is duly regarded by modern aeronauticians, both investigators and designers. The Wright brothers, who, after Lilienthal and Chanute, have been especially active and successful in practical flight, claim for their gliding machine a tow-line horse-load as high as 166 pounds at a speed of eighteen miles an hour, and that, too, including the resistance of the entire framing. Mr. Herring has reported similar good results with a flying model. To secure such efficiency with a plane, either square or shaped, as in table VII, the surface load would have to be much less than one pound per square foot. It seems, therefore, most important to the science of flight to determine accurately the lift and drift of arched surfaces for various speeds and angles of advance.

The frictional resistance of arched surfaces can be determined by the method previously employed for wedges. Thus, resolving the friction on any element, ds, of the surface into components at right angles and parallel to the course and integrating the latter component over the surface, we have

$$R = 2\int_0^s f_s\, ds. \frac{dx}{ds} = 2\, fx,$$

in which f is the average unit friction and x the length of surface fore and aft, the width being unity. Hence the frictional resistance of a plane or arched surface, soaring at small angles on a horizontal course, equals the horizontal projection of the surface multiplied by the average unit friction, as given by table IV; that is,

$$R = 2\, f\, S,$$

in which f is the average friction and S is the projected surface.

The reader may like a practical application of the above formula. Take, for example, the Wright brothers' gliding machine of 1902. Its surface measures 5 feet fore and aft, spreads 320 square feet, and meets a total resistance of 30 pounds when soaring 18 miles an hour. By table IV the average friction is 0.00302 pounds per square foot. Hence by the last formula

$$R = 2 \times 0.00302 \times 320 = 1.9 \text{ lbs.}$$

The friction, therefore, seems to be only about six per cent. of the total resistance.

For spindle-shaped hulls, or surfaces of revolution, the skin resistance is computed in a similar way. Thus resolving the friction on an elementary band of radius r and of width ds, into components at right angles and parallel to the axis, and integrating the latter, we have

$$R = 2\,\pi \int r\, f_s\, ds.\, \frac{dx}{ds} = 2\,\pi \int f_s\, r\, dx = \pi\, f\, A,$$

in which f is the average unit friction and A is the area of the longitudinal section of the solid of revolution.

Let us now apply this formula to compute the resistance of the Zeppelin balloon at a speed of, say, 10 feet a second. The balloon is a cylinder, with ogival ends of 1.5 calibers; the length is 390 feet; the diameter is 39 feet. Hence the longitudinal section may be taken as roughly equivalent to a rectangle 39 feet wide by 350 feet long, the area being 13,650 square feet, approximately. Now, the average friction on a plane surface 350 feet long, at 10 feet a second, is 0.000366 of a pound per square foot. Hence, by the formula just established, $R = \pi f A$, the skin-friction on the entire convex surface is $0.000366 \times 13{,}650 \times 3.1416 = 15.7$ pounds. The pure head resistance of prow and stern is about 61.6 pounds, as determined by the writer's unpublished experiments on spindles. Hence the total resistance of the balloon is 77.3 pounds, approximately, and thus the friction is about 20 per cent. of the whole resistance.

The value just computed of the ratio of friction to total resistance seems very small, but that is because the balloon is so blunt-ended. If, however, the cylindrical part be provided with a two-caliber prow and nine-caliber stern, the resistance, figured as in the last paragraph, would be: Friction, 16.5 pounds; pure head resistance, 15.6 pounds; total resistance, 33.1 pounds. Thus the friction is about one-half of the entire resistance.

Analyzing in a similar way the resistance of street cars and railway trains, it is seen that for a short, blunt car the skin-friction is of small consequence; for a long train it may equal, or exceed, the head resistance. When cars are run at a very high speed, as on the Marienfelde Zossen Electric Railway, the chief resistance is due to the air, since the road-bed has to be very smooth and well balanced. In such cases economy would seem to require that the cars should, like navigable balloons, be designed in accordance with established aerodynamic principles.

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FIG 2.