GREEK MATHEMATICS

II
SELECTIONS
ILLUSTRATING THE HISTORY OF
GREEK MATHEMATICS

WITH AN ENGLISH TRANSLATION BY
IVOR THOMAS
FORMERLY SCHOLAR OF ST. JOHN's AND SENIOR DEMY
OF MAGDALEN COLLEGE, OXFORD

IN TWO VOLUMES
II
FROM ARISTARCHUS TO PAPPUS

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XVI. ARISTARCHUS OF SAMOS
XVI. ARISTARCHUS OF SAMOS

(a) General

Aët. i. 15. 5; Doxographi Graeci, ed. Diels 313. 16-18

'Αρίσταρχος Σάμιος μαθηματικός ἀκουστής Στράτωνος φῶς εἶναι τὸ χρῶμα τὸς ὑποκειμένος ἐπιπίπτον.

Archim. Aren. 1, Archim. ed. Heiberg ii. 218. 7-18

'Αρίσταρχος δὲ ὁ Σάμιος ὑποθεσίων τινῶν ἐξ-ἐδωκεν γραφᾶς, ἐν αἷς ἐκ τῶν ὑποκειμένων συμβαίνει τὸν κόσμον πολλαπλάσιον εἴμεν τοῦ νῦν εἰρημένου. ὑποτίθεται γὰρ τὰ μὲν ἀπλανέα τῶν ἀστρων καὶ τὸν ἀλιον μένειν ἀκίνητον, τὰν δὲ γὰρ περιφέρεσθαι περὶ τῶν ἀλιων κατὰ κύκλου περιφέρειαν, ὃς ἐστιν ἐν μέσῳ τῷ δρόμῳ κείμενος, τὰν δὲ τῶν ἀπλανέων ἀστρων σφαῖραν περὶ τὸ

*a Strato of Lampsacus was head of the Lyceum from 288/287 to 270/269 B.C. The next extract shows that Aristarchus formulated his heliocentric hypothesis before Archimedes wrote the Sand-Reckoner, which can be shown to have been written before 216 B.C. From Ptolemy, Syntaxis iii. 2, Aristarchus is known to have made an observation of the summer solstice in 281/280 B.C. He is ranked by Vitruvius, De Architectura i. 1. 17 among those rare men, such as Philolaus, Archytas, Apollonius, Eratosthenes, 2
XVI. ARISTARCHUS OF SAMOS

(a) General

Aëtius i. 15. 5; Doxographi Graeci, ed. Diels 313. 16-18

ARISTARCHUS of Samos, a mathematician and pupil of Strato, held that colour was light impinging on a substratum.

Archimedes, Sand-Reckoner i, Archim. ed. Heiberg ii. 218. 7-18

Aristarchus of Samos produced a book based on certain hypotheses, in which it follows from the premises that the universe is many times greater than the universe now so called. His hypotheses are that the fixed stars and the sun remain motionless, that the earth revolves in the circumference of a circle about the sun, which lies in the middle of the orbit, and that the sphere of the fixed stars, situated

Archimedes and Scopinas of Syracuse, who were equally proficient in all branches of science. Vitruvius, loc. cit. ix. 8. 1, is also our authority for believing that he invented a sun-dial with a hemispherical bowl. His greatest achievement, of course, was the hypothesis that the earth moves round the sun, but as that belongs to astronomy it can be mentioned only casually here. A full and admirable discussion will be found in Heath, Aristarchus of Samos: The Ancient Copernicus, together with a critical text of Aristarchus’s only extant work.
GREEK MATHEMATICS

αὐτὸ κέντρον τῷ ἀλῶ κείμεναν τῷ μεγέθει ταλικαῦταν εἴμεν, ὡστε τὸν κύκλον, καθ’ ὅν τὰν γὰν ὑποτίθεται περιφέρεσθαι, τοιοῦταν ἔχειν ἀναλογίαν ποτὶ τὰν ἀπλανέων ἀποστασίαν, οἰναν ἔχει τὸ κέντρον τᾶς σφαίρας ποτὶ τὰν ἐπιφάνειαν.

Plut. De facie in orbe lunae 6, 922 f–923 a

Καὶ δ’ Λεύκιος γελάσας, “Μόνον,” εἶπεν, “ὦ τάν, μὴ κρίσων ἦμιν ἀσεβείας ἐπαγγείλης, ὡσπερ Ἀρισταρχον ἠτέ θεὶν Κλεάνθης τὸν Σάμιον ἀσεβείας προσκαλεῖσθαι τοὺς Ἔλληνας, ὡς κυνοῦντα τοῦ κόσμου τὴν ἑστίαν, ὅτι τὰ φαινόμενα σφίξειν ἀνὴρ ἐπειράτο, μένειν τὸν ὑποπανὸν ὑποτιθέμενος, ἐξελίττεσθαι δὲ κατὰ λοξοῦ κύκλου τὴν γῆν, ᾗ μα καὶ περὶ τὸν αὐτῆς ἄξονα διώκουμεν.”

(b) Distances of the Sun and Moon


<Ὑποθέσεις1>

α’. Τὴν σελήνην παρὰ τοῦ ἠλίου τὸ φῶς λαμβάνειν.

β’. Τὴν γῆν σημείου τε καὶ κέντρου λόγον ἔχειν πρὸς τὴν τῆς σελήνης σφαίραν.

γ’. Ἡ σελήνη διχότομος ἡμῖν φαίνεται,

1 ὑποθέσεις add. Heath.

* Aristarchus’s last hypothesis, if taken literally, would mean that the sphere of the fixed stars is infinite. All that he implies, however, is that in relation to the distance of the
ARISTARCHUS OF SAMOS

about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve has such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface.\(^a\)

Plutarch, *On the Face in the Moon* 6, 922 f–923 a

Lucius thereupon laughed and said: "Do not, my good fellow, bring an action against me for impiety after the manner of Cleanthes, who held that the Greeks ought to indict Aristarchus of Samos on a charge of impiety because he set in motion the hearth of the universe; for he tried to save the phenomena by supposing the heaven to remain at rest, and the earth to revolve in an inclined circle, while rotating at the same time about its own axis."\(^b\)

(b) Distances of the Sun and Moon


**Hypotheses**

1. The moon receives its light from the sun.
2. The earth has the relation of a point and centre to the sphere in which the moon moves.\(^c\)
3. When the moon appears to us halved, the great fixed stars the diameter of the earth’s orbit may be neglected. The phrase appears to be traditional (\(v.\) Aristarchus’s second hypothesis, *infra*).

\(^a\) Heraclides of Pontus (along with Ephantus, a Pythagorean) had preceded Aristarchus in making the earth revolve on its own axis, but he did not give the earth a motion of translation as well.

\(^b\) Lit. "sphere of the moon."
neuein eis tην ήμετέραν οφιν τον διορίζοντα το τε σκιερόν καὶ το λαμπρόν της σελήνης μέγιστον κύκλον.

δ’. Ὀταν ἡ σελήνη διχότομος ἡμῖν φαίνεται, τότε αὐτήν ἀπέχειν τοῦ ἥλιου ἕλασον τεταρτημορίον τῷ τοῦ τεταρτημορίου τριακοστῷ.

ε’. Τὸ τῆς σκιᾶς πλάτος σελήνων εἶναι δύο.

ς’. Τὴν σελήνην ὑποτείνειν ὑπὸ πεντεκαίδεκατον μέρος ζῳδίου.

Ἐπιλογίζεται οὖν τὸ τοῦ ἥλιου ἀπόστημα ἀπὸ τῆς γῆς τοῦ τῆς σελήνης ἀποστήματος μείζον μὲν ἡ ὀκτωκαιδεκαπλάσιον, ἑλασον δὲ ἡ εἴκοσαπλάσιον, διὰ τῆς περὶ τὴν διχοτομίαν ὑποθέσεως τοῦ αὐτῶν δὲ λόγου ἔχειν τὴν τοῦ ἥλιου διάμετρον πρὸς τὴν τῆς σελήνης διαμετρον. τὴν δὲ τοῦ ἥλιου διάμετρον πρὸς τὴν τῆς γῆς διάμετρον μείζονα μὲν λόγον ἔχειν ἥ ὅν τὰ ιθ πρὸς γ’, ἑλάσονα δὲ ἡ ὅν μὺ πρὸς 5’, διὰ τοῦ εὐρεθέντος περὶ τὰ ἀποστήματα λόγου, τῆς (τε’) περὶ τὴν σκιᾶν ὑποθέσεως, καὶ τοῦ τῆς σελήνην ὑπὸ πεντεκαίδεκατον μέρος ζῳδίου ὑποτείνειν.

Ibid., Prop. 7, cd. Heath 376. 1–380. 28

Τὸ ἀπόστημα δ’ ἀπέχει ὁ ἥλιος ἀπὸ τῆς γῆς τοῦ

1 τε add. Heath.

Lit. “verges towards our eye.” For “verging,” v. vol. i. p. 244 n. a. Aristarchus means that the observer’s eye lies in the plane of the great circle in question. A great circle is a circle described on the surface of the sphere and having the same centre as the sphere; as the Greek implies, a great circle is the “greatest circle” that can be described on the sphere.
circle dividing the dark and the bright portions of
the moon is in the direction of our eye.\textsuperscript{a}

4. When the moon appears to us halved, its
distance from the sun is less than a quadrant by
one-thirtieth of a quadrant.\textsuperscript{b}

5. The breadth of the earth's shadow is that of
two moons.\textsuperscript{c}

6. The moon subtends one-fifteenth part of a sign
of the zodiac.\textsuperscript{d}

It may now be proved that the distance of the sun
from the earth is greater than eighteen times, but less than
twenty times, the distance of the moon—this follows from
the hypothesis about the halved moon; that the
diameter of the sun has the aforesaid ratio to the diameter
of the moon; and that the diameter of the sun has to the
diameter of the earth a ratio which is greater than 19 : 3
but less than 43 : 6—this follows from the ratio dis-
covered about the distances, the hypothesis about
the shadow, and the hypothesis that the moon sub-
tends one-fifteenth part of a sign of the zodiac.

\textit{Ibid.}, Prop. 7, ed. Heath 376. 1–380. 28

\textbf{The distance of the sun from the earth is greater than}

\textsuperscript{a} i.e., is less than 90° by 3°, and so is 87°. The true value
is 89° 50'.

\textsuperscript{b} i.e., the breadth of the earth's shadow where the moon
traverses it during an eclipse. The figure is presumably
based on records of eclipses. Hipparchus made the figure 2\frac{3}{4}
for the time when the moon is at its mean distance, and
Ptolemy a little less than 2\frac{3}{4} for the time when the moon is at
its greatest distance.

\textsuperscript{c} i.e., the angular diameter of the moon is one-fifteenth
of 30°, or 2°. The true value is about \frac{1}{2}°, and in the \textit{Sand-
Reckoner} (Archim. ed. Heiberg ii. 222. 6–8) Archimedes says
that Aristarchus "discovered that the sun appeared to be
about \frac{1}{40}\textsuperscript{th} part of the circle of the Zodiac"; as he believed
that the sun and moon had the same angular diameter he must, therefore, have found the approximately correct angular diameter of $\frac{1}{8}^\circ$ after writing his treatise *On the Sizes and Distances of the Sun and Moon.*
eighteen times, but less than twenty times, the distance of the moon from the earth.

For let A be the centre of the sun, B that of the earth; let AB be joined and produced; let Γ be the centre of the moon when halved; let a plane be drawn through AB and Γ, and let the section made by it in the sphere on which the centre of the sun moves be the great circle AΔE, let AΓ', Γ'B be joined, and let BΓ be produced to Δ.

Then, because the point Γ is the centre of the moon when halved, the angle AΓ'B will be right.
GREEK MATHEMATICS

ΑΓΒ. ἡ ἕχει δὴ ἀπὸ τοῦ Β τῇ ΒΑ πρὸς ὀρθὰς
ἡ ΒΕ. ἐσται δὴ ἡ ἜΔ ορθεῖα τῆς ἘΔΑ
περιφερείας λ' ὑπόκειται γάρ, ὅταν ἡ σελήνη
dιχότομος ἕμιν φαίνεται, ἀπέχειν ἀπὸ τοῦ ἥλιου
ἐλασσόν τεταρτημορίου τῷ τοῦ τεταρτημορίου λ' ὥστε καὶ ἡ ὑπὸ τῶν ΕΒΓ γωνία ὀρθής ἐστὶ λ'.
συμπεπληρώσω φη τὸ ΑΕ παραλληλόγραμμον,
καὶ ἐπεξεύχθω ἡ ΒΖ. ἐσται δὴ ἡ ὑπὸ τῶν ΖΒΕ
γωνία ἡμίσεια ὀρθῆς. τετμήσων ἡ ὑπὸ τῶν
ΖΒΕ γωνία δίχα τῇ ΒΗ εὐθείᾳ; ἡ ἄρα ὑπὸ τῶν
ΗΒΕ γωνία τεταρτον μέρος ἐστὶν ὀρθῆς. ἀλλὰ καὶ
ἡ ὑπὸ τῶν ΔΒΕ γωνία λ' ἐστι μέρος ὀρθῆς· λόγος
ἀρα τῆς ὑπὸ τῶν ΗΒΕ γωνίας πρὸς τὴν ὑπὸ τῶν
ΔΒΕ γωνίαν ἐστίν1 ὃν ἐστιν ὀρθή γωνία ε', τοιοῦτων ἐστὶν
ἡ μὲν ὑπὸ τῶν ΗΒΕ ε', ἡ δὲ ὑπὸ τῶν ΔΒΕ δύο.
καὶ ἐπει ἡ ΗΕ πρὸς τὴν ΕΘ μείζωνα λόγον ἔχει
ἕπερ ἡ ὑπὸ τῶν ΗΒΕ γωνία πρὸς τὴν ὑπὸ τῶν
ΔΒΕ γωνίαν, ἡ ἄρα ΗΕ πρὸς τὴν ΕΘ μείζωνα
λόγον ἔχει ἕπερ τὰ ὃν ἐστὶν ὀρθή ε', τοιοῦτων ἐστὶν
ἡ στὶν ἡ ΒΕ τῇ ΕΖ, καὶ ἐστὶν ὀρθή ἡ πρὸς τῷ Ε,
τὸ ἄρα ἀπὸ τῆς ΖΒ τοῦ ἀπὸ BE διπλάσιον ἐστίν·
ἕπερ δὲ τὸ ἀπὸ ΖΒ πρὸς τὸ ἀπὸ BE, οὕτως ἐστὶ
τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ HE· τὸ ἄρα ἀπὸ ΖΗ
tοῦ ἀπὸ ΗΕ διπλάσιον ἐστί· τὰ δὲ μὲν τῶν κε
ἐλάσσονά ἐστίν ἡ διπλάσια, ὡστε τὸ ἀπὸ ΖΗ πρὸς
tὸ ἀπὸ ΗΕ μείζωνα λόγον ἔχει ἡ ὃν τὰ ἀπὸ τὸ
KE· καὶ ἡ ΖΗ ἄρα πρὸς τὴν ΗΕ μείζωνα λόγον

1 ἐστίν add. Nizze.
2 ἔχει add. Wallis.

* Lit. "circumference," as in several other places in this proposition.
ARISTARCHUS OF SAMOS

From B let BE be drawn at right angles to BA. Then the arc $E\Delta$ will be one-thirtieth of the arc $E\Delta A$; for, by hypothesis, when the moon appears to us halved, its distance from the sun is less than a quadrant by one-thirtieth of a quadrant [Hypothesis 4]. Therefore the angle $EBI$ is also one-thirtieth of a right angle. Let the parallelogram $AE$ be completed, and let $BZ$ be joined. Then the angle $ZBE$ will be one-half of a right angle. Let the angle $ZBE$ be bisected by the straight line $BH$; then the angle $HBE$ is one-fourth part of a right angle. But the angle $\Delta BE$ is one-thirtieth part of a right angle; therefore angle $HBE : \text{angle } \Delta BE = 15 : 2$; for, of those parts of which a right angle contains 60, the angle $HBE$ contains 15 and the angle $\Delta BE$ contains 2.

Now since

$$HE : E\Theta > \text{angle } HBE : \text{angle } \Delta BE,$$

therefore $HE : E\Theta > 15 : 2$.

And since $BE = EZ$, and the angle at $E$ is right, therefore

$$ZB^2 = 2BE^2.$$

But

$$ZB^2 : BE^2 = ZH^2 : HE^2.$$

Therefore

$$ZH^2 = 2HE^2.$$

Now $49 < 2 \cdot 25$,

so that


Therefore

$$ZH : HE > 7 : 5.$$

Aristarchus's assumption is equivalent to the theorem

$$\frac{\tan a}{\tan \beta} > \frac{a}{\beta},$$

where $\beta < a \leq \frac{1}{2}\pi$. Euclid's proof in *Optics* 8 is given in vol. i. pp. 502-505.
ἐχει ἡ ἂν 1 τὰ ἔ προς τὰ ἕ καὶ συνθεντὶ ἡ ΖΕ ἀρα πρὸς τὴν ΕΗ μείζωνα λόγον ἐχει ἡ ἂν τὰ ἔ προς τὰ ἕ, τουτέστων, ἡ ἂν ἂν τὰ 2 λς πρὸς τὰ ἔ ε. ἐδείχθη δὲ καὶ ἡ ΗΕ πρὸς τὴν ΕΘ μείζωνα λόγον ἐχουσα ἡ ἂν τὰ ἔ προς τὰ δύο. δι’ ἵου ἀρα ἡ ΖΕ πρὸς τὴν ΕΘ μείζωνα λόγον ἐχει ἡ ἂν τὰ λς πρὸς τὰ δύο, τουτέστων, ἡ ἂν τὰ ἔ προς α. ἡ ἀρα ΖΕ τῆς ΕΘ μείζων ἐστὶν ἡ Ἴη. ἡ δὲ ΖΕ ἴση ἐστὶν τῇ ΒΕ καὶ ἡ ΒΕ ἀρα τῆς ΕΘ μείζων ἐστὶν ἡ Ἴη. πολλοὶ ἀρα ἡ ΒΗ τῆς ΘΕ μείζων ἐστὶν ἡ Ἴη. ἀλλ’ ὡς ἡ ΘΘ πρὸς τὴν ΘΕ, οὕτως ἐστὶν ἡ ΑΒ πρὸς τὴν ΒΓ, διὰ τῆν ὁμοιότητα τῶν τριγώνων καὶ ἡ ΑΒ ἀρα τῆς ΒΓ μείζων ἐστὶν ἡ Ἴη. καὶ ἐστὶν ἡ μὲν ΑΒ τὸ ἀπόστημα δ’ ἀπέχει δ’ ἡλιος ἀπὸ τῆς γῆς, ἡ δὲ ΓΒ τὸ ἀπόστημα δ’ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς τὸ ἀρα ἀπόστημα δ’ ἀπέχει ὁ ἡλιος ἀπὸ τῆς γῆς τοῦ ἀποστήματος, οὗ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς, μείζον ἐστιν ἡ Ἴη.

Δέγω δὴ ὅτι καὶ ἐλασσον ἡ ἴκ. ἡχθω γὰρ διὰ τοῦ Δ τῇ ΕΒ παράλληλος ἡ ΔΚ, καὶ περὶ τὸ ΔΚΒ τρίγωνον κύκλος γεγράφθω ὁ ΔΚΒ. ἐσται δὴ αὐτοῦ διαμέτρος ἡ ΔΒ, διὰ τὸ ὀρθὴν εἶναι τὴν πρὸς τῷ Κ γωνιάν. καὶ ἐνημοῦσθω ἡ ΒΛ ἐξαγώνου. καὶ ἐστι μὲν ἡ υπὸ τῶν ΔΒΕ γωνία λ’ ἐστίν ὀρθὴς, καὶ ἡ υπὸ τῶν ΒΔΚ ἀρα λ’ ἐστίν ὀρθὴς. ἡ ἄρα ΒΚ περιφέρεια ξ’ ἐστίν τοῦ ὄλου κύκλου. ἐστὶν δὲ καὶ ἡ ΒΛ ἐκτὸν μέρος τοῦ ὄλου κύκλου. ἡ ἄρα ΒΛ περιφέρεια τῆς ΒΚ περιφέρειας ἐστὶν. καὶ ἐχει ἡ ΒΛ περιφέρεια πρὸς τὴν ΒΚ περιφέρειαν μείζωνα λόγον ἥπερ ἡ ΒΛ

1 ὅν add. Wallis. 2 τὰ add. Wallis.
Therefore, *componendo*, $ZE : EH > 12 : 5,$
that is, $ZE : EH > 36 : 15.$

But it was also proved that

$HE : E\Theta > 15 : 2.$

Therefore, *ex aequali,* $ZE : E\Theta > 36 : 2,$
that is, $ZE : E\Theta > 18 : 1.$

Therefore $ZE$ is greater than eighteen times $E\Theta.$ And $ZE$ is equal to $BE.$ Therefore $BE$ is also greater than eighteen times $E\Theta.$ Therefore $BH$ is much greater than eighteen times $\Theta E.$

But $B\Theta : \Theta E = AB : BG,$
by similarity of triangles. Therefore $AB$ is also greater than eighteen times $BG.$ And $AB$ is the distance of the sun from the earth, while $GB$ is the distance of the moon from the earth; therefore the distance of the sun from the earth is greater than eighteen times the distance of the moon from the earth.

I say now that it is less than twenty times. For through $\Delta$ let $\Delta K$ be drawn parallel to $EB,$ and about the triangle $\Delta KB$ let the circle $\Delta KB$ be drawn; its diameter will be $\Delta B,$ by reason of the angle at $K$ being right. Let $BA,$ the side of a hexagon, be fitted into the circle. Then, since the angle $\Delta BE$ is one-thirtieth of a right angle, therefore the angle $\Delta BK$ is also one-thirtieth of a right angle. Therefore the arc $BK$ is one-sixtieth of the whole circle. But $BA$ is one-sixth part of the whole circle.

Therefore $arc BA = 10 \cdot arc BK.$

And the arc $BA$ has to the arc $BK$ a ratio greater

* For the proportion *ex aequali,* v. vol. i. pp. 448-451.
(c) Continued Fractions (?)

*Εχει δὲ καὶ τὰ ἅγια πρὸς, ὅν μείζονα λόγον ἡπερ τὰ πῆ πρὸς μὲ.

Ibid., Prop. 15, ed. Heath 406. 23-24

*Εχει δὲ καὶ ὁ Μ, ἐως πρὸς Μ, ἐφ μείζονα λόγον ἡ ὁν τὰ μγ πρὸς Λζ.

1 πῆ add. Wallis.

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a This is proved in Ptolemy’s Syntaxis i. 10, v. infra, pp. 435-439.

b If \( \frac{7921}{4050} \) is developed as a continued fraction, we obtain the approximation \( 1 + \frac{1}{1 + \frac{1}{21 + \frac{1}{2}}} \), which is \( \frac{88}{45} \). Similarly, if \( \frac{71755875}{61735500} \) or \( \frac{21261}{18292} \) is developed as a continued fraction, we
ARISTARCHUS OF SAMOS

than that which the straight line BA has to the straight line BK.\(^a\)

Therefore \(BA < 10 \cdot BK\).

And \(BA = 2 \cdot BA\).

Therefore \(BA < 20 \cdot BK\).

But \(BA : BK = AB : BG\).

Therefore \(AB < 20 \cdot BG\).

And \(AB\) is the distance of the sun from the earth, while \(BG\) is the distance of the moon from the earth; therefore the distance of the sun from the earth is less than twenty times the distance of the moon from the earth. And it was proved to be greater than eighteen times.

(c) Continued Fractions (?)

Ibid., Prop. 13, ed. Heath 396. 1-2

But 7921 has to 4050 a ratio greater than that which 88 has to 45.

Ibid., Prop. 15, ed. Heath 406. 23-24

But 71755875 has to 61735500 a ratio greater than that which 43 has to 37.\(^b\)

obtain the approximation \(1 + \frac{1}{6} + \frac{1}{6} \) or \(\frac{43}{37}\). The latter result was first noticed in 1823 by the Comte de Fortia D'Urban (Traité d'Aristarque de Samos, p. 186 n. 1), who added: "Ainsi les Grecs, malgré l'imperfection de leur numération, avaient des méthodes semblables aux nôtres." Though these relations are hardly sufficient to enable us to say that the Greeks knew how to develop continued fractions, they lend some support to the theory developed by D'Arcy W. Thompson in Mind, xxxviii. pp. 43-55, 1929.
XVII. ARCHIMEDES
XVII. ARCHIMEDES

(a) General

Tzetzes, Chil. ii. 103-144

'O Ἀρχιμήδης ὁ σοφός, μηχανητὴς ἐκεῖνος, Τῶ γένει Συρακούσιος ἦν, γέρων γεωμέτρης, Χρόνος τε ἐβδομήκοντα καὶ πέντε παρελαύνων, ὡστὶς εἰργάσατο πολλὰς μηχανικὰς δυνάμεις, Καὶ τῇ τρισπάστῳ μηχανῇ χειρὶ λαῖ ἦ καὶ μόνῃ Πεντεμυριομεδίμνων καθελκυσεν ὀλκάδα Καὶ τοῦ Μαρκέλλου στρατηγοῦ ποτε δὲ τῶν Ἀρωμαίων Τῇ Συρακούσῃ κατὰ γῆν προσβάλλοντος καὶ πόντων, Τώνας μὲν πρῶτον μηχαναῖς ἀνείλκυσεν ὀλκάδας Καὶ πρὸς τὸ Συρακούσιον τεῖχος μετεωρίσας Αὐτάνδρους πάλιν τῷ βυθῷ κατεπέμπεν ἀθρόως, Μαρκέλλου δ᾽ ἀποστήσαντος μικρόν τι τὰς ὀλκάδας Ὁ γέρων πάλιν ἀπαντᾶς ποιεῖ Συρακούσίους

* A life of Archimedes was written by a certain Heraclides—perhaps the Heraclides who is mentioned by Archimedes himself in the preface to his book On Spirals (Archim. ed. Heiberg ii. 2. 3) as having taken his books to Dositheus. We know this from two references by Eutocius (Archim. ed. Heiberg iii. 228. 20, Apollon. ed. Heiberg ii. 163. 3, where, however, the name is given as ‘Ἡράκλειος’), but it has not survived. The surviving writings of Archimedes, together with the commentaries of Eutocius of Ascalon (ft. L.D. 520), have been edited by J. L. Heiberg in three volumes of the Teubner series (references in this volume are to the 2nd ed., Leipzig, 1910-1915). They have been put into mathematical notation by T. L. Heath, The Works of Archimedes (Cam-
XVII. ARCHIMEDES

(a) General

Tzetzes, Book of Histories ii. 103-144

Archimedes the wise, the famous maker of engines, was a Syracusan by race, and worked at geometry till old age, surviving five-and-seventy-years; he reduced to his service many mechanical powers, and with his triple-pulley device, using only his left hand, he drew a vessel of fifty thousand medimni burden. Once, when Marcellus, the Roman general, was assaulting Syracuse by land and sea, first by his engines he drew up some merchant-vessels, lifted them up against the wall of Syracuse, and sent them in a heap again to the bottom, crews and all. When Marcellus had withdrawn his ships a little distance, the old man gave all the Syracusans power to lift bridge, 1897), supplemented by The Method of Archimedes (Cambridge, 1922), and have been translated into French by Paul Ver Eecke, Les Œuvres complètes d'Archimède (Brussels, 1921).

* The lines which follow are an example of the "political" (πολιτικός, popular) verse which prevailed in Byzantine times. The name is given to verse composed by accent instead of quantity, with an accent on the last syllable but one, especially an iambic verse of fifteen syllables. The twelfth-century Byzantine pedant, John Tzetzes, preserved in his Book of Histories a great treasure of literary, historical, theological and scientific detail, but it needs to be used with caution. The work is often called the Chiliades from its arbitrary division by its first editor (N. Gerbel, 1546) into books of 1000 lines each—it actually contains 12,674 lines.

* As he perished in the sack of Syracuse in 212 B.C., he was therefore born about 287 B.C.
Unfortunately, the earliest authority for this story is Lucian, *Hipp. 2*: τὸν δὲ (sc. Ἀρχιμήδην) τὰς τῶν πολεμίων τριήρεις καταφλέξαντα τῇ τέχνῃ. It is also found in Galen, *Περὶ κρασ. iii. 2*, and Zonaras xiv. 3 relates it on the authority of Dion Cassius, but makes Proclus the hero of it.

Further evidence is given by Tzetzes, *Chil. xii. 995* and Eutocius (Archim. ed. Heiberg iii. 132. 5-6) that Archimedes wrote in the Doric dialect, but the extant text of his best-known works, *On the Sphere and Cylinder* and the *Measurement of a Circle*, retains only one genuine trace of its original Doric—the form τῆν. Partial losses have occurred in other books, the *Sand-Reckoner* having suffered least. The subject is fully treated by Heiberg, *Quaestiones Archimedea*, pp. 69-94, and in a preface to the second volume of his edition of Archimedes he indicates the words which he has restored to their Doric form despite the manuscripts; his text is adopted in this selection.

The loss of the original Doric is not the only defect in the

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1 πέμποντας Cary, πέμποντα codd.
stones large enough to load a waggon and, hurling them one after the other, to sink the ships. When Marcellus withdrew them a bow-shot, the old man constructed a kind of hexagonal mirror, and at an interval proportionate to the size of the mirror he set similar small mirrors with four edges, moved by links and by a form of hinge, and made it the centre of the sun's beams—its noon-tide beam, whether in summer or in mid-winter. Afterwards, when the beams were reflected in the mirror, a fearful kindling of fire was raised in the ships, and at the distance of a bow-shot he turned them into ashes. In this way did the old man prevail over Marcellus with his weapons. In his Doric \(b\) dialect, and in its Syracusan variant, he declared: "If I have somewhere to stand, I will move the whole earth with my charistion." The hand of an interpolator—often not particularly skilful—can be repeatedly detected, and there are many loose expressions which Archimedes would not have used, and occasional omissions of an essential step in his argument. Sometimes the original text can be inferred from the commentaries written by Eutocius, but these extend only to the books On the Sphere and Cylinder, the Measurement of a Circle, and On Plane Equilibriums. A partial loss of Doric forms had already occurred by the time of Eutocius, and it is believed that the works most widely read were completely recast a little later in the school of Isidorus of Miletus to make them more easily intelligible to pupils.

\(^c\) The instrument is otherwise mentioned by Simplicius (in Aristot. Phys., ed. Diels 1110, 2-5) and it is implied that it was used for weighing: ταύτη δὲ τῇ ἀναλογίᾳ τοῦ κινοῦντος καὶ τοῦ κινουμένου καὶ τοῦ διαστήματος τὸ σταθμιστικὸν ὄργανον τὸν καλούμενον χαριστίων συντήσας ὁ Ἀρχιμήδης ὡς μέχρι παντὸς τῆς ἀναλογίας προχωρούσης ἐκόμπασεν ἐκείνο τὸ "πά βῶ και κωνῶ τὰν γάν." As Tzetzes in another place (Chil. iii. 61: ὁ γὰρ ἀναστῶν μηχανῆ τῇ τρισπάσσω βοῶν: "οὐ πά βῶ καὶ σαλέως τὴν χθόνα") writes of a triple-pulley device in the same connexion, it may be presumed to have been of this nature.
GREEK MATHEMATICS

Οὔτος, κατὰ Διόδωρον, τῆς Συρακούσης ταύτης Προδότου πρὸς τὸν Μάρκελλον ἀθρόως γενομένης, Ἐτε, κατὰ τὸν Δίωνα, ἮΡωμαιος πορθηθείσης, Ἀρτέμιδι τῶν πολιτῶν τὸτε πανυχζόντων, Τοιοουτοτρόπως τεθυκεν ὑπὸ τινος Ῥωμαίου. Ἡν κεκυψός, διάγραμμα μηχανικὸν τι γράφων, Τής δὲ Ῥωμαίος ἐπιστάς εἶλκεν αἴχμαλωτίζων. ὁ δὲ τοῦ διαγράμματος ὅλος ὑπάρχων τότε, Τῆς ὁ καθέλκων οὐκ εἰδὼς, ἔλεγε πρὸς ἐκεῖνον: "‗Απόστηθι, ὁ ἄνθρωπε, τοῦ διαγράμματός μου." Ὡς δὲ εἶλκε τοῦτον συστραφεῖς καὶ γνοὺς Ῥωμαίον εἶναι, Ῥβόα, "τὶ μηχανήμα τῆς τῶν ἔμων μοι δότω." ὁ δὲ Ῥωμαίος πτοπθείς εὐθὺς ἐκεῖνον κτείνει, "Ἀνδρά σαθρὸν καὶ γέροντα, δαμόνιον τοῖς ἔργοις.

Plut. Marcellus xiv. 7–xvii. 7

Καὶ μέντοι καὶ Ἀρχιμήδης, Ἰέρωνι τῷ βασιλεί συγγενῆς ὃν καὶ φίλος, ἐγραφεν ὡς τῇ δοθείσῃ δυνάμει τὸ δοθὲν βάρος κινήσαι δυνατόν ἐστι καὶ νεανιευσάμενος, ὡς φασί, ρώμη τῆς ἀποδείξεως εἰπεν ὡς, εἰ γὴν εἰχέν ἑτέραν, ἐκύψεν ἣν ταύτην μεταβὰς εἰς ἐκείνην. θαυμάσαντος δὲ τοῦ Ἰέρωνος, καὶ δεσθέντος εἰς ἔργον ἐξαγαγεῖν τὸ πρόβλημα καὶ δειξαὶ τὶ τῶν μεγάλων κινούμενον ὑπὸ σμικρᾶς δυνάμεως, ὅλκαδα τριάρμενον τῶν βασιλικῶν πόνω μεγάλῳ καὶ χειρὶ πολλῇ νεολκηθείσαν, ἐμβαλῶν ἄνθρωπον τε πολλοὺς καὶ τὸν συνήθη φόρτον, αὐτὸς ἀπωθεὶ καθήμενος, οὐ μετὰ σπουδῆς, ἀλλὰ

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b The account of Dion Cassius has not survived.
c Zonaras ix. 5 adds that when he heard the enemy were
Whether, as Diodorus\(^a\) asserts, Syracuse was betrayed and the citizens went in a body to Marcellus, or, as Dion \(^b\) tells, it was plundered by the Romans, while the citizens were keeping a night festival to Artemis, he died in this fashion at the hands of one of the Romans. He was stooping down, drawing some diagram in mechanics, when a Roman came up and began to drag him away to take him prisoner. But he, being wholly intent at the time on the diagram, and not perceiving who was tugging at him, said to the man: "Stand away, fellow, from my diagram." \(^c\) As the man continued pulling, he turned round and, realizing that he was a Roman, he cried, "Somebody give me one of my engines." But the Roman, scared, straightway slew him, a feeble old man but wonderful in his works.

Plutarch, *Marcellus* xiv. 7–xvii. 7

Archimedes, who was a kinsman and friend of King Hiero, wrote to him that with a given force it was possible to move any given weight; and emboldened, as it is said, by the strength of the proof, he averred that, if there were another world and he could go to it, he would move this one. Hiero was amazed and besought him to give a practical demonstration of the problem and show some great object moved by a small force; he thereupon chose a three-masted merchantman among the king's ships which had been hauled ashore with great labour by a large band of men, and after putting on board many men and the usual cargo, sitting some distance away and without any special effort, he pulled gently with his hand at coming "παρ κεφαλάν" ἕφη "καὶ μὴ παρὰ γραμμάν"—"Let them come at my head," he said, "but not at my line."
GREEK MATHEMATICS

ηρέμα τῇ χειρὶ σείων ἀρχὴν τινα πολυσπάστου προσηγάγετο λείως καὶ ἀπταίστως καὶ ωσπερ διὰ θαλάττης ἐπιθέουσαν. ἐκπλαγεῖς οὖν ὁ βασιλεὺς καὶ συννόησας τῆς τέχνης τὴν δύναμιν, ἐπεισε τὸν 'Αρχιμήδην ὅπως αὐτῷ τὰ μὲν ἀμυνομένω, τὰ δ' ἐπιχειροῦντι μηχανήματα κατασκευάσῃ πρὸς πᾶσαν ἰδέαν πολυρκίας, οἷς αὐτὸς μὲν οὐκ ἔχρησατο, τοῦ βίου τὸ πλείοντα ἀπόλεμον καὶ πανηγυρικὸν βιώσας, τότε δ' ὑπῆρχε τοῖς Συρακούσιοις εἰς δέον ἡ παρασκευή καὶ μετὰ τῆς παρασκευῆς ὁ δημιουργός.

'Ως οὖν προσέβαλον οἱ Ῥωμαῖοι διχόθεν, ἐκπληξὶς ἦν τῶν Συρακούσιων καὶ συγὴ διὰ δέος, μηδὲν ἄν άνθέξειν πρὸς βίαν καὶ δύναμιν οἰκομένων τοσαύτην. ὁχάσαντος δὲ τὰς μηχανὰς τοῦ 'Αρχιμήδους ἀμα τοὺς μὲν πεζοῖς ἀπίντα τοξεύματα τε παντοδαπὰ καὶ λίθων ὑπέρογκα μεγέθη, ῥοῖζω καὶ τάχει καταφερομένων ἀπίστω, καὶ μηδενὸς ὅλως τὸ βριθὸς στέγοντος ἀθρόους ἀνατρεπόντων τοὺς υποπίπτοντας καὶ τὰς τάξεις συγχεόντων, ταῖς δὲ ναυοῖς ἀπὸ τῶν τειχῶν ἀφνῶν ὑπεραιωροῦμεναι κεραίαι τὰς μὲν υπὸ βριθῶς στηρίζοντος ἀνωθεν ὣθοῦσα κατέδυνοι εἰς βυθὸν, τὰς δὲ χεριὰ σιδηράς ἢ στόμασιν εἰκασμένοις γεράνων ἀναστῶσαι πρώπαθεν ὀρθὰς ἐπὶ πρύμναν ἐβάπτιζον,
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the end of a compound pulley and drew the vessel smoothly and evenly towards himself as though she were running along the surface of the water. Astonished at this, and understanding the power of his art, the king persuaded Archimedes to construct for him engines to be used in every type of siege warfare, some defensive and some offensive; he had not himself used these engines because he spent the greater part of his life remote from war and amid the rites of peace, but now his apparatus proved of great advantage to the Syracusans, and with the apparatus its inventor.

Accordingly, when the Romans attacked them from two elements, the Syracusans were struck dumb with fear, thinking that nothing would avail against such violence and power. But Archimedes began to work his engines and hurled against the land forces all sorts of missiles and huge masses of stones, which came down with incredible noise and speed; nothing at all could ward off their weight, but they knocked down in heaps those who stood in the way and threw the ranks into disorder. Furthermore, beams were suddenly thrown over the ships from the walls, and some of the ships were sent to the bottom by means of weights fixed to the beams and plunging down from above; others were drawn up by iron claws, or crane-like beaks, attached to the prow and were

p. xx, suggests that the vessel, once started, was kept in motion by the system of pulleys, but the first impulse was given by a machine similar to the κοχλιας described by Pappus viii. ed. Hultsch 1066, 1108 ff., in which a cog-wheel with oblique teeth moves on a cylindrical helix turned by a handle.

b Similar stories of Archimedes' part in the defence are told by Polybius viii. 5. 3-5 and Livy xxiv. 34.
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η δι' αντιτόνων ένδον ἐπιστρεφόμεναι καὶ περιαγό-
μεναι τοῖς ὑπὸ τὸ τείχος πεφυκόσι κρημνοῖς καὶ
σκοπέλοις προσήρασσον, ἁμα φθόρῳ πολλῷ τῶν
ἐπιβατῶν συντριβομένων. πολλάκις δὲ μετέωρος
εξαρθείσα ναῦς ἄπο τῆς θαλάσσης δεύρο κάκεισε
περιδυνομένη καὶ κρεμαμένη θέαμα φρυκώδες ἦν,
mέχρι οὖ τῶν ἀνδρῶν ἀπορριφέντων καὶ διασφε-
dονηθέντων κενὴ προσπέσοι τοῖς τείχεσιν ἦ περι-
ολλόθοι τῆς λαβῆς ἀνείσης. ἦν δὲ ὁ Μάρκελλος
ἀπὸ τοῦ ξεύγματος ἐπῆγε μηχανήν, σαμβύκη μὲν
ἐκαλεῖτο δι' ὅμοιότητά τινα σχήματος πρὸς τὸ
μουσικὸν ὄργανον, ἔτι δὲ ἀπώθησεν αὐτῆς προσφερο-
μένης πρὸς τὸ τείχος ἐξήλατο λίθος δεκατάλαντος
ὀλκῆν, εἶτα ἐτερος ἐπὶ τούτῳ καὶ τρίτος, ὅποι
μεν αὐτῆς ἐμπεσόντες μεγάλων κτύπῳ καὶ κλύδων
τῆς μηχανής τὴν τε βάσιν συγκλόσαν καὶ τὸ
γόμφωμα διέσευσαν καὶ διέσπασαν τοῦ ξεύγματος,
ὡστε τὸν Μάρκελλον ἀποροῦμενον αὐτὸν τε ταῖς
ναυοῖν ἀποπλεῖν κατὰ τάχος καὶ τοῖς πεζοῖς ἀνα-
χώρησιν παρεγγυήσαι.

Βουλευομένοις δὲ ἐδοξεῖν αὐτοῖς ἔτι νυκτός, ἂν
δύνωνται, προσμεῖα τοῖς τεῖχεσιν τοὺς γὰρ τόνους,
οἷς χρῆσθαι τῶν Ἀρχιμήδην, ρύμην ἔχοντας
ὑπερπετείς πουῆσεθαι τὰς τῶν βελῶν ἀφέσεις,
ἐγνύθεν δὲ καὶ τελεώς ἀπράκτους εἶναι διάστημα
τῆς πληγῆς οὐκ ἔχούσης. ὁ δ' ἦν, ὡς ἐσοκεν, ἐπὶ
ταῦτα πάλαι παρεσκευασμένος ὄργανων τε συμ-
μέτρουσ πρὸς πᾶν διάστημα κινήσεις καὶ βέλη
βραχέα, καὶ διὰ (τὸ τείχος2) οὐ μεγάλων, πολλῶν

1 αὐτῆ Coraës, αὐτῆς codd.
2 τὸ τεῖχος add. Sintenis ex Polyb.
plunged down on their sterns, or were twisted round and turned about by means of ropes within the city, and dashed against the cliffs set by Nature under the wall and against the rocks, with great destruction of the crews, who were crushed to pieces. Often there was the fearful sight of a ship lifted out of the sea into mid-air and whirled about as it hung there, until the men had been thrown out and shot in all directions, when it would fall empty upon the walls or slip from the grip that had held it. As for the engine which Marcellus was bringing up from the platform of ships, and which was called samuca from some resemblance in its shape to the musical instrument, a while it was still some distance away as it was being carried to the wall a stone ten talents in weight was discharged at it, and after this a second and a third; some of these, falling upon it with a great crash and sending up a wave, crushed the base of the engine, shook the framework and dislodged it from the barrier, so that Marcellus in perplexity sailed away in his ships and passed the word to his land forces to retire.

In a council of war it was decided to approach the walls, if they could, while it was still night; for they thought that the ropes used by Archimedes, since they gave a powerful impetus, would send the missiles over their heads and would fail in their object at close quarters since there was no space for the cast. But Archimedes, it seems, had long ago prepared for such a contingency engines adapted to all distances and missiles of short range, and through openings in the

a The σαμβύκη was a triangular musical instrument with four strings. Polybius (viii. 6) states that Marcellus had eight quinqueremes in pairs locked together, and on each pair a "sambuca" had been erected; it served as a pent-house for raising soldiers on to the battlements.
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dè καὶ συνεχῶν τριμήτων ὄντων', οἱ σκορπίοι βραχύτων μὲν, ἐγγύθεν δὲ πλῆξαι παρεστήκεσαν ἀόρατοι τοὺς πολεμίους.

'Ως οὖν προσέμεξαν οἰόμενοι λανθάνειν, αὖθις αὖ βέλεσι πολλοῖς ἐνυγχάνοντες καὶ πληγαῖς, πετρῶν μὲν ἐκ κεφαλῆς ἐπ' αὐτοὺς φερομένων ὄσπερ πρὸς κάθετον, τοῦ δὲ τείχους τοξεύματα πανταχόθεν ἀναπέμποντος, ἀνεχώρουν ὀπίσω. κάνταθα πάλιν αὐτῶν εἰς μῆκος ἐκτεταγμένων, βελῶν ἐκθεόντων καὶ καταλαμβανόντων ἀπιόντας ἐγίνετο πολὺς μὲν αὐτῶν φθόρος, πολὺς δὲ τῶν νεὼν συγκρουσμὸς, οὐδὲν ἀντιδρᾶσαι τοὺς πολεμίους δυναμένων. τὰ γὰρ πλεῖστα τῶν ὀργάνων ὑπὸ τὸ τείχος ἐσκευο-ποίητο τῷ Ἀρχιμήδε, καὶ θεομαχοῦσιν ἐφ' ἐκείσαν οἱ Ἰωμαῖοι, μυρίων αὐτοῖς κακῶν εἰς ἀφανοὺς ἐπιχειρομένων.

Οὕ μὴν ἄλλ' ὁ Μάρκελλος ἀπέφυγε τε καὶ τοὺς σὺν ἑαυτῷ σκώπτων τεχνίτας καὶ μηχανοτοιοῦς ἐλεγεν. "οὐ παυσόμεθα πρὸς τὸν γεωμετρικὸν τοῦτον Βριάρεων πολεμοῦντες, δε ταῖς μὲν ναυαῖς ἠμῶν κυαθίζει ἐκ τῆς θαλάσσης, τὴν δὲ σαμβύκτην ῥατίζων μετ' αἰσχύνης ἐκβέβληκε, τοὺς δὲ μυθικοὺς ἐκατόγχειρας ὑπεραιρεὶ τοσαῦτα βάλλων ἀμα βέλη καθ' ἠμῶν;" τῷ γὰρ ὀντι πάντες οἱ λοιποὶ Συρακούσιοι σῶμα τῆς Ἀρχιμήδου παρασκευῆς ἔσαν, ἡ δὲ κυνόσα πάντα καὶ στρέφουσα ψυχὴ μία, τῶν μὲν ἄλλων ὀπλῶν ἀτρέμα κειμένων, μόνοις δὲ τοῖς ἐκείνου τότε τῆς πόλεως χρωμένης καὶ πρὸς ἀμυναν καὶ πρὸς ἀσφάλειαν. τέλος δὲ τοὺς Ἰωμαῖους οὕτω περιφόβους γεγονότας ὅρῶν ὁ Μάρκελλος ὡστ', εἰ καλύσων ἢ ἔνυλον ὑπὲρ τοῦ

1 ὄντων add. Sintenis ex Polyb.
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wall, small in size but many and continuous, short-ranged engines called scorpions could be trained on objects close at hand without being seen by the enemy. When, therefore, the Romans approached the walls, thinking to escape notice, once again they were met by the impact of many missiles; stones fell down on them almost perpendicularly, the wall shot out arrows at them from all points, and they withdrew to the rear. Here again, when they were drawn up some distance away, missiles flew forth and caught them as they were retiring, and caused much destruction among them; many of the ships, also, were dashed together and they could not retaliate upon the enemy. For Archimedes had made the greater part of his engines under the wall, and the Romans seemed to be fighting against the gods, inasmuch as countless evils were poured upon them from an unseen source.

Nevertheless Marcellus escaped, and, twitting his artificers and craftsmen, he said: “Shall we not cease fighting against this geometrical Briareus, who uses our ships like cups to ladle water from the sea, who has whipped our sambuca and driven it off in disgrace, and who outdoes all the hundred-handed monsters of fable in hurling so many missiles against us all at once?” For in reality all the other Syracusans were only a body for Archimedes’ apparatus, and his the one soul moving and turning everything: all other weapons lay idle, and the city then used his alone, both for offence and for defence. In the end the Romans became so filled with fear that, if they saw a little piece of rope or of wood projecting over

2 ταῖς μὲν ναυσίν . . . ἐπιλίζον ἀν ἀνωνύμων καταλέγον πρὸς τὴν θάλασσαν παίζων codd.
Τηλικούτων μέντοι φρόνημα καὶ βάθος ψυχῆς καὶ τοσοῦτον ἐκέκτητο θεωρημάτων πλοῦτον Ἀρχιμήδης ὡστε, ἐφ' ὦς ὄνομα καὶ δόξαν οὐκ ἀνθρωπίνης, ἀλλὰ δαμονίων τινὸς ἐσχε συνέσεως, μηθὲν ἔθελῃσαι σύγγραμμα περὶ τούτων ἀπολιπεῖν, ἀλλὰ τὴν περὶ τὰ μηχανικὰ πραγματείαν καὶ πᾶσαν ἀλλοι τέχνην χρείας ἐφαπτομένῃ ἀγενῆ καὶ βάναυσον ἡγησάμενος, εἰς ἐκεῖνα καταθέσθαι μόνα τὴν αὐτοῦ φιλοτιμίαν οἷς τὸ καλὸν καὶ περιττὸν ἄμυγες τοῦ ἀναγκαῖον πρόσεστιν, ἀσύγκριτα μὲν ὄντα τοῖς ἄλλοις, ἔρων δὲ παρέχοντα πρὸς τὴν ὕλην τῇ ἀποδείξει, τῆς μὲν τὸ μέγεθος καὶ τὸ κάλλος, τῆς δὲ τὴν ἀκρίβειαν καὶ τὴν δύναμιν ὑπερφυῆ παρεχομένης· οὐ γὰρ ἐστὶν ἐν γεωμετρίᾳ χαλεπωτέρας καὶ βαρυτέρας ὑποθέσεις ἐν ἀπλούστεροις λαβεῖν καὶ καθαρωτέροις στοιχείοις γραφομένας. καὶ τοῦθ' οἱ μὲν εὐφυῆ τοῦ ἀνδρὸς προσάπτουσι, οἱ δὲ ὑπερβολῇ τινὶ πόνῳ νομίζουσιν ἀπὸνως πεποιημένως καὶ ῥάδιως ἔκαστον ἐνικὸς γεγονέναι. ξητῶν μὲν γὰρ οὖκ ἀν τις εὕροι δι' αὐτοῦ τὴν ἀπόδειξιν, ἀμα δὲ τῇ μαθῆση παρίσταται δόξα τοῦ κἂν αὐτοῦ εὐρείν· οὕτω λείαν ὅδον ἀγεῖ καὶ ταχείαν ἐπὶ τὸ δεικνύμενον. οὐκον οὐδὲ ἀπιστήσαι τοὺς περὶ αὐτοῦ λεγομένοις ἔστιν, ὡς ὑπ' οἰκείας ὁτί τινος καὶ συνοίκου θελγόμενος ἂει σειρήνος ἐλέληστο καὶ σίτου καὶ θεραπείας σώματος ἔξ- ἐλειπε, βιὰ δὲ πολλάκις ἐλκόμενος ἐπ' ἀλειμμα καὶ
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the wall, they cried, "There it is, Archimedes is training some engine upon us," and fled; seeing this Marcellus abandoned all fighting and assault, and for the future relied on a long siege.

Yet Archimedes possessed so lofty a spirit, so profound a soul, and such a wealth of scientific inquiry, that although he had acquired through his inventions a name and reputation for divine rather than human intelligence, he would not deign to leave behind a single writing on such subjects. Regarding the business of mechanics and every utilitarian art as ignoble and vulgar, he gave his zealous devotion only to those subjects whose elegance and subtlety are untrammelled by the necessities of life; these subjects, he held, cannot be compared with any others; in them the subject-matter vies with the demonstration, the former possessing strength and beauty, the latter precision and surpassing power; for it is not possible to find in geometry more difficult and weighty questions treated in simpler and purer terms. Some attribute this to the natural endowments of the man, others think it was the result of exceeding labour that everything done by him appeared to have been done without labour and with ease. For although by his own efforts no one could discover the proof, yet as soon as he learns it, he takes credit that he could have discovered it: so smooth and rapid is the path by which he leads to the conclusion. For these reasons there is no need to disbelieve the stories told about him—how, continually bewitched by some familiar siren dwelling with him, he forgot his food and neglected the care of his body; and how, when he was dragged by main force, as often happened, to the

1 ἄγεῖ Bryan, ἄγεν codd.
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λοντρόν, ἐν ταῖς ἐσχάραις ἐγραφεῖ σχήματα τῶν γεωμετρικῶν, καὶ τοῦ σώματος ἀληθιμένου διήγε τῷ δακτύλῳ γραμμάς, ὑπὸ ἠδονῆς μεγάλης κάτοχος ὅν καὶ μουσόληπτος ἀλήθως. πολλῶν δὲ καὶ καλῶν εὐρετῆς γεγονός λέγεται τῶν φιλῶν δευ-θήναι καὶ τῶν συγγενῶν ὅπως αὐτοῦ μετὰ τὴν τελευτὴν ἐπιστήσωσι τῷ τάφῳ τὸν περιλαμβάνοντα τὴν σφαίραν ἐντὸς κύλινδρον, ἐπιγράμματος τὸν λόγον τῆς ὑπορεχῆς τοῦ περιέχοντος στερεοῦ πρὸς τὸ περιεχόμενον.

Ibid. xix. 4-6

Μάλιστα δὲ τὸ Ἀρχιμῆδος πάθος ἢνίασε Μάρκελλον. ἔτυχε μὲν γὰρ αὐτὸς τι καθ’ ἐαυτὸν ἀνασκοπῶν ἐπὶ διαγράμματος καὶ τῇ θεωρίᾳ δεδωκὼς ἀμα τὴν τε διάνοιαν καὶ τὴν πρόσοψιν οὐ προῆσθετο τὴν καταδρομὴν τῶν Ἐρωμαίων οὐδὲ τὴν ἀλωσιν τῆς πόλεως, ἄφων δὲ ἐπιστάντος αὐτῆς στρατιώτου καὶ κελεύοντος ἀκολουθεῖν πρὸς Μάρκελλον οὐκ ἐβούλετο πρὶν ἢ τελέσα τὸ πρόβλημα καὶ καταστῆσαι πρὸς τὴν ἀπόδειξιν. ὅ δὲ ὀργισθεῖς καὶ σπασάμενος τὸ ξίφος ἀνείλεν αὐτοῦ. ἔτεροι μὲν οὖν λέγουσιν ἐπιστήμα τις εὐθύς ὅς ἀποκτενοῦσα τοῦ Ἐρωμαίον, ἐκείνων δ’ ἰδόντα δεύσασι καὶ ἀντιβολέων ἀναμείνω βραχὺν χρόνον, ὡς μὴ καταλήπτη τὸ ξητούμενον ἀτέλεις καὶ ἀθεώρητων, τὸν δὲ οὐ φρονίσασα διαχρῆ-σασθαι. καὶ τρίτος ἔστι λόγος, ὡς κομίζοντι πρὸς Μάρκελλον αὐτῷ τῶν μαθηματικῶν ὀργάνων σκιόθηρα καὶ σφαίρας καὶ γωνίας, αὐτὰ ἔναρμόττει

* Cicero, when quaestor in Sicily, found this tomb over-
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place for bathing and anointing, he would draw geometrical figures in the hearths, and draw lines with his finger in the oil with which his body was anointed, being overcome by great pleasure and in truth inspired of the Muses. And though he made many elegant discoveries, he is said to have besought his friends and kinsmen to place on his grave after his death a cylinder enclosing a sphere, with an inscription giving the proportion by which the including solid exceeds the included.²

*Ibid.* xix. 4-6

But what specially grieved Marcellus was the death of Archimedes. For it chanced that he was alone, examining a diagram closely; and having fixed both his mind and his eyes on the object of his inquiry, he perceived neither the inroad of the Romans nor the taking of the city. Suddenly a soldier came up to him and bade him follow to Marcellus, but he would not go until he had finished the problem and worked it out to the demonstration. Thereupon the soldier became enraged, drew his sword and dispatched him. Others, however, say that the Roman came upon him with drawn sword intending to kill him at once, and that Archimedes, on seeing him, besought and entreated him to wait a little while so that he might not leave the question unfinished and only partly investigated; but the soldier did not understand and slew him. There is also a third story, that as he was carrying to Marcellus some of his mathematical instruments, such as sundials, spheres and
grown with vegetation, but still bearing the cylinder with the sphere, and he restored it (*Tusc. Disp.* v. 64-66). The theorem proving the proportion is given *infra*, pp. 124-127.
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to tou hliou megéthos pros tyn ópyyn, stratwta peiruxontes kai xhpsiôn en tò teûchei doxantnes férein apékteinaven. òti méntoi Márkellos ÷lýnse kai tòn autóxeira toû anadròs apestráfhi kataûper énagh, toûs de oikeîous ánveurôn étîmhsen, ómologeîta.

Papp. Coll. viii. 11. 19, ed. Hultsch 1060. 1-4

Tìs autìs de èstw thewriàs to dothèn báròs tì dotheióu dunamei kînhsai: tòouto gar 'Arxhmìdous mévn éýrhma [légetai]1 µhxanikôn, êf' ò légetai eîrnhkénav: "dòs moî (φησι) poû sw kai kìnî tìn ýnh."

Diod. Sic. i. 34. 2

Potaímoxwstos gar oûsa kai katárrutos pol-loûs kai pantadapous èkfére karpoiûs, toû mévn potamou dià tìn kat' ètos ánábasin nearavn ilûn ðei kataxéontos, tûn ð' anðrwotûn rádîws èppasan árdeonîntwn dià tûnos µhxanhês, ðn èppenòhsie mévn 'Arxhmìdhês õ Sùrakósios, ónomyaíte toû sêmmatoû koxliàs.

Ibid. v. 37. 3

To pânthn paraðoûstatoû, áparrûtouû tás rûseis tûn ùdâtûn toûs Aìgnutpiakoûs legeoménoûs koxliàs, òûs 'Arxhmìdhês õ Sùrakósios èdren, óste parébalen eîs Aìgnuttoû.

1 légetai om. Hultsch.

a Diodorus is writing of the island in the delta of the Nile.
b It may be inferred that he studied with the successors of Euclid at Alexandria, and it was there perhaps that he made the acquaintance of Conon of Samos, to whom, as
angles adjusted to the apparent size of the sun, some soldiers fell in with him and, under the impression that he carried treasure in the box, killed him. What is, however, agreed is that Marcellus was distressed, and turned away from the slayer as from a polluted person, and sought out the relatives of Archimedes to do them honour.

Pappus, Collection viii. 11. 19, ed. Hultsch 1060. 1-4

To the same type of inquiry belongs the problem: *To move a given weight by a given force.* This is one of Archimedes' discoveries in mechanics, whereupon he is said to have exclaimed: "Give me somewhere to stand and I will move the earth."

Diodorus Siculus i. 34. 2

As it is made of silt watered by the river after being deposited, it bears an abundance of fruits of all kinds; for in the annual rising the river continually pours over it fresh alluvium, and men easily irrigate the whole of it by means of a certain instrument conceived by Archimedes of Syracuse, and which gets its name because it has the form of a spiral or screw.

*Ibid.* v. 37. 3

Most remarkable of all, they draw off streams of water by the so-called Egyptian screws, which Archimedes of Syracuse invented when he went by ship to Egypt.  

the preface to his books *On the Sphere and Cylinder* shows, he used to communicate his discoveries before publication, and Eratosthenes of Cyrene, to whom he sent the *Method* and probably the *Cattle Problem.*
Archimedis vero cum multa miranda inventa et varia fuerint, ex omnibus etiam infinita sollertia id quod exponam videtur esse expressum. Nimium Hiero Syracusis auctus regia potestate, rebus bene gestis cum auream coronam votivam diis immortalibus in quodam fano constituisset ponendam, manupretio locavit faciendum et aurum ad sacoma appendit re-demptori. Is ad tempus opus manu factum subtiliter regi adprobavit et ad sacoma pondus coronae visus est praestitisse. Posteaquam indicium est factum dempto auro tantundem argenti in id coronarium opus admixtum esse, indignatus Hiero se contemptum esse neque inveniendum qua ratione id furtum deprehenderet, rogavit Archimeden uti insumeret sibi de eo cogitationem. Tunc is cum haberet eius rei curam, casu venit in balineum ibique cum in solium descendideret, animadvertit quantum corporis sui in eo insideret tantum aquae extra solium effluere. Idque cum eius rei rationem explicationis ostendisset, non est moratus sed exsiluit gaudio motus de solio et nudus vadens domum versus significabat clara voce invenisse quod quaereret. Nam currens identidem gaece clamabat εὑρηκα εὑρηκα.

Tum vero ex eo inventionis ingressu duas fecisse dicitur massas aequo pondere quo etiam fuerat corona, unam ex auro et alteram ex argento. Cum ita fecisset, vas amplum ad summa labra implevit

* "I have found, I have found."
Archimedes made many wonderful discoveries of different kinds, but of all these that which I shall now explain seems to exhibit a boundless ingenuity. When Hiero was greatly exalted in the royal power at Syracuse, in return for the success of his policy he determined to set up in a certain shrine a golden crown as a votive offering to the immortal gods. He let out the work for a stipulated payment, and weighed out the exact amount of gold for the contractor. At the appointed time the contractor brought his work skilfully executed for the king's approval, and he seemed to have fulfilled exactly the requirement about the weight of the crown. Later information was given that gold had been removed and an equal weight of silver added in the making of the crown. Hiero was indignant at this disrespect for himself, and, being unable to discover any means by which he might unmask the fraud, he asked Archimedes to give it his attention. While Archimedes was turning the problem over, he chanced to come to the place of bathing, and there, as he was sitting down in the tub, he noticed that the amount of water which flowed over the tub was equal to the amount by which his body was immersed. This indicated to him a means of solving the problem, and he did not delay, but in his joy leapt out of the tub and, rushing naked towards his home, he cried out with a loud voice that he had found what he sought. For as he ran he repeatedly shouted in Greek, *heureka, heureka.*

Then, following up his discovery, he is said to have made two masses of the same weight as the crown, the one of gold and the other of silver. When he had so done, he filled a large vessel right up to the brim
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aqua, in quo demisit argenteam massam. Cuius quanta magnitudo in vase depressa est, tantum aquae effluxit. Ita exempta massa quanto minus factum fuerat refudit sextario mensus, ut eodem modo quo prius fuerat ad labra aequaretur. Ita ex eo invenit quantum pondus argenti ad certam aquae mensuram responderet.

Cum id expertus esset, tum auream massam similiter pleno vase demisit et ea exempta eadem ratione mensura addita invenit deesse aquae non tantum sed minus, quanto minus magno corpore eodem pondere auri massa esset quam argenti. Postea vero repleto vase in eadem aqua ipsa corona demissa invenit plus aquae defluxisse in coronam quam in auream eodem pondere massam, et ita ex eo quod defuerit plus aquae in corona quam in massa, ratiocinatus deprehendit argenti in auro mixtionem et manifestum furtum redemptoris.

* The method may be thus expressed analytically.

Let $w$ be the weight of the crown, and let it be made up of a weight $w_1$ of gold and a weight $w_2$ of silver, so that $w = w_1 + w_2$.

Let the crown displace a volume $v$ of water.

Let the weight $w$ of gold displace a volume $v_1$ of water; then a weight $w_1$ of gold displaces a volume $\frac{w_1}{w} \cdot v_1$ of water.

Let the weight $w$ of silver displace a volume $v_2$ of water;
with water, into which he dropped the silver mass. The amount by which it was immersed in the vessel was the amount of water which overflowed. Taking out the mass, he poured back the amount by which the water had been depleted, measuring it with a pint pot, so that as before the water was made level with the brim. In this way he found what weight of silver answered to a certain measure of water.

When he had made this test, in like manner he dropped the golden mass into the full vessel. Taking it out again, for the same reason he added a measured quantity of water, and found that the deficiency of water was not the same, but less; and the amount by which it was less corresponded with the excess of a mass of silver, having the same weight, over a mass of gold. After filling the vessel again, he then dropped the crown itself into the water, and found that more water overflowed in the case of the crown than in the case of the golden mass of identical weight; and so, from the fact that more water was needed to make up the deficiency in the case of the crown than in the case of the mass, he calculated and detected the mixture of silver with the gold and the contractor's fraud stood revealed.  

then a weight \( w_2 \) of silver displaces a volume \( \frac{w_2}{w} \cdot v_2 \) of water.

It follows that

\[
v = \frac{w_1}{w} \cdot v_1 + \frac{w_2}{w} \cdot v_2 \]

\[
= \frac{w_1 v_1 + w_2 v_2}{w_1 + w_2},
\]

so that

\[
\frac{w_1}{w_2} = \frac{v_2 - v}{v - v_1}.
\]

For an alternative method of solving the problem, \( v. \ infra, \) pp. 248-251.
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(b) **Surface and Volume of the Cylinder and Sphere**

Archim. *De Sphaera et Cyl.* i., Archim. ed. Heiberg i. 2–132. 3

'Αρχιμήδης Δοσιθέω χαίρειν

Πρότερον μέν ἀπέσταλκά σοι τῶν ὑφ’ ἡμῶν τεθεωρημένων γράψας μετὰ ἀποδείξεως, ὅτι πᾶν τμῆμα τὸ περιεχόμενον ὑπὸ τε εὐθείας καὶ ὀρθογωνίου κώνου τομῆς ἐπίτριτον ἐστὶ τριγώνου τοῦ βάσιν τὴν αὐτὴν ἔχοντος τῷ τμῆματι καὶ ύψους ἰσον. ὤστερον δὲ ἡμῖν ὑποπεσόντων θεωρημάτων ἀξίων λόγου πεπραγματεύμεθα περὶ τὰς ἀποδείξεις αὐτῶν. ἔστων δὲ τάδε πρώτων μέν, ὅτι πάσης σφαίρας ἡ ἐπιφάνεια τετραπλασία ἐστὶν τοῦ μεγίστου κύκλου τῶν ἐν αὐτῇ ἔπειτα δὲ, ὅτι πάντος τμῆματος σφαίρας τῇ ἐπιφάνεια ἱσος ἐστὶ κύκλος, οὗ ἡ ἐκ τοῦ κέντρου ἴση ἐστὶ τῇ εὐθείᾳ τῇ ἀπὸ τῆς κορυφῆς τοῦ τμῆματος ἀγομένη ἐπὶ τὴν περιφέρειαν τοῦ κύκλου, ὅς ἐστι βάσις τοῦ τμῆματος.

1 ἀξίων λόγου cod., ἀνελέγκτων coni. Heath.

* The chief results of this book are described in the prefatory letter to Dositheus. In this selection as much as possible is given of what is essential to finding the proportions between the surface and volume of the sphere and the surface and volume of the enclosing cylinder, which Archimedes regarded as his crowning achievement (*supra*, p. 32). In the case of the surface, the whole series of propositions is reproduced so that the reader may follow in detail the majestic chain of reasoning by which Archimedes, starting from seemingly remote premises, reaches the desired conclusion; in the case of the volume only the final proposition (34) can be given, for reasons of space, but the reader will be able to prove the omitted theorems for himself. *Pari passu* with 40
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(b) SURFACE AND VOLUME OF THE CYLINDER AND SPHERE

Archimedes. On the Sphere and Cylinder i., Archim. ed. Heiberg i. 2-132. 3

Archimedes to Dositheus greeting

On a previous occasion I sent you, together with the proof, so much of my investigations as I had set down in writing, namely, that any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle having the same base as the segment and equal height. Subsequently certain theorems deserving notice occurred to me, and I have worked out the proofs. They are these: first, that the surface of any sphere is four times the greatest of the circles in it; then, that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of the circle which is the base of the segment; and, this demonstration, Archimedes finds the surface and volume of any segment of a sphere. The method in each case is to inscribe in the sphere or segment of a sphere, and to circumscribe about it, figures composed of cones and frusta of cones. The sphere or segment of a sphere is intermediate in surface and volume between the inscribed and circumscribed figures, and in the limit, when the number of sides in the inscribed and circumscribed figures is indefinitely increased, it would become identical with them. It will readily be appreciated that Archimedes' method is fundamentally the same as integration, and on p. 116 n. 6 this will be shown trigonometrically.

* This is proved in Props. 17 and 24 of the Quadrature of the Parabola, sent to Dositheus of Pelusium with a prefatory letter, v. pp. 228-243, infra.

* De Sphaera et Cyl. i. 30. "The greatest of the circles," here and elsewhere, is equivalent to "a great circle."

* Ibid. i. 42, 43.
πρὸς δὲ τούτοις, ὅτι πάσης σφαίρας ὁ κύλινθρος ὁ βάσις μὲν ἐξων ἢσθη τῷ μεγίστῳ κύκλῳ τῶν ἐν τῇ σφαίρᾳ, ὕψος δὲ ἢσθη τῇ διαμέτρῳ τῆς σφαίρας αὐτὸς τῇ ἡμιολίθῳ ἐστὶν τῆς σφαίρας, καὶ ἡ ἐπι-φάνεια αὐτοῦ τῆς ἐπιφανείας τῆς σφαίρας. ταύτα δὲ τὰ συμπτώματα τῇ φύσει προσήρχειν περὶ τὰ εἰρημένα σχήματα, ἦγονείτο δὲ ὑπὸ τῶν πρὸ ἡμῶν περὶ γεωμετρίαν ἀνεστραμμένων ύδενος αὐτῶν ἐπινενοηκότος, ὅτι τούτων τῶν σχημάτων ἐστὶν συμμετρία. . . εξέσται δὲ περὶ τούτων ἐπι-σκέψασθαι τοῖς δυνησμένοις. ὥφειλε μὲν ὡς Κόνωνος ἐτὶ ξύντος ἐκδίδοθαι ταύτα· τῇν γὰρ ύπολαμβάνομεν που μάλιστα ἄν δύνασθαι κατα-
νοῆσαι ταύτα καὶ τὴν ἁρμόζουσαν ὑπὲρ αὐτῶν ἀπόφασιν ποιήσασθαι· δοκιμάζοντες δὲ καλῶς ἔχειν μεταδιδόναι τοῖς οἰκείοις τῶν μαθημάτων ἀποστέλλομεν σοι τὰς ἀποδείξεις ἀναγράφαντες, ὑπὲρ δὲν εξέσται τοῖς περὶ τὰ μαθήματα ἀναστρε-φομένοις ἐπισκέψασθαι. ἐρρωμένως.
Γράφονται πρῶτον τὰ τὰ ἀξιώματα καὶ τὰ λαμβα-νόμενα εἰς τὰς ἀποδείξεις αὐτῶν.

'Αξιώματα

α'. Εἰσὶ τινὲς ἐν ἐπιτεδὼ καμπύλαι γραμμαὶ πεπερασμέναι, αἱ τῶν τὰ πέρατα ἐπιζευγνυούσων αὐτῶν εὐθειῶν ἦτοι ὄλαι ἐπὶ τὰ αὐτὰ εἰσὶν ἢ οὐδὲν ἔχουσιν ἐπὶ τὰ ἔτερα.

β'. Ἐπὶ τὰ αὐτὰ δὴ κοίλην καλῶ τὴν τοιαύτην γραμμῆν, εὖ ἢ ἐὰν δύο σχημεῖν λαμβανομένων

* De Sphaera et Cyl. i. 34 coroll. The surface of the cylinder here includes the bases.
further, that, in the case of any sphere, the cylinder having its base equal to the greatest of the circles in the sphere, and height equal to the diameter of the sphere, is one-and-a-half times the sphere, and its surface is one-and-a-half times the surface of the sphere. Now these properties were inherent in the nature of the figures mentioned, but they were unknown to all who studied geometry before me, nor did any of them suspect such a relationship in these figures. But now it will be possible for those who have the capacity to examine these discoveries of mine. They ought to have been published while Conon was still alive, for I opine that he more than others would have been able to grasp them and pronounce a fitting verdict upon them; but, holding it well to communicate them to students of mathematics, I send you the proofs that I have written out, which proofs will now be open to those who are conversant with mathematics. Farewell.

In the first place, the axioms and the assumptions used for the proofs of these theorems are here set out.

AXIOMS

1. There are in a plane certain finite bent lines which either lie wholly on the same side of the straight lines joining their extremities or have no part on the other side.

2. I call concave in the same direction a line such that, if any two points whatsoever are taken on it, either

*b In the omitted passage which follows, Archimedes compares his discoveries with those of Eudoxus; it has already been given, vol. i. pp. 408-411.
*c These so-called axioms are more in the nature of definitions.
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όποιωνοιν αἱ μεταξὺ τῶν σημειῶν εὐθεῖαι ἦτοι πᾶσαι ἐπὶ τὰ αὐτὰ πίπτουσιν τῆς γραμμῆς, ἥ τινες μὲν ἐπὶ τὰ αὐτὰ, τινὲς δὲ κατʼ αὐτῆς, ἐπὶ τὰ ἔτερα δὲ μηθείμα.

γ'. Ὁμοίως δὴ καὶ ἐπιφάνειαι τινὲς εἰσὶν πεπερασμέναι, αὐταὶ μὲν οὐκ ἐν ἐπιπέδῳ, τὰ δὲ πέρατα ἔχουσαι ἐν ἐπιπέδῳ, αἱ τοῦ ἐπιπέδου, ἐν οἷς τὰ πέρατα ἔχουσιν, ήτοι ὅλαι ἐπὶ τὰ αὐτὰ ἔσονται ἡ οὐδὲν ἔχουσιν ἐπὶ τὰ ἔτερα.

δ'. Ἐπὶ τὰ αὐτὰ δὴ κοίλας καλῶ τὰς τουαύτας ἐπιφανείας, ἐν αἷς ἀν δύο σημείων λαμβανομένων αἱ μεταξὺ τῶν σημείων εὐθείαι ἦτοι πᾶσαι ἐπὶ τὰ αὐτὰ πίπτουσιν τῆς ἐπιφανείας, ἥ τινες μὲν ἐπὶ τὰ αὐτὰ, τινὲς δὲ κατʼ αὐτῆς, ἐπὶ τὰ ἔτερα δὲ μηθείμα.

ε'. Τομέα δὲ στερεῶν καλῶ, ἐπειδὰν σφαῖραν κώνος τέμνῃ κορυφήν ἔχων πρὸς τῷ κέντρῳ τῆς σφαίρας, τὸ ἐμπεριεχόμενον σχῆμα ὑπὸ τε τῆς ἐπιφανείας τοῦ κώνου καὶ τῆς ἐπιφανείας τῆς σφαίρας εὐτὸς τοῦ κώνου.

ζ'. Ῥόμβον δὲ καλῶ στερεῶν, ἐπειδὰν δύο κώνωι τὴν αὐτὴν βάσιν ἔχουσι τὰς κορυφὰς ἔχωσιν ἐφʼ ἐκάτερα τοῦ ἐπιπέδου τῆς βάσεως, ὅπως οἱ ἄξονες αὐτῶν ἐπὶ εὐθείας ὄσι κείμενοι, τὸ ἐξ ἀμφότερον τῶν κώνων συγκείμενον στερεῶν σχῆμα.

Λαμβανόμενα

Λαμβάνω δὲ ταῦτα:

α'. Τῶν τὰ αὐτὰ πέρατα ἔχουσῶν γραμμῶν ἑλαχίστην εἶναι τῆν εὐθείαν.

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all the straight lines joining the points fall on the same side of the line, or some fall on one and the same side while others fall along the line itself, but none fall on the other side.

3. Similarly also there are certain finite surfaces, not in a plane themselves but having their extremities in a plane, and such that they will either lie wholly on the same side of the plane containing their extremities or will have no part on the other side.

4. I call concave in the same direction surfaces such that, if any two points on them are taken, either the straight lines between the points all fall upon the same side of the surface, or some fall on one and the same side while others fall along the surface itself, but none falls on the other side.

5. When a cone cuts a sphere, and has its vertex at the centre of the sphere, I call the figure comprehended by the surface of the cone and the surface of the sphere within the cone a solid sector.

6. When two cones having the same base have their vertices on opposite sides of the plane of the base in such a way that their axes lie in a straight line, I call the solid figure formed by the two cones a solid rhombus.

postulates

i make these postulates:

1. of all lines which have the same extremities the straight line is the least.a

[a] proclus (in eucl., ed. friedlein 110. 10-14) saw in this statement a connexion with euclid's definition of a straight line as lying evenly with the points on itself: δ' αὐτ' αὔρχιμὴν τὴν εὐθείαν ὑφιστατο γραμμὴν ἐλαχίστην τῶν τὰ αὐτὰ πέρατα ἐξουσίων. διὸ γὰρ, ὡς ὁ εὐκλείδειος λόγος φησίν, εἴ ἵσον κείται τοῖς ἐφ' ἐαυτῆς σημείοις, διὰ τούτο ἐλαχίστη ἐστὶν τῶν τὰ αὐτὰ πέρατα ἐξουσίων.
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β'. Τῶν δὲ ἄλλων γραμμῶν, ἐὰν ἐν ἑπιπέδῳ 
οὐδὲ τὰ αὐτὰ πέρατα ἐχωσιν, ἀνίσους εἶναι τὸς 
τοιαύτας, ἐπειδὰν ὅσιν ἀμφότεραι ἐπὶ τὰ αὐτὰ 
κολλαί, καὶ ἦτοι οὐλη περιλαμβάνεται ἢ ἐτέρα 
αὐτῶν ὑπὸ τῆς ἐτέρας καὶ τῆς εὐθείας τῆς τὰ 
αὐτὰ πέρατα ἐχουσῆς αὐτῆ, ἢ τινὰ μὲν περιλαμ-
βάνεται, τινὰ δὲ κοινὰ ἐχη, καὶ ἐλάσσονα εἶναι 
τὴν περιλαμβανομένην.

g'. Ὅμοιως δὲ καὶ τῶν ἐπιφανειῶν τῶν τὰ 
αὐτὰ πέρατα ἐχουσῶν, ἐὰν ἐν ἑπιπέδῳ τὰ πέρατα 
ἐχουσιν, ἐλάσσονα εἶναι τὴν ἑπίπεδον.

d'. Τῶν δὲ ἄλλων ἐπιφανειῶν καὶ τὰ αὐτὰ πέ-
ρατα ἐχουσῶν, ἐὰν ἐν ἑπιπέδῳ τὰ πέρατα ἢ, 
ἀνίσους εἶναι τὰς τοιαύτας, ἐπειδὰν ὅσιν ἀμφότεραι 
ἐπὶ τὰ αὐτὰ κολλαί, καὶ ἦτοι οὐλη περιλαμβάνεται 
ὑπὸ τῆς ἐτέρας ἢ ἐτέρα ἐπιφάνεια καὶ τῆς ἑπιπέδου 
τῆς τὰ αὐτὰ πέρατα ἐχουσῆς αὐτῆ, ἢ τινὰ μὲν 
περιλαμβάνεται, τινὰ δὲ κοινὰ ἐχη, καὶ ἐλάσσονα 
eἶναι τὴν περιλαμβανομένην.

e'. Ἐτι δὲ τῶν ἀνίσων γραμμῶν καὶ τῶν ἀνίσων 
ἐπιφανειῶν καὶ τῶν ἀνίσων στερεῶν τὸ μείζον τοῦ 
ἐλάσσονος ὑπέρεχειν τοιοῦτω, ὅ συντιθὲμενον αὐτὸ 
ἐαυτῷ δυνατόν ἐστιν ὑπερέχειν παντὸς τοῦ προ-
τεθέντος τῶν πρὸς ἄλληλα λεγομένων.

Τούτων δὲ ὑποκειμένων, ἐὰν εἰς κύκλον πολύγω-
νον ἐγγραφῆ, φανερῶν, ὅτι ἡ περίμετρος τοῦ 
ἐγγραφέντος πολυγώνου ἐλάσσων ἐστὶν τῆς τοῦ 
κύκλου περιφερείας· ἐκάστη γὰρ τῶν τοῦ πολυ-
γώνου πλευρῶν ἐλάσσων ἐστὶ τῆς τοῦ κύκλου 
περιφερείας τῆς ὑπὸ τῆς αὐτῆς ἀποτελομένης.

a This famous “Axiom of Archimedes” is, in fact, gener-
ally used by him in the alternative form in which it is proved
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2. Of other lines lying in a plane and having the same extremities, [any two] such are unequal when both are concave in the same direction and one is either wholly included between the other and the straight line having the same extremities with it, or is partly included by and partly common with the other; and the included line is the lesser.

3. Similarly, of surfaces which have the same extremities, if those extremities be in a plane, the plane is the least.

4. Of other surfaces having the same extremities, if the extremities be in a plane, [any two] such are unequal when both are concave in the same direction, and one surface is either wholly included between the other and the plane having the same extremities with it, or is partly included by and partly common with the other; and the included surface is the lesser.

5. Further, of unequal lines and unequal surfaces and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those comparable with one another.

With these premises, if a polygon be inscribed in a circle, it is clear that the perimeter of the inscribed polygon is less than the circumference of the circle; for each of the sides of the polygon is less than the arc of the circle cut off by it.

in Euclid x. 1, for which v. vol. i. pp. 452-455. The axiom can be shown to be equivalent to Dedekind's principle, that a section of the rational points in which they are divided into two classes is made by a single point. Applied to straight lines, it is equivalent to saying that there is a complete correspondence between the aggregate of real numbers and the aggregate of points in a straight line; v. E. W. Hobson, The Theory of Functions of a Real Variable, 2nd ed., vol. i. p. 55.
'Εαν περὶ κύκλου πολύγωνον περιγραφῆ, ἢ τοῦ περιγραφέντος πολυγώνου περίμετρος μεῖζων ἐστὶν τῆς περιμέτρου τοῦ κύκλου.

Περὶ γὰρ κύκλου πολύγωνον περιγεγραφθὼ τὸ ὑποκείμενον. λέγω, ὅτι ἢ περίμετρος τοῦ πολυγώνου μεῖζων ἐστὶν τῆς περιμέτρου τοῦ κύκλου.

'Επεὶ γὰρ συναμφότερος ἡ ΒΔΑ μεῖζων ἐστὶ τῆς ΒΔ περιφερείας διὰ τὸ τὰ αὐτὰ πέρατα ἔχοντας περιλαμβάνειν τὴν περιφέρειαν, ὡμοίως δὲ καὶ συναμφότερος μὲν ἡ ΔΓ, ΓΒ τῆς ΔΒ, συναμφότερος δὲ ἡ ΛΚ, ΚΘ τῆς ΛΘ, συναμφότερος δὲ ἡ ΖΗΘ τῆς ΖΘ, ἐτὶ δὲ συναμφότερος ἡ ΔΕ, ΕΖ τῆς ΔΖ, ὅλη ἄρα ἢ περίμετρος τοῦ πολυγώνου μεῖζων ἐστὶ τῆς περιφερείας τοῦ κύκλου.

* It is here indicated, as in Prop. 3, that Archimedes added a figure to his own demonstration.
If a polygon be circumscribed about a circle, the perimeter of the circumscribed polygon is greater than the circumference of the circle.

For let the polygon be circumscribed about the circle as below. I say that the perimeter of the polygon is greater than the circumference of the circle.

For since $BA + AA > \text{arc } BA$, owing to the fact that they have the same extremities as the arc and include it, and similarly

$$\Delta \Gamma + \Gamma \Theta > [\text{arc}] \Delta \Theta,$$
$$\Delta K + K\Theta > [\text{arc}] \Delta \Theta,$$
$$ZH + H\Theta > [\text{arc}] Z\Theta,$$

and further $\Delta E + EZ > [\text{arc}] \Delta Z$,

therefore the whole perimeter of the polygon is greater than the circumference of the circle.
Δύο μεγεθῶν ἀνίσων δοθέντων δυνατὸν ἔστω εὑρεῖν δύο εὐθείας ἀνίσους, ὥστε τὴν μεῖζον εὐθείαν πρὸς τὴν ἐλάσσονα λόγων ἔχειν ἐλάσσονα ἤ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλάσσον.

"Εστω δύο μεγέθη ἀνίσα τὰ AB, Δ, καὶ ἐστώ μεῖζὸν τὸ AB. λέγω, ὅτι δυνατὸν ἔστι δύο εὐθείας ἀνίσους εὑρεῖν τὸ εἰρήμενον ἐπίταγμα ποιούσας.

Κείσθω διὰ τὸ β' τοῦ α' τῶν Εὐκλείδου τῶ Δ ἵσον τὸ ΒΓ, καὶ κείσθω τις εὐθεία γραμμὴ ἦ ZΗ· τὸ δὴ ΓΑ ἑαυτῷ ἐπισυντιθέμενον ὑπερέξει τοῦ Δ. πεπολλαπλασιάσθω νῦν, καὶ ἔστω τὸ ΑΘ, καὶ ὀσαπλασιόν ἐστὶ τὸ ΑΘ τοῦ ΑΓ, τοσαυταπλασίας ἐστώ ἢ ZΗ τῆς HE· ἔστων ἄρα, ὡς τὸ ΘΑ πρὸς ΑΓ, οὔτως ἢ ZΗ πρὸς HE· καὶ ἀνάπαλιν ἔστων, ὡς ἢ EH πρὸς ΗΖ, οὔτως τὸ ΑΓ πρὸς ΑΘ. καὶ ἐπεὶ μεῖζὸν ἐστὶν τὸ ΑΘ τοῦ Δ, τούτῃ τοῦ τὸ ΓΒ, τὸ ἄρα ΓΑ πρὸς τὸ ΑΘ λόγων ἐλάσσονα ἔχει ἦπερ τὸ ΓΑ πρὸς ΓΒ. ἀλλ' ὡς τὸ ΓΑ πρὸς ΑΘ, οὔτως ἢ EH πρὸς ΗΖ· ἢ EH ἄρα πρὸς ΗΖ ἐλάσσονα λόγων ἔχει ἦπερ τὸ ΓΑ πρὸς ΓΒ· καὶ συνθέτι ἦ EZ [ἄρα]¹ πρὸς ZΗ ἐλάσσονα λόγων ἔχει ἦπερ τὸ AB πρὸς ΒΓ [διὰ λήμμα].² ἵσον δὲ τὸ ΒΓ τῷ Δ· ἢ EZ ἄρα πρὸς ZΗ ἐλάσσονα λόγων ἔχει ἦπερ τὸ AB πρὸς τὸ Δ.

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Prop. 2

Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less.

Let $AB$, $\Delta$ be two unequal magnitudes, and let $AB$ be the greater. I say that it is possible to find two unequal straight lines satisfying the aforesaid requirement.

By the second proposition in the first book of Euclid let $BG$ be placed equal to $\Delta$, and let $ZH$ be any straight line; then $GA$, if added to itself, will exceed $\Delta$. [Post. 5.] Let it be multiplied, therefore, and let the result be $A\Theta$, and as $A\Theta$ is to $AG$, so let $ZH$ be to $HE$; therefore

$$\Theta A : AG = ZH : HE$$  \[cf. \ Eucl. \ v. \ 15\]

and conversely, $EH : HZ = AG : A\Theta$.

[Eucl. v. 7, coroll.

And since $A\Theta > \Delta$

$$> GB,$$

therefore $GA : A\Theta < GA : GB.$  \[Eucl. \ v. \ 8\]

But $GA : A\Theta = EH : HZ$;

therefore $EH : HZ < GA : GB$;

componendo, $EZ : ZH < AB : BG$.

Now $BG = \Delta$

therefore $EZ : ZH < AB : \Delta$.

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*a* This and related propositions are proved by Eutocius [Archim. ed. Heiberg iii. 16. 11-18. 22] and by Pappus, Coll. ed. Hultsch 684. 20 ff. It may be simply proved thus. If $a : b < c : d$, it is required to prove that $a + b : b < c + d : d$. Let $e$ be taken so that $a : b = e : d$. Then $e : d < c : d$. Therefore $e < c$, and $e + d : d < c + d : d$. But $e : d = a + b : b$ (ex hypothesi, componendo). Therefore $a + b : b < c + d : d$.

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1 ἀρα om. Heiberg.
2 διὰ λῆμμα om. Heiberg.
Εὑρημέναι εἰςίν ἄρα δύο εὐθείαι ἀνίσοι ποιοῦσαι τὸ εἰρημένον ἐπίταγμα [τούτεστι τὴν μείζονα πρὸς τὴν ἐλάσσονα λόγον ἔχειν ἐλάσσονα ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλάσσον].

γ

Δύο μεγεθῶν ἀνίσων δοθέντων καὶ κύκλου δυνατόν ἔστιν εἰς τὸν κύκλον πολύγωνον ἐγγράψαι καὶ ἄλλο περιγράψαι, ὡς ἢ τοῦ περιγραμμένου πολυγώνου πλευρά πρὸς τὴν τοῦ ἐγγραμμένου πολυγώνου πλευρὰν ἐλάσσονα λόγον ἔχῃ ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλαττον.

"Εστω τὰ δοθέντα δύο μεγέθη τὰ Α, Β, ὁ δὲ δοθεῖς κύκλος ὁ ὑποκείμενος. λέγω οὖν, ὅτι δυνατὸν ἐστι ποιεῖν τὸ ἐπίταγμα.

Εὑρήσθωσαν γὰρ δύο εὐθείαι αἱ Θ, ΚΛ, ὅν μεῖζον ἔστω ἢ Θ, ὡστε τὴν Θ πρὸς τὴν ΚΛ.
Accordingly there have been discovered two unequal straight lines fulfilling the aforesaid requirement.

Prop. 3

Given two unequal magnitudes and a circle, it is possible to inscribe a [regular] polygon in the circle and to circumscribe another, in such a manner that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that which the greater magnitude has to the less.

Let $A, B$ be the two given magnitudes, and let the given circle be that set out below. I say then that it is possible to do what is required.

For let there be found two straight lines $\Theta, K\Lambda$, of which $\Theta$ is the greater, such that $\Theta$ has to $K\Lambda$ a ratio

$1$ τοντέστω ... ἕλασσον verba subditiva esse suspicatur Heiberg.
Ελάσσονα λόγον ἔχειν ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλαττον, καὶ ἡχθω ἀπὸ τοῦ Λ τῇ ΛΚ πρὸς ὁρθὰς ἢ ΛΜ, καὶ ἀπὸ τοῦ Κ τῇ Θ ἵνα κατήχθω ἢ ΚΜ [δυνατὸν γὰρ τοῦτο], καὶ ἡχθωσαι τοῦ κύκλου δύο διάμετροι πρὸς ὁρθὰς ἀλλήλας αἱ ΓΕ, ΔΖ. τεμνοντες οὖν τὴν ὑπὸ τῶν ΔΗΓ γωνίαν δίχα καὶ τὴν ἡμίσειαν αὐτῆς δίχα καὶ αἰεὶ τοῦτο ποιοῦντες λειψομέν τινα γωνίαν ἐλάσσονα ἢ διπλασίαν τῆς ὑπὸ ΛΚΜ. λειψῖθω καὶ ἐστὶν ἢ ὑπὸ ΝΗΓ, καὶ ἐπεζεύχθω ἢ ΝΓ· ἢ ἁρα ΝΓ πολυγώνου ἐστὶ πλευρά ἰσοπλεύρου [ἐπείπερ ἢ ὑπὸ ΝΗΓ γωνία μετρεῖ τὴν ὑπὸ ΔΗΓ ὀρθὴν οὖσαν, καὶ ἢ ΝΓ ἁρα περιφέρεια μετρεῖ τὴν ΓΔ τέταρτον οὖσαν κύκλου· ὠστε καὶ τὸν κύκλου μετρεῖ. πολυγώνου ἁρα ἐστὶ πλευρά ἰσοπλεύρου· φανερῶν γάρ ἐστὶ τοῦτο]. καὶ τετμήθω ἢ ὑπὸ ΓΗΝ γωνία δίχα τῇ ΗΞ εὐθεία, καὶ ἀπὸ τοῦ Ξ ἔφαστεσθω τοῦ κύκλου ἢ ΩΠ, καὶ ἐκβεβλήσωσαν αἱ ΗΝΠ, ΗΓΩ· ὠστε καὶ ἢ ΠΟ πολυγώνου ἐστὶ πλευρά τοῦ περιγραφομένου περὶ τὸν κύκλου καὶ ἰσοπλεύρον [φανερῶν, ὅτι καὶ ὁμοίου τῶ ἐγγραφομένων, οὐ πλευρὰ ἢ ΝΓ]. ἐπεὶ δὲ ἐλάσσων ἐστὶν ἢ διπλασία ἢ ὑπὸ ΝΗΓ τῆς ὑπὸ ΛΚΜ, διπλασία δὲ τῆς ὑπὸ ΤΗΓ, ἐλάσσων ἁρα ἢ ὑπὸ ΤΗΓ τῆς ὑπὸ ΛΚΜ. καὶ εἰσὶν ὀρθὰς αἱ πρὸς τοὺς Λ, Τ· ἢ ἁρα ΜΚ πρὸς ΛΚ μεῖζονα λόγον ἔχει ήπερ ἢ ΓΗ πρὸς ΗΤ. ἵνα δὲ ἢ ΓΗ τῇ ΗΞ· ὠστε ἢ ΗΞ πρὸς ΗΤ ἐλάσσονα λόγον ἔχει, τουτέστιν ἢ ΠΟ πρὸς ΝΓ, ἡπερ ἢ ΜΚ πρὸς ΚΛ· ἔτι δὲ ἢ ΜΚ πρὸς ΚΛ ἐλάσσονα λόγον ἔχει ἡπερ τὸ Α πρὸς τὸ Β. καὶ ἐστὶν ἢ μὲν ΠΟ πλευρά

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less than that which the greater magnitude has to the less [Prop. 2], and from A let $\Delta M$ be drawn at right angles to $\Delta K$, and from K let $K M$ be drawn equal to $\Theta$, and let there be drawn two diameters of the circle, $G E, \Delta Z$, at right angles one to another. If we bisect the angle $\Delta H \Gamma$ and then bisect the half and so on continually we shall leave a certain angle less than double the angle $\Delta K M$. Let it be left and let it be the angle $N H \Gamma$, and let $N \Gamma$ be joined; then $N \Gamma$ is the side of an equilateral polygon. Let the angle $\Gamma H N$ be bisected by the straight line $H \Xi$, and through $\Xi$ let the tangent $O \Xi \Pi$ be drawn, and let $H N \Pi, H \Gamma O$ be produced; then $\Pi O$ is a side of an equilateral polygon circumscribed about the circle. Since the angle $N H \Gamma$ is less than double the angle $\Delta K M$ and is double the angle $T H \Gamma$, therefore the angle $T H \Gamma$ is less than the angle $\Delta K M$. And the angles at $\Lambda, T$ are right; therefore

$$MK : \Delta K > \Gamma H : HT.$$  

But $$\Gamma H = H \Xi.$$  

Therefore $$H \Xi : HT < MK : KA,$$  

that is, $$\Pi O : N \Gamma < MK : KA.$$  

Further, $$MK : KA < A : B.$$  

[Therefore $$\Pi O : N \Gamma < A : B.$$]

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1. This is proved by Eutocius and is equivalent to the assertion that if $a < \beta \leq \frac{\pi}{2}$, cosec $\beta >$ cosec $a$.

2. For $H \Xi : HT = \Pi O : N \Gamma$, since $H \Xi : HT = O \Xi : \Gamma \Gamma = 2 \Xi \Xi : 2 \Gamma \Gamma = \Pi O : \Gamma \Gamma$.

3. For by hypothesis $\Theta : KA < A : B$, and $\Theta = MK$.

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1. δυνάτων . . . τούτο om. Heiberg.
2. ἐπείπερ . . . τούτο om. Heiberg.
3. φανερῶν . . . η ΝΓ om. Heiberg.
τοῦ περιγραφομένου πολυγώνου, ἢ δὲ ΓΝ τοῦ ἐγγραφομένου· ὅπερ προέκειτο εὐρεῖν.

ε' 
Κύκλου δοθέντος καὶ δύο μεγεθῶν ἀνίσων περιγράφαι περὶ τὸν κύκλον πολύγωνον καὶ ἄλλο ἐγγράψαι, ὥστε τὸ περιγραφέν πρὸς τὸ ἐγγραφέν ἐλάσσονα λόγον ἔχειν ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλασσόν.

'Εκκείσθω κύκλος ὁ Α καὶ δύο μεγέθη ἀνίσα

τὰ Ε, Ζ καὶ μεῖζον τὸ Ε. δεῖ οὖν πολύγωνον ἐγγράψαι εἰς τὸν κύκλον καὶ ἄλλο περιγράψαι, ἵνα γενηται τὸ ἑπταχθὲν.

Λαμβάνω γὰρ δύο εὐθείας ἀνίσους τὰς Γ, Δ, ὃν μεῖζων ἔστω ἢ Γ. ὥστε τὴν Γ πρὸς τὴν Δ

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And \( \Pi O \) is a side of the circumscribed polygon, \( \Gamma N \) of the inscribed; which was to be found.

Prop. 5

Given a circle and two unequal magnitudes, to circumscribe a polygon about the circle and to inscribe another, so that the circumscribed polygon has to the inscribed polygon a ratio less than the greater magnitude has to the less.

Let there be set out the circle \( A \) and the two unequal magnitudes \( E, Z \), and let \( E \) be the greater; it is therefore required to inscribe a polygon in the circle and to circumscribe another, so that what is required may be done.

For I take two unequal straight lines \( \Gamma, \Delta \), of which let \( \Gamma \) be the greater, so that \( \Gamma \) has to \( \Delta \) a ratio
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ελάσσονα λόγον ἐχεῖν ἢ τὴν Ε πρὸς τὴν Ζ· καὶ τῶν Γ, Δ μέσης ἀνάλογον ληφθείσης τῆς Η μείζων ἢ Αρὰ καὶ ἢ Γ τῆς Η. περιγεγράφθω δὴ περὶ κύκλου πολὺγων καὶ ἄλλο ἐγγεγράφθω, ὡστε τὴν τού περιγραφέντος πολυγώνου πλευρὰν πρὸς τὴν τού ἐγγραφέντος ελάσσονα λόγον ἐχεῖν ἢ τὴν Γ πρὸς τὴν Η [καθὼς ἐμάθομεν]· διὰ τοῦτο δὴ καὶ ὁ διπλάσιος λόγος τοῦ διπλασίου ελάσσων ἐστί. καὶ τοῦ μὲν τῆς πλευρᾶς πρὸς τὴν πλευρὰν διπλάσιος ἐστι ὁ τοῦ πολυγώνου πρὸς τὸν πολυγώνον [ὁμοία γάρ]. πῆς δὲ Γ πρὸς τὴν Η ὁ τῆς Γ πρὸς τὴν Δ· καὶ τὸ περιγραφέν ἢ ἁρὰ πολὺγων πρὸς τὸ εγγραφέν ελάσσονα λόγον ἐχεῖ ἢπερ ἢ Γ πρὸς τὴν Δ· πολλῷ ἢρα τὸ περιγραφέν πρὸς τὸ εγγραφέν ελάσσονα λόγον ἐχεῖ ἢπερ τὸ Ε πρὸς τὸ Ζ.

η

Ἐὰν περὶ κῶνου ἴσοσκελὴ πυραμίς περιγραφῆ, ἢ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς βάσεως ἦσθ' ἐστὶν τριγώνῳ βάσιν μὲν ἔχοντι τὴν ἴσην τῇ περιμέτρῳ τῆς βάσεως, ὕψος δὲ τὴν πλευρὰν τοῦ κῶνου. . . .

θ

Ἐὰν κῶνου τινὸς ἴσοσκελοῦς εἰς τῶν κύκλων, ὡς ἐστὶ βάσις τοῦ κῶνου, εὐθεῖα γραμμὴ ἐμπέσῃ, ἀπὸ δὲ τῶν περάτων αὐτῆς εὐθεία γραμμαί ἀχθῶσιν ἐπὶ τὴν κορυφὴν τοῦ κῶνου, τὸ περιληψθὲν τρίγωνον ὑπὸ τε τῆς ἐμπεσούσης καὶ τῶν ἐπιζευχθεισῶν ἐπὶ τὴν κορυφὴν ἐλασσὸν ἔσται τῆς 58
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less than that which E has to Z [Prop. 2]; if a mean proportional H be taken between Γ, Δ, then Γ will be greater than H [Eucl. vi. 13]. Let a polygon be circumscribed about the circle and another inscribed, so that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that which Γ has to H [Prop. 3]; it follows that the duplicate ratio is less than the duplicate ratio. Now the duplicate ratio of the sides is the ratio of the polygons [Eucl. vi. 20], and the duplicate ratio of Γ to H is the ratio of Γ to Δ [Eucl. v. Def. 9]; therefore the circumscribed polygon has to the inscribed polygon a ratio less than that which Γ has to Δ; by much more therefore the circumscribed polygon has to the inscribed polygon a ratio less than that which E has to Z.

Prop. 8

If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid without the base is equal to a triangle having its base equal to the perimeter of the base [of the pyramid] and its height equal to the side of the cone. . . .

Prop. 9

If in an isosceles cone a straight line [chord] fall in the circle which is the base of the cone, and from its extremities straight lines be drawn to the vertex of the cone, the triangle formed by the chord and the lines joining it to

* The “side of the cone” is a generator. The proof is obvious.

1 καθὼς ἐμάθομεν om. Heiberg.
2 οὗτοι γάρ om. Heiberg.
επιφανείας τοῦ κύκλου τῆς μεταξὺ τῶν ἐπὶ τὴν κορυφὴν ἐπιζευγθεῖσῶν.

"Εστω κύκλος ισοσκελὴς βάσις ὁ ΑΒΓ κύκλος, κορυφὴ δὲ τὸ Δ, καὶ διήθη τις εἰς αὐτὸν εὐθεία ἡ ΑΓ, καὶ ἀπὸ τῆς κορυφῆς ἐπὶ τὰ Α, Γ ἐπεζεύχθησαν αἱ ΑΔ, ΔΓ· λέγω, ὅτι τὸ ΑΔΓ τρίγωνον ἔλασσον ἐστὶν τῆς ἐπιφανείας τῆς κωνικῆς τῆς μεταξὺ τῶν ΑΔΓ.

Τετμησθῶ ἡ ΑΒΓ περιφέρεια δῖχα κατὰ τὸ Β, καὶ ἐπεζεύχθησαν αἱ ΑΒ, ΓΒ, ΔΒ· ἔσται δὴ τὰ ΑΒΔ, ΒΓΔ τρίγωνα μείζονα τοῦ ΑΔΓ τριγώνου. Ὄ δὴ ὑπερέχει τὰ εἰρημένα τρίγωνα τοῦ ΑΔΓ τριγώνου, ἐστὶν τὸ Θ. τὸ δὴ Θ ἦτοι τῶν ΑΒ, ΒΓ τμημάτων ἔλασσον ἐστὶν ἡ οὐ.
the vertex will be less than the surface of the cone between the lines drawn to the vertex.

Let the circle $A\Gamma$ be the base of an isosceles cone, let $\Delta$ be its vertex, let the straight line $A\Gamma$ be drawn in it, and let $A\Delta$, $\Delta\Gamma$ be drawn from the vertex to $A$, $\Gamma$; I say that the triangle $A\Delta\Gamma$ is less than the surface of the cone between $A\Delta$, $\Delta\Gamma$.

Let the arc $A\Gamma$ be bisected at $B$, and let $AB$, $GB$, $DB$ be joined; then the triangles $AB\Delta$, $B\Gamma\Delta$ will be greater than the triangle $A\Delta\Gamma$. Let $\Theta$ be the excess by which the aforesaid triangles exceed the triangle $A\Delta\Gamma$. Now $\Theta$ is either less than the sum of the segments $AB$, $B\Gamma$ or not less.

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* For if $h$ be the length of a generator of the isosceles cone, triangle $AB\Delta = \frac{1}{2}h \cdot AB$, triangle $B\Gamma\Delta = \frac{1}{2}h \cdot B\Gamma$, triangle $A\Delta\Gamma = \frac{1}{2}h \cdot A\Gamma$, and $AB + B\Gamma > A\Gamma$.

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1 $\varepsilon \sigma t \alpha \ldots \tau \rho \gamma \nu \nu \omicron o\nu$: ex Eutoci videtur Archim edem scrip sisse: $\mu \epsilon i'\zeta o\nu a \delta r a \varepsilon t \iota \tau \alpha \ A B \Delta$, $B \Delta \Gamma \tau \rho \gamma \nu \nu \nu a \tau o\nu A \Delta \Gamma$ $\tau \rho \gamma \nu \nu \nu o\nu$. 61
"Εστώ μή ἔλασσον πρότερον. ἔπειτο σὲν δύο εἴσιν ἐπιφάνειαι ἡ τε κωνική ἡ μεταξὺ τῶν AΔΒ μετὰ τοῦ ΑΕΒ τμῆματος καὶ ἡ τοῦ AΔΒ τριγώνου τὸ αὐτὸ πέρας ἔχουσαι τὴν περίμετρον τοῦ τριγώνου τοῦ AΔΒ, μείζων ἔσται ἡ περιλαμβάνουσα τῆς περιλαμβανομένης· μείζων ἀρα ἔστιν ἡ κωνικὴ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΒ μετὰ τοῦ ΑΕΒ τμῆματος τοῦ AΒΔ τριγώνου. ὅμως δὲ καὶ ἡ μεταξὺ τῶν ΒΔΓ μετὰ τοῦ ΓΖΒ τμῆματος μείζων ἔστιν τοῦ ΒΔΓ τριγώνου· ὅλη ἀρα ἡ κωνικὴ ἐπιφάνεια μετὰ τοῦ Θ χωρίου μείζων ἐστὶ τῶν εἰρημένων τριγώνων. τὰ δὲ εἰρημένα τρίγωνα ἵσα ἐστὶν τῷ τε ΑΔΓ τριγώνῳ καὶ τῷ Θ χωρίῳ. κοινὸν ἀφηρήσθω τὸ Θ χωρίον· λοιπὴ ἀρα ἡ κωνικὴ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μείζων ἔστιν τοῦ ΑΔΓ τριγώνου.

"Εστὼ δὴ τὸ Θ ἔλασσον τῶν ΑΒ, ΒΓ τμῆμάτων. τέμνοντες δὴ τὰς ΑΒ, ΒΓ περιφερείας δίχα καὶ τὰς ἡμισειάς αὐτῶν δίχα λειψομεν τμῆματα ἔλασσονα ὅντα τοῦ Θ χωρίου. λειείφθω τὰ ἑπὶ τῶν ΑΕ, ΕΒ, ΒΖ, ΖΓ εὐθεῖων, καὶ ἐπεζεύγθωσαν αἱ ΔΕ, ΔΖ. πάλιν τούτων κατὰ τὰ αὐτὰ ἡ μὲν ἐπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν ΑΔΕ μετὰ τοῦ ἑπὶ τῆς ΑΕ τμῆματος μείζων ἔστιν τοῦ ΑΔΕ τριγώνου, ἡ δὲ μεταξὺ τῶν ΕΔΒ μετὰ τοῦ ἑπὶ τῆς ΕΒ τμῆματος μείζων ἔστιν τοῦ ΕΔΒ τριγώνου· ἡ ἀρα ἐπιφάνεια ἡ μεταξὺ τῶν AΔΒ μετὰ τῶν ἑπὶ τῶν ΑΕ, ΕΒ τμημάτων μείζων ἔστιν τῶν AΔΕ, EΒΔ τριγώνων. ἔπει δὲ τά ΑΕΔ, ΔΕΒ τρίγωνα μείζονα ἐστὶν τοῦ ΑΒΔ τριγώνου, καθὼς δέδεικται, πολλῶ ἀρα ἡ ἐπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν ΑΔΒ μετὰ τῶν ἑπὶ τῶν ΑΕ,
Firstly, let it be not less. Then since there are two surfaces, the surface of the cone between $\Delta$, $\Delta B$ together with the segment $AEB$ and the triangle $A\Delta B$, having the same extremity, that is, the perimeter of the triangle $A\Delta B$, the surface which includes the other is greater than the included surface [Post. 3]; therefore the surface of the cone between the straight lines $A\Delta, \Delta B$ together with the segment $AEB$ is greater than the triangle $A\Delta B$. Similarly the [surface of the cone] between $B\Delta, \Delta \Gamma$ together with the segment $\Gamma ZB$ is greater than the triangle $B\Delta \Gamma$; therefore the whole surface of the cone together with the area $\theta$ is greater than the aforesaid triangles. Now the aforesaid triangles are equal to the triangle $A\Delta \Gamma$ and the area $\theta$. Let the common area $\theta$ be taken away; therefore the remainder, the surface of the cone between $A\Delta, \Delta \Gamma$ is greater than the triangle $A\Delta \Gamma$.

Now let $\theta$ be less than the segments $AB$, $B\Gamma$. Bisecting the arcs $AB$, $B\Gamma$ and then bisecting their halves, we shall leave segments less than the area $\theta$ [Eucl. xii. 2]. Let the segments so left be those on the straight lines $AE$, $EB$, $BZ$, $Z\Gamma$, and let $\Delta E, \Delta Z$ be joined. Then once more by the same reasoning the surface of the cone between $A\Delta, \Delta E$ together with the segment $AE$ is greater than the triangle $A\Delta E$, while that between $E\Delta, \Delta B$ together with the segment $EB$ is greater than the triangle $E\Delta B$; therefore the surface between $A\Delta, \Delta B$ together with the segments $AE, EB$ is greater than the triangles $A\Delta E$, $EB\Delta$. Now since the triangles $AE\Delta, \Delta EB$ are greater than the triangle $A\Delta B$, as was proved, by much more therefore the surface of the cone between $A\Delta, \Delta B$ together with the segments $AE, EB$ is
ΕΒ τμημάτων μειζών ἐστὶ τοῦ ΑΔΒ τριγώνου. διὰ τὰ αὐτὰ δὴ καὶ ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΒΔΓ μετὰ τῶν ἐπὶ τῶν ΒΖ, ΖΓ τμημάτων μειζών ἐστὶν τοῦ ΒΔΓ τριγώνου. ὅλη ἀρὰ ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μετὰ τῶν εἰρημένων τμημάτων μειζών ἐστὶ τῶν ΑΒΔ, ΔΒΓ τριγώνων. ταύτα δὲ ἐστὶν ἵσα τῷ ΑΔΓ τριγώνῳ καὶ τῷ Θ χωρίῳ. ὅτι τὰ εἰρημένα τμήματα ἐλάσσονα τοῦ Θ χωρίου· λοιπὴ ἀρὰ ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μειζών ἐστὶν τοῦ ΑΔΓ τριγώνου.

Εάν ἐπιφανεύσαι αχθῶσιν τοῦ κύκλου, ὅς ἐστι βάσις τοῦ κόνου, ἐν τῷ αὐτῷ ἐπιπέδῳ οὕσα τῷ κύκλῳ καὶ συμπόστουσα ἀλλήλαις, ἀπὸ δὲ τῶν ἀφῶν καὶ τῆς συμπτώσεως ἐπὶ τὴν κορυφὴν τοῦ κόνου εὐθεῖα αχθῶσιν, τα περιεχόμενα τρίγωνα ὑπὸ τῶν ἐπιφανοῦσῶν καὶ τῶν ἐπὶ τὴν κορυφὴν τοῦ κόνου ἐπιζευγχεισῶν εὐθείῶν μειζώνα ἐστὶν τῆς τοῦ κόνου ἐπιφανείας τῆς ἀπολαμβανομένης ὑπ’ αὐτῶν. . . .

. . . Τούτων δὴ δεδειγμένων φανερῶν [ἐπὶ μὲν τῶν προειρημένων], ὅτι, ἐάν εἰς κόνων ἵσοσκελῆ πυραμίς ἐγγραφῇ, ἡ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς βάσεως ἐλάσσων ἐστὶ τῆς κοινῆς ἐπιφανείας [ἐκαστον γὰρ τῶν περιεχόντων τῆν πυραμίδα τριγώνων ἐλασσὼν ἐστὶν τῆς κοινῆς ἐπιφανείας τῆς μεταξὺ τῶν τοῦ τριγώνου πλευρῶν· ὥστε καὶ ὅλη ἡ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς
greater than the triangle \( \triangle \Delta B \). By the same reasoning the surface between \( \triangle \Delta, \Delta \Gamma \) together with the segments \( BZ, Z \Gamma \) is greater than the triangle \( \triangle B \Delta \Gamma \); therefore the whole surface between \( \triangle \Delta, \Delta \Gamma \) together with the aforesaid segments is greater than the triangles \( \triangle \Delta \Delta \Gamma, \Delta B \Gamma \). Now these are equal to the triangle \( \triangle \Delta \Gamma \) and the area \( \Theta \); and the aforesaid segments are less than the area \( \Theta \); therefore the remainder, the surface between \( \triangle \Delta, \Delta \Gamma \) is greater than the triangle \( \triangle \Delta \Gamma \).

**Prop. 10**

*If tangents be drawn to the circle which is the base of an [isosceles] cone, being in the same plane as the circle and meeting one another, and from the points of contact and the point of meeting straight lines be drawn to the vertex of the cone, the triangles formed by the tangents and the lines drawn to the vertex of the cone are together greater than the portion of the surface of the cone included by them.*...

**Prop. 12**

... From what has been proved it is clear that, if a pyramid is inscribed in an isosceles cone, the surface of the pyramid without the base is less than the surface of the cone [Prop. 9], and that, if a pyramid

\footnote{The proof is on lines similar to the preceding proposition.}

1 \( \epsilon \pi \) ... \( \pi \rho \rho \rho \rho \rho \mu \varepsilon \nu \nu \) om. Heiberg.
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βάσεως ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κώνου χωρίς τῆς βάσεως, καὶ ὅτι, ἐὰν περὶ κώνου ἰσοσκελῆ πυραμίδας περιγραφῆ, ἡ ἐπιφάνεια τῆς πυραμίδος χωρίς τῆς βάσεως μεῖζων ἐστὶν τῆς ἐπιφανείας τοῦ κώνου χωρίς τῆς βάσεως [κατὰ τὸ συνεχὲς ἐκεῖνῳ].

Φανερῶν δὲ ἐκ τῶν ἀποδεδειγμένων, ὅτι τε, ἐὰν εἰς κυλίνδρον ὀρθόν πρίσμα ἐγγραφῆ, ἡ ἐπιφάνεια τοῦ πρίσματος ἢ ἐκ τῶν παραλληλογράμμων συγκειμένη ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κυλίνδρου χωρίς τῆς βάσεως [ἐλάσσον γὰρ ἐκαστὸν παραλληλογράμμον τοῦ πρίσματος ἐστὶ τῆς καθ’ αὐτὸ τοῦ κυλίνδρου ἐπιφανείας], καὶ ὅτι, ἐὰν περὶ κυλίνδρον ὀρθόν πρίσμα περιγραφῆ, ἡ ἐπιφάνεια τοῦ πρίσματος ἢ ἐκ τῶν παραλληλογράμμων συγκειμένη μεῖζων ἐστὶ τῆς ἐπιφανείας τοῦ κυλίνδρου χωρίς τῆς βάσεως.

η'

Παντὸς κυλίνδρου ὀρθοῦ ἡ ἐπιφάνεια χωρίς τῆς βάσεως ᾗ οὗ ἐστὶ κύκλῳ, ο进程中 ἐν τῳ κεντρῳ μέσου λόγου ἔχει τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς διαμέτρου τῆς βάσεως τοῦ κυλίνδρου.

Ἔστω κυλίνδρου τυρὸς ὀρθοῦ βάσις ὁ Α κύκλος, καὶ ἔστω τῇ μὲν διαμέτρῳ τοῦ Α κύκλου ᾗ ἡ ΓΔ, τῇ δὲ πλευρᾷ τοῦ κυλίνδρου ἡ EZ, ἔχετω δὲ μέσον λόγων τῶν ΔΓ, EZ ᾗ ἡ Η, καὶ κείσθω κύκλος, οὗ ἡ ἐκ τῳ κέντρῳ ᾗ ἐστὶ τῇ ἡ Η, ὁ Β· δεικτέον, ὅτι ὁ Β κύκλος ἰσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κυλίνδρου χωρίς τῆς βάσεως.

Εἰ γὰρ μὴ ἐστὶν ἰσος, ἦτοι μεῖζων ἐστὶ ἡ 66.
is circumscribed about an isosceles cone, the surface of the pyramid without the base is greater than the surface of the cone without the base [Prop. 10].

From what has been demonstrated it is also clear that, if a right prism be inscribed in a cylinder, the surface of the prism composed of the parallelograms is less than the surface of the cylinder excluding the bases \(^a\) [Prop. 11], and if a right prism be circumscribed about a cylinder, the surface of the prism composed of the parallelograms is greater than the surface of the cylinder excluding the bases.

Prop. 13

The surface of any right cylinder excluding the bases \(^b\) is equal to a circle whose radius is a mean proportional between the side of the cylinder and the diameter of the base of the cylinder.

Let the circle \(A\) be the base of a right cylinder, let \(\Gamma \Delta\) be equal to the diameter of the circle \(A\), let \(EZ\) be equal to the side of the cylinder, let \(H\) be a mean proportional between \(\Delta \Gamma\), \(EZ\), and let there be set out a circle, \(B\), whose radius is equal to \(H\); it is required to prove that the circle \(B\) is equal to the surface of the cylinder excluding the bases.\(^b\)

For if it is not equal, it is either greater or less.

\(^a\) Here, and in other places in this and the next proposition, Archimedes must have written \(\chi\rho\iota\varsigma\ \tau\alpha\nu\ \beta\alpha\sigma\epsilon\omega\nu\), not \(\chi\rho\iota\varsigma\ \tau\iota\varsigma\ \beta\alpha\sigma\epsilon\omega\nu\).

\(^b\) See preceding note.

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1 \(\varepsilon\kappa\alpha\sigma\tau\nu\) . . . \(\beta\alpha\sigma\epsilon\omega\varsigma\). Heiberg suspects that this demonstration is interpolated. Why give a proof of what is \(\phi\alpha\nu\epsilon\rho\omicron\) ?

2 \(\kappa\alpha\tau\alpha\) . . . \(\varepsilon\kappa\epsilon\iota\nu\) om. Heiberg.

3 \(\varepsilon\lambda\alpha\sigma\sigma\nu\) . . . \(\epsilon\pi\phi\alpha\nu\epsilon\alpha\lambda\alpha\varsigma\). Heiberg suspects that this proof is interpolated.
ελάσσων. ἔστω πρότερον, εἰ δυνατόν, ἑλάσσων. δύο δὴ μεγεθῶν ὄντων ἀνίσων τῆς τε ἐπιφανείας τοῦ κυλίνδρου καὶ τοῦ Β κύκλου δυνατόν ἔστω εἰς τὸν Β κύκλον ἰσόπλευρον πολύγωνον ἐγγράψαι καὶ ἄλλο περιγράψαι, ὡστε τὸ περιγραφὲν πρὸς τὸ ἐγγραφὲν ἑλάσσων λόγον ἔχειν τοῦ, διὸ ἔχει ἡ ἐπιφάνεια τοῦ κυλίνδρου πρὸς τὸν Β κύκλον. νοείσθω δὴ περιγεγραμμένον καὶ ἐγγεγραμμένον, καὶ περὶ τὸν Α κύκλον περιγεγράφου εὐθύγραμμον ὀμοίον τῷ περὶ τὸν Β περιγεγραμμένῳ, καὶ ἀναγεγράφῳ ἀπὸ τοῦ εὐθυγράμμου πρίσματος ἐσται δὴ περὶ τὸν κύλινδρον περιγεγραμμένον. ἔστω δὲ καὶ τῇ περιμέτρῳ τοῦ εὐθυγράμμου τοῦ περὶ

- One ms. has the marginal note, “equalis altitudinis chylindro,” on which Heiberg comments: “nee hoc omiserat Archimedes.” Heiberg notes several places in which the text is clearly not that written by Archimedes.
Let it first be, if possible, less. Now there are two unequal magnitudes, the surface of the cylinder and the circle B, and it is possible to inscribe in the circle B an equilateral polygon, and to circumscribe another, so that the circumscribed has to the inscribed a ratio less than that which the surface of the cylinder has to the circle B [Prop. 5]. Let the circumscribed and inscribed polygons be imagined, and about the circle A let there be circumscribed a rectilineal figure similar to that circumscribed about B, and on the rectilineal figure let a prism be erected $^a$; it will be circumscribed about the cylinder. Let $K\Delta$ be equal
τὸν Α κύκλον ἵση ἡ ΚΔ καὶ τῇ ΚΔ ὑση ἡ ΛΖ, τῆς δὲ ΓΔ ἡμίσεια ἔστω ἡ ΓΓ. ἔσται δὴ τὸ ΚΔΤ τρίγωνον ἵσον τῷ περιγεγραμμένῳ εὐθυγράμμῳ περὶ τὸν Α κύκλον [ἐπειδὴ βάσιν μὲν ἔχει τῇ περιμέτρῳ ὑση, ὕψος δὲ ἵσον τῇ ἑκ τοῦ κέντρου τοῦ Α κύκλου],¹ τὸ δὲ ΕΔ παραλληλόγραμμόν τῇ ἐπιφανείᾳ τοῦ πρίσματος τοῦ περὶ τὸν κυλινδρον περιγεγραμμένου [ἐπειδὴ περιέχεται ὑπὸ τῆς πλευρᾶς τοῦ κυλινδρον καὶ τῆς ὑσης τῇ περιμέτρῳ τῆς βάσεως τοῦ πρίσματος].² κείσθω δὴ τῇ ΕΖ ὑση ἡ ΕΡ· ἵσον ἀρα ἔστιν τὸ ΖΡΔ τρίγωνον τῷ ΕΔ παραλληλόγραμμῷ, ὥστε καὶ τῇ ἐπιφανείᾳ τοῦ πρίσματος. καὶ ἐπεὶ ὄμοιά ἔστιν τὰ εὐθύγραμμα· τὰ περὶ τοῦς Α, Β κύκλους περιγεγραμμένα, τὸν αὐτὸν ἔχει λόγον [τὰ εὐθύγραμμα],³ ὀντερ αὐτῷ ἔχει λόγον [τὰ εὐθύγραμμα]. αὐτὸς δὲ τὸν κέντρου δυνάμει. ἐξεῖ ἀρα τὸ ΚΤΔ τρίγωνον πρὸς τὸ περὶ τὸν Β κύκλον εὐθύγραμμον λόγον, ὧν ἡ ΤΔ πρὸς Η δυνάμει [αὐτῷ οὖρ ΤΔ, Η ἵσαι εἰσὶν ταῖς ἑκ τῶν κέντρων]. ἀλλ’ ὁν ἔχει λόγον ἡ ΤΔ πρὸς Η δυνάμει, τούτων ἔχει τὸν λόγον ἡ ΤΔ πρὸς ΡΖ μήκει [ἡ γαρ Η τῶν ΤΔ, ΡΖ μέση ἔστι ἀνάλογον διὰ τὸ καὶ τῶν ΓΔ, ΕΖ· πῶς δὲ τούτο; ἐπεὶ γαρ ὑση ἔστιν ἡ μὲν ΔΤ τῇ ΤΓ, ἡ δὲ ΡΕ τῇ ΕΖ, διπλασία ἀρα ἔστιν ἡ ΠΔ τῆς ΤΔ, καὶ ἡ ΡΖ τῆς ΡΕ· ἔστιν ἀρα, ὡς ἡ ΔΓ πρὸς ΔΤ, οὐτως ἡ ΡΖ πρὸς ΖΕ. τὸ ἀρα ὑπὸ τῶν ΓΔ, ΕΖ ἵσον ἔστιν τῷ ὑπὸ τῶν ΤΔ, ΡΖ. τῷ δὲ ὑπὸ τῶν ΓΔ, ΕΖ ἵσον ἔστιν τὸ ἀπὸ Η· καὶ τῷ ὑπὸ τῶν ΤΔ, ΡΖ ἀρα ἵσον ἔστι τὸ ἀπὸ τῆς Η. ἔστιν ἀρα,
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to the perimeter of the rectilineal figure about the circle A, let $\Delta Z$ be equal to $K\Delta$, and let $\Gamma T$ be half of $\Gamma \Delta$; then the triangle $K\Delta T$ will be equal to the rectilineal figure circumscribed about the circle A, while the parallelogram $EA$ will be equal to the surface of the prism circumscribed about the cylinder. Let $EP$ be set out equal to $EZ$; then the triangle $ZPA$ is equal to the parallelogram $EA$ [Eucl. i. 41], and so to the surface of the prism. And since the rectilineal figures circumscribed about the circles A, B are similar, they will stand in the same ratio as the squares on the radii; therefore the triangle $KT\Delta$ will have to the rectilineal figure circumscribed about the circle B the ratio $T\Delta^2 : H^2$.

But $T\Delta^2 : H^2 = T\Delta : PZ$.

a Because the base $K\Delta$ is equal to the perimeter of the polygon, and the altitude $\Delta T$ is equal to the radius of the circle A, i.e., to the perpendiculars drawn from the centre of A to the sides of the polygon.

b Because the base $\Delta Z$ is made equal to $\Delta K$ and so is equal to the perimeter of the polygon forming the base of the prism, while the altitude $EZ$ is equal to the side of the cylinder and therefore to the height of the prism.

c Eutocius supplies a proof based on Eucl. xii. 1, which proves a similar theorem for inscribed figures.

d For, by hypothesis, $H^2 = \Delta \Gamma \cdot EZ = 2T\Delta \cdot \frac{1}{2}PZ = T\Delta \cdot PZ$

Heiberg would delete the demonstration in the text on the ground of excessive verbosity, as Nizze had already perceived to be necessary.
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ός η ΤΔ προς Η, οὗτος η Η προς ΡΖ· ἕστιν ἄρα, ός η ΤΔ προς ΡΖ, τὸ ἀπὸ τῆς ΤΔ προς τὸ ἀπὸ τῆς Η· εὰν γὰρ τρεῖς εὐθείαι ἀνάλογον ὄσιν, ἕστιν, ός η πρώτῃ προς τὴν τρίτην, τὸ ἀπὸ τῆς δευτέρας εἴδος πρὸς τὸ ἀπὸ τῆς ὁμοίου καὶ ὁμολογος ἀναγεγραμμένον] 1. ὃν δὲ λόγον ἔχει ἡ ΤΔ προς ΡΖ μῆκος, τούτον ἔχει τὸ ΚΤΔ τρίγωνον πρὸς τὸ ΡΔΖ [ἐπειδήπερ ἵσαι εἰσίν αἱ ΚΔ, ΛΖ] 2. τὸν αὐτὸν ἄρα λόγον ἔχει τὸ ΚΤΔ τρίγωνον πρὸς τὸ εὐθυγραμμὸν τὸ περὶ τὸν Β κύκλον περιγεγραμμένον, ὃνπερ τὸ ΤΚΔ τρίγωνον πρὸς τὸ ΡΖΔ τρίγωνον. ἴσον ἄρα ἕστιν τὸ ΖΔΡ τρίγωνον τῷ περὶ τὸν Β κύκλον περιγεγραμμένῳ εὐθυγράμμῳ· ὡστε καὶ ἡ ἐπιφάνεια τοῦ πρίσματος τοῦ περὶ τὸν Α κύλινδρον περιγεγραμμένον τῷ εὐθυγράμμῳ τῷ περὶ τὸν Β κύκλον ἴσῃ ἕστιν. καὶ ἐπεὶ ἐλάσσονα λόγον ἔχει τὸ εὐθύγραμμον τὸ περὶ τὸν Β κύκλον πρὸς τὸ ἐγγεγραμμένον ἐν τῷ κύκλῳ τού, ὃν ἔχει ἡ ἐπιφάνεια τοῦ Α κυλίνδρου πρὸς τὸν Β κύκλον, ἐλάσσονα λόγον ἔξει καὶ ἡ ἐπιφάνεια τοῦ πρίσματος τοῦ περὶ τὸν κύλινδρον περιγεγραμμένου πρὸς τὸ εὐθύγραμμον τὸ ἐν τῷ κύκλῳ τῷ Β ἐγγεγραμμένον ἣπερ ἡ ἐπιφάνεια τοῦ κυλίνδρου πρὸς τὸν Β κύκλον· καὶ ἐναλλάξ· ὅπερ ἀδύνατον [ἡ μὲν γὰρ ἐπιφάνεια τοῦ πρίσματος τοῦ περιγεγραμμένου περὶ τὸν κυλίνδρον μεῖωσ᾽ οὕσα δεδεικτα τῆς ἐπιφανείας τοῦ κυλίνδρου, τὸ δὲ ἐγγεγραμμένον εὐθύγραμμον ἐν τῷ Β κύκλῳ ἔλασσον ἕστιν τοῦ Β κύκλου]. 3

οὐκ ἄρα ἕστιν ὁ Β κύκλος ἐλάσσων τῆς ἐπιφανείας τοῦ κυλίνδρου.

1 ἡ γὰρ . . . ὁμολογος ἀναγεγραμμένον om. Heiberg.
2 ἐπειδήπερ . . . ΚΔ, ΛΖ om. Heiberg.

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And \( T\Delta : PZ = \text{triangle} \quad K\Delta : \text{triangle} \quad P\Lambda Z \).\textsuperscript{a}

Therefore the ratio which the triangle \( K\Delta \) has to the rectilineal figure circumscribed about the circle \( B \) is the same as the ratio of the triangle \( TK\Delta \) to the triangle \( PZ\Delta \). Therefore the triangle \( TK\Delta \) is equal to the rectilineal figure circumscribed about the circle \( B \) [Eucl. v. 9]; and so the surface of the prism circumscribed about the cylinder \( A \) is equal to the rectilineal figure about \( B \). And since the rectilineal figure about the circle \( B \) has to the inscribed figure in the circle a ratio less than that which the surface of the cylinder \( A \) has to the circle \( B \) [ex hypothesist], the surface of the prism circumscribed about the cylinder will have to the rectilineal figure inscribed in the circle \( B \) a ratio less than that which the surface of the cylinder has to the circle \( B \); and, permutando, [the prism will have to the cylinder a ratio less than that which the rectilineal figure inscribed in the circle \( B \) has to the circle \( B \)]\textsuperscript{b}; which is absurd.\textsuperscript{c} Therefore the circle \( B \) is not less than the surface of the cylinder.

\textsuperscript{a} By Eucl. vi. 1, since \( \Delta Z = K\Delta \).

\textsuperscript{b} From Eutocius’s comment it appears that Archimedes wrote, in place of \( \kappa i \epsilon \nu \alpha l l \alpha \zeta : \delta \pi e r \ \alpha d \gamma n \alpha t o n \) in our text: \( \epsilon \nu \alpha l l \alpha \zeta \ \alpha r a \ \epsilon l \alpha s s o n a \ \lambda \gamma n o n \ \epsilon x e i \ \tau \circ \ \pi \rho \iota \circ \mu a \ \pi r o s \ \tau \circ \nu \ k \gamma l n d r o n \ \eta \pi e r \ \tau \circ \ \epsilon \gamma g e g r a m m \epsilon n o n \ \epsilon i s \ \tau \circ \nu \ \kappa \circ k l o n \ \pi o l \gamma w n o n \ \pi r o s \ \tau \circ \nu \ \kappa \circ k l o n \ \kappa \circ k l o n \ \delta \pi e r \ \alpha t o p o n \). This is what I translate.

\textsuperscript{c} For the surface of the prism is greater than the surface of the cylinder [Prop. 12], but the inscribed figure is less than the circle \( B \); the explanation in our text to this effect is shown to be an interpolation by the fact that Eutocius supplies a proof in his own words.

\textsuperscript{3} \( \eta \ \mu \epsilon n \ldots \tau o \nu \ \kappa \circ k l o n \ \circ m . \ \text{Heiberg ex Eutocio.} \)

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"Εστι δὴ, εἰ δυνατὸν, μείζων. πάλιν δὴ νοείσθω εἰς τὸν Β κύκλον εὐθύγραμμον ἐγγεγραμ-μένον καὶ ἄλλο περιγεγραμμένον, ὡστε τὸ περι-γεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἔχειν ἢ τὸν Β κύκλον πρὸς τὴν ἐπιφάνειαν τοῦ κυλίνδρου, καὶ ἐγγεγράφθω εἰς τὸν Α κύκλον πολύγωνον ὁμοιον τῷ εἰς τὸν Β κύκλον ἐγγεγραμ-μένω, καὶ πρίσμα ἀναγεγράφθω ἀπὸ τοῦ ἐν τῷ κύκλῳ ἐγγεγραμμένον πολυγώνου. καὶ πάλιν ἢ ΚΔ ἢς ἐστώ τῇ περιμέτρῳ τοῦ εὐθυγράμμου τοῦ ἐν τῷ Α κύκλῳ ἐγγεγραμμένου, καὶ ἢ ΖΔ ἢς αὐτῇ ἐστώ. ἐσται δὴ τὸ μὲν ΚΔ τριγώνου μείζον τοῦ εὐθυγράμμου τοῦ ἐν τῷ Α κύκλῳ ἐγ-γεγραμμένου [διότι βάσιν μὲν ἔχει τῇ περιμετρῶν αὐτοῦ, ύψος δὲ μεῖζον τῆς ἀπὸ τοῦ κέντρου ἐπὶ μίαν πλευρὰν τοῦ πολυγώνου ἀγομένης καθέτου], 1 τὸ δὲ ΕΔ παραλληλόγραμμον ἢς τῇ ἐπιφανείᾳ τοῦ πρίσματος τῇ ἐκ τῶν παραλληλόγραμμων συγκεκριμένη [διότι περιέχεται ὑπὸ τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς ἢς τῇ περιμέτρῳ τοῦ εὐθυ-γράμμου, ὡς τὸ βάσιν τοῦ πρίσματος]. ὡστε καὶ τὸ ΡΔΖ τριγώνου ἢς ἐστὶ τῇ ἐπιφανείᾳ τοῦ πρίσματος. καὶ ἔστι ὁμοιά ἐστὶ τὰ εὐθύγραμμα τὰ ἐν τοῖς Α, Β κύκλοις ἐγγεγραμμένα, τὸν αὐτὸν ἐχει λόγον πρὸς ἀλληλα, διὸ αἱ ἐκ τῶν κέντρων αὐτῶν δυνάμει. ἔχει δὲ καὶ τὰ ΚΔ, ΖΡΔ τριγώνων πρὸς ἀλληλα λόγον, διὸ αἱ ἐκ τῶν κέντρων τῶν κύκλων δυνάμει. τὸν αὐτὸν ἀρα λόγον ἔχει

1 διότι . . . καθέτου om. Heiberg.

a For the base ΚΔ is equal to the perimeter of the polygon and the altitude ΔΤ, which is equal to the radius of the
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Now let it be, if possible, greater. Again, let there be imagined a rectilineal figure inscribed in the circle B, and another circumscribed, so that the circumscribed figure has to the inscribed a ratio less than that which the circle B has to the surface of the cylinder [Prop. 5], and let there be inscribed in the circle A a polygon similar to the figure inscribed in the circle B, and let a prism be erected on the polygon inscribed in the circle [A]; and again let $K\Delta$ be equal to the perimeter of the rectilineal figure inscribed in the circle A, and let $Z\Lambda$ be equal to it. Then the triangle $K\Delta T\Delta$ will be greater than the rectilineal figure inscribed in the circle A, and the parallelogram $E\Lambda$ will be equal to the surface of the prism composed of the parallelograms $\pmb{b}$; and so the triangle $P\Delta Z$ is equal to the surface of the prism. And since the rectilineal figures inscribed in the circles A, B are similar, they have the same ratio one to the other as the squares of their radii [Eucl. xii. 1]. But the triangles $K\Delta T\Delta$, $Z\Delta P\Lambda$ have one to the other the same ratio as the squares of the radii $\pmb{c}$; therefore the rectilineal figure inscribed in circle A, is greater than the perpendiculars drawn from the centre of the circle to the sides of the polygon; but Heiberg regards the explanation to this effect in the text as an interpolation.

$\pmb{b}$ Because the base $Z\Delta$ is made equal to $K\Delta$, and so is equal to the perimeter of the polygon forming the base of the prism, while the altitude $EZ$ is equal to the side of the cylinder and therefore to the height of the prism.

$\pmb{c}$ For triangle $K\Delta T\Delta$: triangle $Z\Delta P\Lambda = T\Delta : ZP$

$$= T\Delta^2 : H^2$$

[cf. p. 71 n. d.]

But $T\Delta$ is equal to the radius of the circle A, and $H$ to the radius of the circle B.
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τὸ εὐθύγραμμον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον πρὸς τὸ εὐθύγραμμον τὸ ἐν τῷ Β ἐγγεγραμμένον καὶ τὸ ΚΤΔ τρίγωνον πρὸς τὸ ΛΖΠ τρίγωνον. Ἐλάσσων δὲ ἐστὶ τὸ εὐθύγραμμον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον τοῦ ΚΤΔ τριγώνου. Ἐλάσσων ἀρα καὶ τὸ εὐθύγραμμον τὸ ἐν τῷ Β κύκλῳ ἐγγεγραμμένον τοῦ ΖΠΔ τριγώνου. ὡστε καὶ τῆς ἐπιφανείας τοῦ πρίσματος τοῦ ἐν τῷ κυλίνδρῳ ἐγγεγραμμένον ὃπερ ἀδύνατον [ἐπεὶ γὰρ ἠλάσσων λόγον ἔχει τὸ περιγεγραμμένον εὐθύγραμμον περὶ τὸν Β κύκλον πρὸς τὸ ἐγγεγραμμένον ἦ ὁ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κυλίνδρου, καὶ ἑναλλάξ, μεῖζον δὲ ἐστὶ τὸ περιγεγραμμένον περὶ τὸν Β κύκλον τοῦ Β κύκλου, μεῖζον ἀρα. ἔστιν τὸ ἐγγεγραμμένον ἐν τῷ Β κύκλῳ τῆς ἐπιφανείας τοῦ κυλίνδρου. ὡστε καὶ τῆς ἐπιφανείας τοῦ πρίσματος]. ὡς ἀρα μεῖζων ἔστιν ὁ Β κύκλος τῆς ἐπιφανείας τοῦ κυλίνδρου. ἐδείχθη δὲ, ὅτι οὐδὲ ἠλάσσων· ἰσος ἀρα ἔστιν.

id'

Παντὸς κόνων ἰσοσκελοὺς χωρὶς τῆς βάσεως ἢ ἐπιφάνεια ἢ ἐστὶ κύκλω, ὅ ἦ ἐκ τοῦ κέντρου μέσον λόγον ἐχει τῆς πλευρᾶς τοῦ κόνων καὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ὅσ ἔστιν βάσις τοῦ κόνων.

"Εστω κόνως ἰσοσκελῆς, οὗ βάσις ὁ Α κύκλος, ἢ δὲ ἐκ τοῦ κέντρου ἔστω ἢ Π, τῇ δὲ πλευρᾷ τοῦ

1 ἐπεὶ . . . πρίσματος om. Heiberg.

* For since the figure circumscribed about the circle B has to the inscribed figure a ratio less than that which the circle B has to the surface of the cylinder [ex hypothesi], and the circle B is less than the circumscribed figure, therefore the 76
the circle A has to the rectilineal figure inscribed in the circle B the same ratio as the triangle KTΔ has to the triangle ΔZP. But the rectilineal figure inscribed in the circle A is less than the triangle KTΔ; therefore the rectilineal figure inscribed in the circle B is less than the triangle ZPΔ; and so it is less than the surface of the prism inscribed in the cylinder; which is impossible. Therefore the circle B is not greater than the surface of the cylinder. But it was proved not to be less. Therefore it is equal.

Prop. 14

The surface of any cone without the base is equal to a circle, whose radius is a mean proportional between the side of the cone and the radius of the circle which is the base of the cone.

Let there be an isosceles cone, whose base is the circle A, and let its radius be Γ, and let Δ be equal inscribed figure is greater than the surface of the cylinder, and a fortiori is greater than the surface of the prism [Prop. 12]. An explanation on these lines is found in our text, but as the corresponding proof in the first half of the proposition was unknown to Eutocius, this also must be presumed an interpolation.
κόντων έστω ἵπτης ἡ Δ, τῶν δὲ Γ, Δ μέση ἀνάλογον ἡ Ε, ὁ δὲ Β κύκλος ἐχέτω τὴν ἐκ τοῦ κέντρου τῆς Ε ἵπτης· λέγω, ὅτι ὁ Β κύκλος ἐστίν ἴσος τῆς ἐπιφάνειας τοῦ κόντων χωρίς τῆς βάσεως.

Εἰ γάρ μὴ ἐστὶν ἴσος, ἦτοι μείζων ἐστίν ἡ ἐλάσσων. ἐστω πρότερον ἐλάσσων. ἔστι δὴ δύο μεγέθη ἀνίσα ἡ τε ἐπιφάνεια τοῦ κόντων καὶ ὁ Β κύκλος, καὶ μείζων ἡ ἐπιφάνεια τοῦ κόντων· δύνατον ἀρα εἰς τὸν Β κύκλον πολύγωνον ἴσοπλευρον ἐγγράψαι καὶ ἄλλο περιγράψαι ὦμοιον τῷ ἐγγεγραμμένῳ, ὡστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἔχειν τοῦ, διὰ ἔχει ἡ ἐπιφάνεια τοῦ κόντων πρὸς τὸν Β κύκλον. νοεῖσθω δὴ καὶ περὶ τὸν Α κύκλον πολύγωνον περιγεγραμμένον ὦμοιον τῷ περὶ τὸν Β κύκλον περιγεγραμμένῳ, καὶ ἀπὸ τοῦ περὶ τὸν Α κύκλον περιγεγραμμένου πολυγώνου πυραμίδος ἀνεστάτω ἀναγεγραμμένη τῇ ἀυτῇ κορυφῆς ἔχουσα τῷ κόντῳ. ἐπεὶ οὖν ὦμοιὰ ἐστὶν τὰ πολύγωνα τὰ περὶ 78
to the side of the cone, and let E be a mean proportional between \( \Gamma, \Delta \), and let the circle B have its radius equal to E; I say that the circle B is equal to the surface of the cone without the base.

For if it is not equal, it is either greater or less. First let it be less. Then there are two unequal magnitudes, the surface of the cone and the circle B, and the surface of the cone is the greater; it is therefore possible to inscribe an equilateral polygon in the circle B and to circumscribe another similar to the inscribed polygon, so that the circumscribed polygon has to the inscribed polygon a ratio less than that which the surface of the cone has to the circle B [Prop. 5]. Let this be imagined, and about the circle A let a polygon be circumscribed similar to the polygon circumscribed about the circle B, and on the polygon circumscribed about the circle A let a pyramid be raised having the same vertex as the cone. Now since the polygons circumscribed about
τούς A, B κύκλους περιγεγραμμένα, τον αὐτὸν ἔχει λόγον πρὸς ἀλληλα, δι' αἰ ἐκ τοῦ κέντρου δυνάμει πρὸς ἀλλήλας, τουτέστιν δι' ἔχει ἡ Γ πρὸς Ε δύναμει, τουτέστιν ἡ Γ πρὸς Δ μὴ κει. δι' δὲ λόγον ἔχει ἡ Γ πρὸς Δ μὴ κει, τοῦτον ἔχει τὸ περιγεγραμμένον πολύγωνον περὶ τὸν A κύκλον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς περιγεγραμμένης περὶ τὸν κώνον [ἡ μὲν γὰρ Γ ἵση ἔστι τῇ ἀπὸ τοῦ κέντρου καθέτῳ ἐπὶ μίαν πλευρὰν τοῦ πολυγώνου, δὲ Δ τῇ πλευρᾷ τοῦ κώνου· κοινὸν δὲ ύπος ἡ περίμετρος τοῦ πολυγώνου πρὸς τὰ ἡμίσθι τῶν ἐπιφανειῶν]. τὸν αὐτὸν ἄρα λόγον ἔχει τὸ εὐθύγραμμον τὸ περὶ τὸν A κύκλον πρὸς τὸ εὐθύγραμμον τὸ περὶ τὸν B κύκλον καὶ αὐτὸ τὸ εὐθύγραμμον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς περιγεγραμμένης περὶ τὸν κώνον· ὡστε ἵσῃ ἐστὶν ἡ ἐπιφάνεια τῆς πυραμίδος τῷ εὐθύγραμμῳ τῷ περὶ τὸν B κύκλον περιγεγραμμένῳ. ἔπει οὖν ἐλάσσονα λόγον ἔχει τὸ εὐθύγραμμον τὸ περὶ τὸν B κύκλον περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἦπερ ἡ ἐπιφάνεια τοῦ κώνου πρὸς τὸν B κύκλον, ἐλάσσονα λόγον ἔχει ἡ ἐπιφάνεια τῆς πυραμίδος τῆς περὶ τὸν κώνον περιγεγραμμήσις πρὸς τὸ εὐθύγραμμον τὸ ἐν τῷ B κύκλῳ ἐγγεγραμμένον ἦπερ ἡ ἐπιφάνεια τοῦ κώνου πρὸς τὸν B κύκλον· ὅπερ ἀδύνατον [ἡ μὲν γὰρ ἐπιφάνεια τῆς πυραμίδος μείζων οὐσα δέδεκται τῆς ἐπιφανείας τοῦ κώνου, τὸ δὲ ἐγγεγραμμένον εὐθύγραμμον ἐν τῷ B κύκλῳ ἐλάσσον έσται τοῦ B κύκλου]. οὐκ ἄρα ὁ B κύκλος ἐλάσσων ἐσται τῆς ἐπιφανείας τοῦ κώνου.
the circles A, B are similar, they have the same ratio one toward the other as the square of the radii have one toward the other, that is \( \Gamma^2 : E^2 \), or \( \Gamma : \Delta \) [Eucl. vi. 20, coroll. 2]. But \( \Gamma : \Delta \) is the same ratio as that of the polygon circumscribed about the circle A to the surface of the pyramid circumscribed about the cone \( a \); therefore the rectilineal figure about the circle A has to the rectilineal figure about the circle B the same ratio as this rectilineal figure [about A] has to the surface of the pyramid circumscribed about the cone; therefore the surface of the pyramid is equal to the rectilineal figure circumscribed about the circle B. Since the rectilineal figure circumscribed about the circle B has towards the inscribed [rectilineal figure] a ratio less than that which the surface of the cone has to the circle B, therefore the surface of the pyramid circumscribed about the cone will have to the rectilineal figure inscribed in the circle B a ratio less than that which the surface of the cone has to the circle B; which is impossible.\(^b\) Therefore the circle B will not be less than the surface of the cone.

\(^a\) For the circumscribed polygon is equal to a triangle, whose base is equal to the perimeter of the polygon and whose height is equal to \( \Gamma \), while the surface of the pyramid is equal to a triangle having the same base and height \( \Delta \) [Prop. 8]. There is an explanation to this effect in the Greek, but so obscurely worded that Heiberg attributes it to an interpolator.

\(^b\) For the surface of the pyramid is greater than the surface of the cone [Prop. 12], while the inscribed polygon is less than the circle B.

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1 η μὲν ... ἐπιφανείαν om. Heiberg.
2 η μὲν ... τοῦ Β κύκλου om. Heiberg.
Λέγω δή, οτι ουδε μείζων. ει γαρ δυνατον 
εστιν, εστω μείζων. παλαι δη νοείσθω εις τον Β
κυκλον πολυγωνον εγγεγραμμενον και άλλο περι
γεγραμμενον, οστε το περιγεγραμμενον προς το
εγγεγραμμενον έλασσονα λόγον έχειν του, ον έχει
ο Β κύκλος προς την έπιφάνειαν του κώνου, και
εις τον Α κύκλον νοείσθω εγγεγραμμενον πολυ
γωνον ομοιον τω εις τον Β κύκλον εγγεγραμμενος,
και αναγεγράφθω απ' αυτων πυραμίς την αυτην
κορυφην έχουσα τω κώνω. έπει οδυ ομοια έστι
τα εν τοις Α, Β κύκλοις εγγεγραμμενα, τον αυτον
έξει λόγον προς άλληλα, ον αι εκ των κέντρων
dυναμει προς άλληλας· τον αυτον άρα λόγον έχει
το πολυγωνον προς το πολυγωνον και η Γ προς
την Δ μήκει. η δε Γ προς την Δ μείζονα λόγον
έχει η το πολυγωνον το εν τω Α κύκλω εγγεγραμ-
μενον προς την έπιφανειαν της πυραμίδος της
εγγεγραμμενης εις τον κώνον [η γαρ εκ του κέν-
tρου του Α κύκλου προς την πλευραν του κώνου
μείζονα λόγον έχει ήπερ η άπο του κέντρου αγο-
méνη κάθετος επί μιαν πλευραν του πολυγωνου
προς την επι την πλευραν του πολυγώνου κάθε-
tου αγομένην απο της κορυφῆς του κώνου]1· μεί-

1 η γαρ ... του κώνου om. Heiberg.

* Eutocius supplies a proof. ΖΘΚ is the polygon inscribed in the circle Α (of centre Α), ΑΗ is drawn perpendicular to 82
I say now that neither will it be greater. For if it is possible, let it be greater. Then again let there be imagined a polygon inscribed in the circle B and another circumscribed, so that the circumscribed has to the inscribed a ratio less than that which the circle B has to the surface of the cone [Prop. 5], and in the circle A let there be imagined an inscribed polygon similar to that inscribed in the circle B, and on it let there be drawn a pyramid having the same vertex as the cone. Since the polygons inscribed in the circles A, B are similar, therefore they will have one toward the other the same ratio as the squares of the radii have one toward the other; therefore the one polygon has to the other polygon the same ratio as \( \Gamma \) to \( \Delta \) [Eucl. vi. 20, coroll. 2]. But \( \Gamma \) has to \( \Delta \) a ratio greater than that which the polygon inscribed in the circle A has to the surface of the pyramid inscribed in the cone \( a \); therefore the polygon in-

\[ K\theta \text{ and meets the circle in } M, A \text{ is the vertex of the isosceles cone (so that } AH \text{ is perpendicular to } K\theta), \]

and \( HN \) is drawn parallel to \( MA \) to meet \( AA \) in \( N \). Then the area of the polygon inscribed in the circle = \( \frac{1}{3} \) perimeter of polygon \( \cdot AH \), and the area of the pyramid inscribed in the cone = \( \frac{1}{3} \) perimeter of polygon \( \cdot AH \), so that the area of the polygon has to the area of the pyramid the ratio \( AH : \Delta H \). Now, by similar triangles, \( AM : MA = AH : HN \), and \( AH : HN > AH : HA \), for \( HA > HN \). Therefore \( AM : MA > AH : HA \); that is, \( \Gamma : \Delta \) exceeds the ratio of the polygon to the surface of the pyramid.
ζονά ἄρα λόγον ἔχει τὸ πολύγωνον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραµµένον πρὸς τὸ πολύγωνον τὸ ἐν τῷ Β ἐγγεγραµµένον ἢ αὐτὸ τὸ πολύγωνον πρὸς τὴν ἐπιφάνειαν τῆς πυραµίδος· µεῖζον ἄρα ἐστὶν ἢ ἐπιφάνεια τῆς πυραµίδος τοῦ ἐν τῷ Β πολυγώνου ἐγγεγραµµένου. Ἐλάσσονα δὲ λόγον ἔχει τὸ πολύγωνον τὸ περὶ τὸν Β κύκλον περιγεγραµµένον πρὸς τὸ ἐγγεγραµµένον ἢ οἱ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου· πολλῷ ἄρα τὸ πολύγωνον τὸ περὶ τὸν Β κύκλον περιγεγραµµένον πρὸς τὴν ἐπιφάνειαν τῆς πυραµίδος τῆς ἐν τῷ κώνῳ ἐγγεγραµµένης Ἐλάσσονα λόγον ἔχει ἢ οἱ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου· ὅπερ ἀδύνατον τὸ µὲν γὰρ περιγεγραµµένον πολύγωνον µεῖζόν ἐστὶν τοῦ Β κύκλου, ἢ δὲ ἐπιφάνεια τῆς πυραµίδος τῆς ἐν τῷ κώνῳ ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κώνου]. ὡς ἄρα οὐδὲ µεῖζων ἐστὶν οἱ κύκλος τῆς ἐπιφανείας τοῦ κώνου. Ἐδείχθη δὲ, ὅτι οὐδὲ Ἐλάσσων ἢγος ἄρα.

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Ἤστω κώνος ἱσοσκελῆς ἐπιπέδῳ τηµήθη παραλλήλῳ τῇ βάσει, τῇ µεταξὗ τῶν παραλληλῶν ἐπιπέδων ἐπιφανεία τοῦ κώνου ἵσος ἐστὶ κύκλος, οὔ ἢ ἐκ τοῦ κέντρου µέσον λόγον ἔχει τῆς τε πλευρᾶς τοῦ κώνου τῆς µεταξὗ τῶν παραλληλῶν ἐπιπέδων καὶ τῆς ἴσης ἀµφοτέραις ταῖς ἐκ τῶν κέντρων τῶν κύκλων τῶν ἐν τοῖς παραλληλοῖς ἐπιπέδοις.

Ἤστω κώνος, οὔ τοῦ διὰ τοῦ ἄξονος τρίγωνον ἵσον τῷ ἈΒΓ, καὶ τετµήθω παραλλήλῳ ἐπιπέδῳ τῇ βάσει, καὶ ποιεῖτω τοµὴν τῆν ΔΕ, ἄξων δὲ τοῦ κώνου ἐστὶ ὁ ΒΗ κύκλος δὲ τις ἐκκεῖσθω, οὔ ἢ 84
scribed in the circle A has to the polygon inscribed in the circle B a ratio greater than that which the same polygon [inscribed in the circle A] has to the surface of the pyramid; therefore the surface of the pyramid is greater than the polygon inscribed in B. Now the polygon circumscribed about the circle B has to the inscribed polygon a ratio less than that which the circle B has to the surface of the cone; by much more therefore the polygon circumscribed about the circle B has to the surface of the pyramid inscribed in the cone a ratio less than that which the circle B has to the surface of the cone; which is impossible.\(^a\) Therefore the circle is not greater than the surface of the cone. And it was proved not to be less; therefore it is equal.

Prop. 16

If an isosceles cone be cut by a plane parallel to the base, the portion of the surface of the cone between the parallel planes is equal to a circle whose radius is a mean proportional between the portion of the side of the cone between the parallel planes and a straight line equal to the sum of the radii of the circles in the parallel planes.

Let there be a cone, in which the triangle through the axis is equal to \(\triangle \alpha \beta \gamma\), and let it be cut by a plane parallel to the base, and let [the cutting plane] make the section \(\Delta \varepsilon\), and let BH be the axis of the cone,

For the circumscribed polygon is greater than the circle B, but the surface of the inscribed pyramid is less than the surface of the cone [Prop. 12]; the explanation to this effect in the text is attributed by Heiberg to an interpolator.

\(^a\) For the circumscribed polygon is greater than the circle B, but the surface of the inscribed pyramid is less than the surface of the cone [Prop. 12]; the explanation to this effect in the text is attributed by Heiberg to an interpolator.
ἔκ τοῦ κέντρου μέση ἀνάλογον ἐστὶ τῆς τῆς ΑΔ καὶ συναμφοτέρου τῆς ΔΖ, ΗΑ, ἐστὶ δὲ κύκλος ὁ Θ.

λέγω, ὅτι ὁ Θ κύκλος ἰσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΔΕ, ΑΓ.

'Εκκείσθωσαν γὰρ κύκλοι οἱ Λ, Κ, καὶ τού μὲν Κ κύκλου ή ἐκ τοῦ κέντρου δυνάσθω τὸ ὑπὸ ΒΔΖ, τοῦ δὲ Λ ή ἐκ τοῦ κέντρου δυνάσθω τὸ ὑπὸ ΒΑΗ· ὁ μὲν ἄρα Λ κύκλος ίσος ἐστὶν τῇ ἐπιφανείᾳ τοῦ ΑΒΓ κώνου, ὁ δὲ Κ κύκλος ἰσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ ΔΕΒ. καὶ επει τὸ ὑπὸ τῶν BA, ΑΗ ἰσον ἐστὶ τῷ τῷ τῶν ΒΔ, ΔΖ καὶ τῷ ὑπὸ τῆς ΑΔ καὶ συναμφοτέρου τῆς ΔΖ, ΑΗ διὰ τὸ παράλληλον εἶναι τῇ ΔΖ τῇ ΑΗ, ἀλλὰ τὸ μὲν ὑπὸ ΑΒ, ΑΗ δύναται ή ἐκ τοῦ κέντρου τοῦ Λ κύκλου, τὸ δὲ ὑπὸ ΒΔ, ΔΖ δύναται ή ἐκ τοῦ κέντρου τοῦ Κ κύκλου, τὸ δὲ ὑπὸ τῆς ΔΑ καὶ συναμφοτέρου τῆς ΔΖ, ΑΗ δύναται ή ἐκ τοῦ κέντρου τοῦ Θ, τὸ άρα ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Λ κύκλου ἰσον ἐστὶ τοῖς ἀπὸ τῶν ἐκ τῶν κέντρων τῶν Κ, Θ κύκλων· ὅστε καὶ ο Λ κύκλος ἰσος ἐστὶ τοῖς Κ, Θ κύκλωσ. 86
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and let there be set out a circle whose radius is a mean proportional between $A\Delta$ and the sum of $\Delta Z$, $HA$, and let $\Theta$ be the circle; I say that the circle $\Theta$ is equal to the portion of the surface of the cone between $\Delta E$, $A\Gamma$.

For let the circles $\Lambda$, $K$ be set out, and let the square of the radius of $K$ be equal to the rectangle contained by $B\Delta$, $\Delta Z$, and let the square of the radius of $\Lambda$ be equal to the rectangle contained by $BA$, $AH$; therefore the circle $\Lambda$ is equal to the surface of the cone $AB\Gamma$, while the circle $K$ is equal to the surface of the cone $\Delta EB$ [Prop. 14]. And since

$$BA \cdot AH = B\Delta \cdot \Delta Z + A\Delta \cdot (\Delta Z + AH)$$

because $\Delta Z$ is parallel to $AH$, while the square of the radius of $\Lambda$ is equal to $AB \cdot AH$, the square of the radius of $K$ is equal to $B\Delta \cdot \Delta Z$, and the square of the radius of $\Theta$ is equal to $\Delta A \cdot (\Delta Z + AH)$, therefore the square on the radius of the circle $\Lambda$ is equal to the sum of the squares on the radii of the circles $K$, $\Theta$; so that the circle $\Lambda$ is equal to the sum of the circles.

* The proof is given by Eutocius as follows:

$$BA : AH = B\Delta : \Delta Z$$

$$\therefore$$

$$BA \cdot \Delta Z = B\Delta \cdot AH.$$  \hspace{1cm} [Eucl. vi. 16

But $$BA \cdot \Delta Z = B\Delta \cdot \Delta Z + A\Delta \cdot \Delta Z.$$  \hspace{1cm} [Eucl. ii. 1

$$\therefore$$

$$B\Delta \cdot AH = B\Delta \cdot \Delta Z + A\Delta \cdot \Delta Z.$$  

Let $\Delta A \cdot AH$ be added to both sides. Then$$B\Delta \cdot AH + \Delta A + AH,$$

i.e.  \hspace{1cm} $$BA \cdot AH = B\Delta \cdot \Delta Z + A\Delta \cdot \Delta Z + A\Delta \cdot AH.$$
ἀλλ’ ὁ μὲν Α ἵσος ἐστὶ τῇ ἐπιφάνειᾳ τοῦ ΒΑΓ κώνου, ὁ δὲ Κ τῇ ἐπιφάνειᾳ τοῦ ΔΒΕ κώνου· λοιπῇ ἂρα ἡ ἐπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν παραλλήλων ἐπιπέδων τῶν ΔΕ, ΑΓ ἵση ἐστὶ τῷ Θ Κύκλῳ.

κα’

’Εὰν εἰς κύκλον πολύγωνον ἐγγραφῇ ἀρτιωπλευρόν τε καὶ ἰσόπλευρον, καὶ διαχώσῃ εὐθεία ἐπιξευγνύουσαι τὰς πλευρὰς τοῦ πολυγώνου, ὥστε αὐτὰς παραλλήλους εἶναι μιὰ ὀποιοὶν τῶν ὑπὸ δύο πλευρὰς τοῦ πολυγώνου ύποτεινούσων, αἱ ἐπιξευγνύουσαι πάσαι πρὸς τὴν τοῦ κύκλου διάμετρον τοῦτον ἔχουσι τὸν λόγον, ὅπερ ἦ ὑποτείνουσα τὰς μιὰ ἐλάσσονας τῶν ἡμίσεων πρὸς τὴν πλευρὰν τοῦ πολυγώνου.

"Εστώ κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ πολύγωνον ἐγγεγράφθω τὸ ΑΕΖΒΗΘΟΓΜΝΔΛΚ, καὶ ἐπεζευχθῶσαν αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ· δὴλον δὴ, ὅτι παράλληλοι εἰσὶν τῇ ὑπὸ δύο πλευρὰς τοῦ πολυγώνου ύποτεινούσῃ· λέγων οὖν, ὅτι αἱ εἰρημέναι πάσαι πρὸς τὴν τοῦ κύκλου διάμετρον τὴν ΑΓ τὸν αὐτὸν λόγον ἔχουσι τῷ τῆς ΓΕ πρὸς ΕΑ.

’Ἐπεζεύχθωσαν γὰρ αἱ ΖΚ, ΛΒ, ΗΔ, ΘΝ· παράλληλος ἄρα ἡ μὲν ΖΚ τῇ ΕΑ, ἡ δὲ ΒΛ τῇ ΖΚ, καὶ ἐτι ἡ μὲν ΔΗ τῇ ΒΛ, ἡ δὲ ΘΝ τῇ ΔΗ, καὶ ἡ ΓΜ τῇ ΘΝ [καὶ ἐπεὶ δύο παράλληλοι εἰσὶν αἱ ΕΑ, ΚΖ, καὶ δύο διηγοῦμεν εἰσὶν αἱ ΕΚ, ΑΟ]1· ἐστων ἄρα, ὅπερ ἡ ΕΞ πρὸς ΕΑ, ὁ ΚΞ πρὸς ΞΟ. ὡς δ’ ἡ ΚΞ πρὸς ΞΟ, ἡ ΖΠ πρὸς ΠΟ, ὡς δὲ

1 καὶ ἐπεὶ ... ΕΚ, ΑΟ om. Heiberg.
K, Θ. But Λ is equal to the surface of the cone BAT, while Κ is equal to the surface of the cone ΔBE; therefore the remainder, the portion of the surface of the cone between the parallel planes ΔE, ΑΓ, is equal to the circle Θ.

Prop. 21

If a regular polygon with an even number of sides be inscribed in a circle, and straight lines be drawn joining the angles a of the polygon, in such a manner as to be parallel to any one whatsoever of the lines subtended by two sides of the polygon, the sum of these connecting lines bears to the diameter of the circle the same ratio as the straight line subtended by half the sides less one bears to the side of the polygon.

Let ABΓΔ be a circle, and in it let the polygon AEZBHΘΓMNΔΛΚ be inscribed, and let EK, ΖΛ, BA, ΗΝ, ΘΜ be joined; then it is clear that they are parallel to a straight line subtended by two sides of the polygon b; I say therefore that the sum of the aforementioned straight lines bears to ΑΓ, the diameter of the circle, the same ratio as ΖΕ bears to ΖΑ.

For let ΖΚ, ΔΒ, ΗΔ, ΘΝ be joined; then ΖΚ is parallel to ΖΑ, ΒΛ to ΖΚ, also ΔΗ to ΒΛ, ΘΝ to ΔΗ and ΓΜ to ΘΝ c; therefore

\[ EΞ : ΕΑ = KΞ : ΕΟ. \]

But \[ KΞ : ΕΟ = ZΠ : ΠΟ, \] [Eucl. vi. 4]

a "Sides" according to the text, but Heiberg thinks Archimedes probably wrote γωνίας where we have πλευράς.

b For, because the arcs KA, EZ are equal, \( \angle EKZ = \angle KZΛ \) [Eucl. iii. 27]; therefore EK is parallel to AZ; and so on.

c For, as the arcs AK, EZ are equal, \( \angle AEK = \angle EKZ \), and therefore AE is parallel to ZK; and so on.
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ή ΖΠ προς ΠΟ, ή ΛΠ προς ΠΡ, ώς δὲ ή ΛΠ προς ΠΡ, οὕτως η ΒΣ προς ΣΡ, καὶ ἐτι, ώς ἡ μὲν ΒΣ προς ΣΡ, ή ΔΣ προς ΣΙ, ώς δὲ ή ΔΣ προς ΣΙ, ή ΗΥ προς ΥΤ, καὶ ἐτι, ώς ἡ μὲν ΗΥ προς ΥΤ, ἡ ΝΥ προς ΥΦ, ώς δὲ ή ΝΥ προς ΥΦ, ή ΘΧ προς ΧΦ, καὶ ἐτι, ώς μὲν ή ΘΧ προς ΧΦ, ή MX προς ΧΓ [καὶ πάντα ἄρα προς πάντα ἐστίν, ὡς εἰς τῶν λόγων προς ἑνα]¹. ὡς ἄρα η ΕΕ προς ΞΑ, οὕτως αἱ ΕΚ, ΖΑ, ΒΔ, ΗΝ, ΘΜ πρὸς τὴν ΑΓ διάμετρον, ὡς δὲ ή ΕΕ προς ΞΑ, οὕτως ή ΓΕ προς ΞΑ. ἔσται ἄρα καὶ, ώς ἡ ΓΕ προς ΞΑ, οὕτω πάσαι αἱ ΕΚ, ΖΑ, ΒΔ, ΗΝ, ΘΜ πρὸς τὴν ΑΓ διάμετρον.

¹ καὶ ... ἑνα om. Heiberg.

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while $Z\Pi : \Pi O = \Lambda \Pi : \Pi R$, [ibid.]
and $\Lambda \Pi : \Pi R = B\Sigma : \Sigma R$. [ibid.]
Again, $B\Sigma : \Sigma R = \Delta \Sigma : \Sigma T$, [ibid.]
while $\Delta \Sigma : \Sigma T = H\Upsilon : \Upsilon T$. [ibid.]
Again, $H\Upsilon : \Upsilon T = N\Upsilon : \Upsilon \Phi$, [ibid.]
while $N\Upsilon : \Upsilon \Phi = \Theta X : X\Phi$. [ibid.]
Again, $\Theta X : X\Phi = M\Xi : X\Gamma$, [ibid.]
therefore $E\Xi : \Xi A = EK + Z\Lambda + B\Delta + HN + 
\Theta M : A\Gamma$. [Eucl. v. 12]

But $E\Xi : \Xi A = GE : EA$; [Eucl. vi. 4]
therefore $GE : EA = EK + Z\Lambda + B\Delta + HN + 
\Theta M : A\Gamma$. b

* By adding all the antecedents and consequents, for $E\Xi : \Xi A = E\Xi + K\Xi + Z\Pi + \Lambda \Pi + B\Sigma + \Delta \Sigma + H\Upsilon + N\Upsilon + \Theta X + 
M\Xi : \Xi A + \Xi O + \Pi O + \Pi R + \Sigma R + \Sigma T + \Upsilon T + \Upsilon \Phi 
+ X\Phi + X\Gamma$
$= EK + Z\Lambda + B\Delta + HN + \Theta M : A\Gamma$.

* If the polygon has $4n$ sides, then

$\angle E\Gamma K = \frac{\pi}{2n}$ and $EK : AG = \sin \frac{\pi}{2n}$;

$\angle Z\Gamma \Lambda = \frac{2\pi}{2n}$ and $Z\Lambda : AG = \sin \frac{2\pi}{2n}$;

$\angle \Theta \Gamma M = (2n - 1) \frac{\pi}{2n}$ and $\Theta M : AG = \sin (2n - 1) \frac{\pi}{2n}$.

Further, $\angle A\Gamma E = \frac{\pi}{4n}$ and $GE : EA = \cot \frac{\pi}{4n}$.

Therefore the proposition shows that

$\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin (2n - 1) \frac{\pi}{2n} = \cot \frac{\pi}{4n}$. 
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κυ'

"Εστω ἐν σφαίρα μέγιστος κύκλος ὁ ΑΒΓΔ, καὶ ἕγγεγράφθω εἰς αὐτὸν πολύγωνον ἰσόπλευρον, τὸ

δὲ πλῆθος τῶν πλευρῶν αὐτοῦ μετρεῖσθω ὑπὸ τετράδος, αἱ δὲ ΑΓ, ΔΒ διάμετροι ἐστώσαν. ἕαν δὴ μενοῦσης τῆς ΑΓ διαμέτρου περιενεχθῇ ὁ ΑΒΓΔ κύκλος ἔχων τὸ πολύγωνον, δῆλον, ὅτι ἡ μὲν περιφέρεια αὐτοῦ κατὰ τὴν ἐπιφάνειαν τῆς σφαίρας ἐνεχθήσεται, αἱ δὲ τοῦ πολυγώνου γωνίαι χωρὶς τῶν πρὸς τὸσ Α, Γ σημείοις κατὰ κύκλων περιφερεῖῶν ἐνεχθήσονται ἐν τῇ ἐπιφάνεια τῆς σφαίρας γεγραμμένων ὀρθῶν πρὸς τὸν ΑΒΓΔ κύκλον διάμετροι δὲ αὐτῶν ἔσονται αἱ ἐπίζευγνόνται τὰς γωνίας τοῦ πολυγώνου παρὰ τὴν ΒΔ οὖσαι. αἱ δὲ τοῦ πολυγώνου πλευραὶ κατὰ τινῶν κώνων ἐνεχθήσονται, αἱ μὲν ΑΖ, ΑΝ κατ᾽ ἐπίφανειας κώνου, οὐ βάσις μὲν ὁ κύκλος ὁ περὶ διάμετρον τῆς ΖΝ, κορυφῇ δὲ τὸ Α σημεῖον, αἱ δὲ

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Let $AB\Gamma\Delta$ be the greatest circle in a sphere, and let there be inscribed in it an equilateral polygon, the number of whose sides is divisible by four, and let $A\Gamma, \Delta B$ be diameters. If the diameter $A\Gamma$ remain stationary and the circle $AB\Gamma\Delta$ containing the polygon be rotated, it is clear that the circumference of the circle will traverse the surface of the sphere, while the angles of the polygon, except those at the points $A, \Gamma$, will traverse the circumferences of circles described on the surface of the sphere at right angles to the circle $AB\Gamma\Delta$; their diameters will be the [straight lines] joining the angles of the polygon, being parallel to $B\Delta$. Now the sides of the polygon will traverse certain cones; $AZ, AN$ will traverse the surface of a cone whose base is the circle about the diameter $ZN$ and whose vertex is the point $A$; $ZH,$
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ΖΗ, ΜΝ κατά τως κωνικῆς ἐπιφανείας οἰσθήσουνται, ἣς βάσις μὲν ὁ κύκλος ὁ περὶ διάμετρον τὴν ΜΗ, κορυφή δὲ τὸ σημεῖον, καθ’ ὁ συμβάλλουσιν ἐκβαλλόμεναι αἱ ΖΗ, ΜΝ ἀλλήλαις τε καὶ τῇ ΑΓ, αἱ δὲ ΒΗ, ΔΜ πλευραὶ κατὰ κωνικῆς ἐπιφανείας οἰσθήσουνται, ἣς βάσις μὲν ἐστὶν ὁ κύκλος ὁ περὶ διάμετρον τὴν ΒΔ ὀρθὸς πρὸς τὸν ΑΒΓΔ κύκλον, κορυφή δὲ τὸ σημεῖον, καθ’ ὁ συμβάλλουσιν ἐκβαλλόμεναι αἱ ΒΗ, ΔΜ ἀλλήλαις τε καὶ τῇ ΓΑ· ὁμοίως δὲ καὶ αἱ ἐν τῷ ἑτέρῳ ἡμικυκλίῳ πλευραὶ κατὰ κωνικῶν ἐπιφανειῶν οἰσθήσουνται πάλιν ὁμοίων ταύταις. ἔσται δὴ τὶ σχῆμα ἐγγεγραμμένον ἐν τῇ σφαῖρᾳ ὑπὸ κωνικῶν ἐπιφανειῶν περιεχόμενον τῶν προερημένων, οὗ ἡ ἐπιφάνεια ἐλάσσων ἐσται τῆς ἐπιφανείας τῆς σφαίρας.

Διαφερεῖσθαι γὰρ τῆς σφαίρας ὑπὸ τοῦ ἐπιπέδου τοῦ κατὰ τὴν ΒΔ ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον ἡ ἐπιφάνεια τοῦ ἑτέρου ἡμισφαιρίου καὶ ἡ ἐπιφάνεια τοῦ σχήματος τοῦ ἐν αὐτῷ ἐγγεγραμμένου τὰ αὐτὰ πέρατα ἔχουσιν ἐν ἐνὶ ἐπιπέδῳ ἀμφοτέρων γὰρ τῶν ἐπιφανειῶν πέρας ἐστὶν τοῦ κύκλου ἡ περιφέρεια τοῦ περὶ διάμετρον τὴν ΒΔ ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον· καὶ εἰς ἀμφότερα ἐπὶ τὰ αὐτὰ κολλαῖ, καὶ περιλαμβάνεται αὐτῶν ἡ ἑτέρα ὑπὸ τῆς ἑτέρας ἐπιφανείας καὶ τῆς ἐπιπέδου τῆς τὰ αὐτὰ πέρατα ἔχουσις αὐτῆς· ὁμοίως δὲ καὶ τοῦ ἐν τῷ ἑτέρῳ ἡμισφαιρίῳ σχῆματος ἡ ἐπιφάνεια ἐλάσσων ἐστὶν τῆς τοῦ ἡμισφαιρίου ἐπιφανείας· καὶ ὅλη ὁδὸν ἡ ἐπιφάνεια τοῦ σχήματος τοῦ ἐν τῇ σφαίρᾳ ἐλάσσων ἐστὶν τῆς ἐπιφανείας τῆς σφαίρας.

* Archimedes would not have omitted to make the deduc-
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MN will traverse the surface of a certain cone whose base is the circle about the diameter MH and whose vertex is the point in which ZH, MN produced meet one another and with AT; the sides BH, MΔ will traverse the surface of a cone whose base is the circle about the diameter BΔ at right angles to the circle ABΓΔ and whose vertex is the point in which BH, ΔM produced meet one another and with TA; in the same way the sides in the other semicircle will traverse surfaces of cones similar to these. As a result there will be inscribed in the sphere and bounded by the aforesaid surfaces of cones a figure whose surface will be less than the surface of the sphere.

For, if the sphere be cut by the plane through BΔ at right angles to the circle ABΓΔ, the surface of one of the hemispheres and the surface of the figure inscribed in it have the same extremities in one plane; for the extremity of both surfaces is the circumference of the circle about the diameter BΔ at right angles to the circle ABΓΔ; and both are concave in the same direction, and one of them is included by the other surface and the plane having the same extremities with it. Similarly the surface of the figure inscribed in the other hemisphere is less than the surface of the hemisphere; and therefore the whole surface of the figure in the sphere is less than the surface of the sphere.

a] From Postulate 4, that the surface of the figure inscribed in the hemisphere is less than the surface of the hemisphere.
'Η τοῦ ἐγγραφομένου σχήματος εἰς τὴν σφαῖραν ἐπιφάνεια ἵση ἐστὶ κύκλω, οὐ ἢ ἐκ τοῦ κέντρου δύναται τὸ περιεχόμενον ὑπὸ τε τῆς πλευρᾶς τοῦ σχήματος καὶ τῆς ἵσης πάσαις ταῖς ἐπιζευγνυόσας τὰς πλευρὰς τοῦ πολυγώνου παράλληλαις οὖσαι τῇ ὑπὸ δύο πλευρὰς τοῦ πολυγώνου ὑποτευνοῦση εὐθεία.

'Εστώ ἐν σφαῖρα μέγιστος κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ πολύγωνον ἐγγεγράφθη ἴσοπλευρον, οὐ αἱ πλευραὶ ὑπὸ τετράδος μετροῦνται, καὶ ἀπὸ τοῦ πολυγώνου τοῦ ἐγγεγραμμένου νοεῖσθω τι εἰς τὴν σφαῖραν ἐγγραφέν σχῆμα, καὶ ἐπεζευγνύσωσαν αἰ ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ παράλληλαι οὖσαι τῇ ὑπὸ δύο πλευρὰς ὑποτευνοῦσῃ εὐθείᾳ, κύκλος δὲ τὶς ἐκκεῖσθω ὁ Ξ, οὐ ἢ ἐκ τοῦ κέντρου δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΑΕ καὶ τῆς ἱσῆς ταῖς ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ· λέγω, ὅτι ὁ κύκλος οὕτος ἱσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ εἰς τὴν σφαῖραν ἐγγραφομένου σχήματος.

'Εκκεῖσθωσαν γὰρ κύκλοι οἱ Ο, Π, Ρ, Σ, Τ, Υ, καὶ τοῦ μὲν Ο ἢ ἐκ τοῦ κέντρου δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΕΑ καὶ τῆς ἴμμεισιάς τῆς ΕΖ, ἢ δὲ ἐκ τοῦ κέντρου τοῦ Π δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΕΑ καὶ τῆς ἴμμεισιάς τῶν ΕΖ, ΗΘ, ΓΔ, ἢ δὲ ἐκ τοῦ κέντρου τοῦ Ρ δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΕΑ καὶ τῆς ἴμμεισιάς τῶν ΗΘ, ΓΔ, ΚΛ, ἢ δὲ ἐκ τοῦ κέντρου τοῦ Τ δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΑΕ καὶ τῆς ἴμμεισιάς

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The surface of the figure inscribed in the sphere is equal to a circle, the square of whose radius is equal to the rectangle contained by the side of the figure and a straight line equal to the sum of the straight lines joining the angles of the polygon, being parallel to the straight line subtended by two sides of the polygon.

Let $AB\Gamma\Delta$ be the greatest circle in a sphere, and in it let there be inscribed an equilateral polygon, the number of whose sides is divisible by four, and, starting from the inscribed polygon, let there be imagined a figure inscribed in the sphere, and let $EZ$, $H\Theta$, $\Gamma\Delta$, $K\Lambda$, $MN$ be joined, being parallel to the straight line subtended by two sides; now let there be set out a circle $E$, the square of whose radius is equal to the rectangle contained by $AE$ and a straight line equal to the sum of $EZ$, $H\Theta$, $\Gamma\Delta$, $K\Lambda$, $MN$; I say that this circle is equal to the surface of the figure inscribed in the sphere.

For let the circles $O$, $\Pi$, $P$, $\Sigma$, $T$, $Y$ be set out, and let the square of the radius of $O$ be equal to the rectangle contained by $EA$ and the half of $EZ$, let the square of the radius of $\Pi$ be equal to the rectangle contained by $EA$ and the half of $EZ + H\Theta$, let the square of the radius of $P$ be equal to the rectangle contained by $EA$ and the half of $H\Theta + \Gamma\Delta$, let the square of the radius of $\Sigma$ be equal to the rectangle contained by $EA$ and the half of $\Gamma\Delta + K\Lambda$, let the square of the radius of $T$ be equal to the rectangle
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tων ΚΑ, ΜΝ, ἥ δὲ ἐκ τοῦ κέντρου τοῦ Υ δυνάσθω τὸ περιεχόμενον ὑπὸ τε τῆς ΑΕ καὶ τῆς ἡμισείας τῆς ΜΝ. διὰ δὴ ταύτα ὁ μὲν Ο κύκλος ἰσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ ΑΕΖ κώνου, ὁ δὲ Π τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΕΖ, ΗΘ, ὁ δὲ Ρ τῇ μεταξὺ τῶν ΗΘ, ΓΔ, ὁ δὲ Σ τῇ μεταξὺ τῶν

ΔΓ, ΚΑ, καὶ ἔτι ὁ μὲν Τ ἰσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΚΑ, ΜΝ, ὁ δὲ Υ τῇ τοῦ ΜΒΝ κώνου ἐπιφανείᾳ ἰσος ἐστίν· οἱ πάντες ἀρα κύκλοι ἰσοι εἰσὶν τῇ τοῦ ἐγγεγραμμένου σχήματος ἐπιφανείᾳ. καὶ φανερῶν, ὅτι αἱ ἐκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Υ κύκλων δύνανται τὸ περιεχόμενον ὑπὸ τε τῆς ΑΕ καὶ δις τῶν ἡμισεῶν τῆς ΕΖ, ΗΘ, ΓΔ, ΚΑ, ΜΝ, αἱ ὅλαι εἰσὶν

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contained by AE and the half of KA + MN, and let the square of the radius of Y be equal to the rectangle contained by AE and the half of MN. Now by these constructions the circle O is equal to the surface of the cone AEZ [Prop. 14], the circle Π is equal to the surface of the conical frustum between EZ and HΘ, the circle P is equal to the surface of the conical frustum between HΘ and ΓΔ, the circle Σ is equal to the surface of the conical frustum between ΔΓ and KA, the circle T is equal to the surface of the conical frustum between KA, MN [Prop. 16], and the circle Y is equal to the surface of the cone MBN [Prop. 14]; the sum of the circles is therefore equal to the surface of the inscribed figure. And it is manifest that the sum of the squares of the radii of the circles O, Π, P, Σ, T, Y is equal to the rectangle contained by AE and twice the sum of the halves of EZ, HΘ, ΓΔ, KA, MN, that is to say, the sum of EZ,
If the radius of the sphere is \( a \) this proposition shows that

Surface of inscribed figure = circle \( \Xi \)

\[
= \pi \cdot AE \cdot (EZ + H\Theta + \Gamma\Delta + K\Lambda + MN).
\]

Now \( AE = 2a \sin \frac{\pi}{4n} \), and by p. 91 n. b
Prop. 25

The surface of the figure inscribed in the sphere and bounded by the surfaces of cones is less than four times the greatest of the circles in the sphere.

Let $AB\Gamma\Delta$ be the greatest circle in a sphere, and in it let there be inscribed an equilateral polygon, the number of whose sides is divisible by four, and, starting from it, let a surface bounded by surfaces of

$$EZ + H\Theta + \Gamma\Lambda + K\Lambda + MN = 2a \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \frac{(2n-1)\pi}{2n} \right].$$

\[ \text{Surface of inscribed figure} = 4\pi a^2 \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin (2n-1) \frac{\pi}{2n} \right] \]

$$= 4\pi a^2 \cos \frac{\pi}{4n}$$

(by p. 91 n. b.)

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κωνικῶν ἐπιφανειών περιεχομένη· λέγω, ὅτι ἡ ἐπιφάνεια τοῦ ἐγγραφέντος ἐλάσσον· ἔστιν ἡ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

'Ἐπεξεύθωσαν γάρ αἱ ὑπὸ δύο πλευράς ὑποτείνουσαι τοῦ πολυγώνου αἱ ΕΙ, ΘΜ καὶ ταῦτας παράλληλοι αἱ ΖΚ, ΔΒ, ΗΛ, ἐκκείσθω δὲ τις κύκλος ὁ Ρ, οὗ ἡ ἐκ τοῦ κέντρου δύναται τὸ ὑπὸ τῆς ΕΑ καὶ τῆς ἱσης πάσαις ταῖς ΕΙ, ΖΚ, ΒΔ, ΗΛ, ΘΜ· διὰ δὲ τὸ προδειχθὲν ἱσοῦ ἔστιν ὁ κύκλος τῇ τού εἰρημένου σχῆματος ἐπιφανεία. καὶ ἔπει ἐδείχθη, ὅτι ἔστιν, ὡς ἡ ἱση πάσαις ταῖς ΕΙ, ΖΚ, ΒΔ, ΗΛ, ΘΜ πρὸς τὴν διάμετρον τοῦ κύκλου τῆς ΑΓ, οὕτως ἡ ΓΕ πρὸς ΕΑ, τὸ ἀρα ὑπὸ τῆς ἱσης πάσαις ταῖς εἰρημέναις καὶ τῆς ΕΑ, τούτεστιν τὸ ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Ρ κύκλου, ἵσον ἔστιν τῷ ὑπὸ τῶν ΑΓ, ΓΕ. ἀλλὰ καὶ τὸ ὑπὸ ΑΓ, ΓΕ ἐλάσσον ἐστὶ τοῦ ἀπὸ τῆς ΑΓ· ἐλάσσον ἄρα ἔστιν τὸ ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Ρ ἀπὸ τῆς ΑΓ [ἐλάσσον ἄρα ἐστὶν ἡ ἐκ τοῦ κέντρου τοῦ Ρ τῆς ΑΓ· ὥστε ἡ διάμετρος τοῦ Ρ κύκλου ἐλάσσων ἔστιν ἡ διάμετρος τῆς διαμέτρου τοῦ ΑΒΓΔ κύκλου, καὶ δύο ἄρα τοῦ ΑΒΓΔ κύκλου διάμετρου μεῖζον εἰσὶ τῆς διαμέτρου τοῦ Ρ κύκλου, καὶ τὸ τετράκις ἀπὸ τῆς διαμέτρου τοῦ ΑΒΓΔ κύκλου, τούτεστι τῆς ΑΓ, μεῖζόν ἔστι τοῦ ἀπὸ τῆς τοῦ Ρ κύκλου διαμέτρου. ὡς δὲ τὸ τετράκις ἀπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς τοῦ Ρ κύκλου διαμέτρου, οὕτως τέσσαρες κύκλοι οἱ ΑΒΓΔ πρὸς τὸν Ρ κύκλον· τέσσαρες ἄρα κύκλοι οἱ ΑΒΓΔ μεῖζον εἰσίν τοῦ Ρ κύκλου]². ὁ ἄρα κύκλος ὁ Ρ ἐλάσσων ἔστιν ἡ τετραπλασία τοῦ

² ἐλάσσων . . . κύκλου om. Heiberg.
cones be imagined; I say that the surface of the inscribed figure is less than four times the greatest of the circles inscribed in the sphere.

For let $EI, \Theta M$, subtended by two sides of the polygon, be joined, and let $ZK, \Delta B, H\Lambda$ be parallel to them, and let there be set out a circle $P$, the square of whose radius is equal to the rectangle contained by $EA$ and a straight line equal to the sum of $EI, ZK, B\Delta, H\Lambda, \Theta M$; by what has been proved above, the circle is equal to the surface of the aforesaid figure. And since it was proved that the ratio of the sum of $EI, ZK, B\Delta, H\Lambda, \Theta M$ to $\Gamma P$, the diameter of the circle, is equal to the ratio of $\Gamma E$ to $EA$ [Prop. 21], therefore

$$EA \cdot (EI + ZK + B\Delta + H\Lambda + \Theta M)$$

that is, the square on the radius of the circle $P$

$$= \Gamma P \cdot \Gamma E. \quad \text{[Eucl. vi. 16]}$$

But $\Gamma P \cdot \Gamma E < \Gamma P^2$. \quad \text{[Eucl. iii. 15]}

Therefore the square on the radius of $P$ is less than the square on $\Gamma P$; therefore the circle $P$ is less...
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μεγιστον κύκλου. ὁ δὲ Ρ κύκλος ἵσος ἐδείχθη τῇ εἰρημένῃ ἐπιφανείᾳ τοῦ σχήματος· ἢ ἀρα ἐπιφάνεια τοῦ σχήματος ἐλάσσων ἐστὶν ἢ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

κη'

"Εστω ἐν σφαίρᾳ μέγιστος κύκλος ὁ ΑΒΓΔ, περὶ δὲ τὸν ΑΒΓΔ κύκλον περιγεγράφθω πολύγωνον ἰσόπλευρον τε καὶ ἰσογώνιον, τὸ δὲ πλῆθος τῶν πλευρῶν αὐτοῦ μετρεῖσθω ὑπὸ τετράδος, τὸ δὲ περὶ τὸν κύκλον περιγεγραμμένον πολύγωνον κύκλος περιγεγραμμένος περιλαμβανέτω περὶ τὸ αὐτὸ κέντρον γυνόμενος τῷ ΑΒΓΔ. μενούσης δὴ τῆς ΕΗ περιενεχθῆτω τὸ ΕΖΗΘ ἐπίπεδον, ἐν ὑ γ τὸ τε πολύγωνον καὶ ὁ κύκλος· δήλον οὖν, ὅτι ἢ μὲν περιφέρεια τοῦ ΑΒΓΔ κύκλου κατὰ τῆς ἐπι-

φανείας τῆς σφαίρας οὐσθησεται, ἢ δὲ περιφέρεια 

tου ΕΖΗΘ κατ' ἄλλης ἐπιφανείας σφαίρας τὸ 104
than four times the greatest circle. But the circle $P$ was proved equal to the aforesaid surface of the figure; therefore the surface of the figure is less than four times the greatest of the circles in the sphere.

Prop. 28

Let $AB\Gamma\Delta$ be the greatest circle in a sphere, and about the circle $AB\Gamma\Delta$ let there be circumscribed an equilateral and equiangular polygon, the number of whose sides is divisible by four, and let a circle be described about the polygon circumscribing the circle, having the same centre as $AB\Gamma\Delta$. While $EH$ remains stationary, let the plane $EZ\Theta\Omega$, in which lie both the polygon and the circle, be rotated; it is clear that the circumference of the circle $AB\Gamma\Delta$ will traverse the surface of the sphere, while the circumference of $EZ\Theta\Omega$ will traverse the surface of another
αυτὸ κέντρον ἔχουσις τῇ ἐλάσσουν οἰσθήσεται, αἱ δὲ ἀφαί, καθ' ἄς ἐπιφανοῦσιν αἱ πλευραὶ, γρά-
φουσιν κύκλου ὀρθοὺς πρὸς τὸν ΑΒΓΔ κύκλον ἐν τῇ ἐλάσσουν σφαίρα, αἱ δὲ γωνίαι τοῦ πολυγώ-
νου χωρίς τῶν πρὸς τοῖς Ε, Η σημείοις κατὰ κύ-
κλων περιφερείων οἰσθήσονται ἐν τῇ ἐπιφανείᾳ τῆς
μείζονος σφαίρας γεγραμμένων ὀρθῶν πρὸς τὸν
ΕΖΗΘ κύκλον, αἱ δὲ πλευραὶ τοῦ πολυγώνου κατὰ
κωνικῶν ἐπιφανειῶν οἰσθήσονται, καθάπερ ἐπὶ τῶν
πρὸ τοῦτον ἔσται οὖν τὸ σχῆμα τὸ περιεχόμενον
ὑπὸ τῶν ἐπιφανείων τῶν κωνικῶν περὶ μὲν τὴν
ἐλάσσονα σφαῖραν περιγεγραμμένον, ἐν δὲ τῇ
μείζων ἐγγεγραμμένον. ὃτι δὲ ἡ ἐπιφάνεια τοῦ
περιγεγραμμένου σχῆματος μείζων ἔστι τῆς ἐπι-
φανείας τῆς σφαίρας, οὕτως δειχθήσεται.

'Εστω γὰρ ἡ KD διάμετρος κύκλου τυός τῶν ἐν
τῇ ἐλάσσουν σφαίρα τῶν K, Δ σημείων οὗτων,
καθ' ἄ ἀπτονται τοῦ ΑΒΓΔ κύκλον αἱ πλευραὶ
tοῦ περιγεγραμμένου πολυγώνου. διηρημένης δὲ
τῆς σφαίρας ὑπὸ τοῦ ἐπιπέδου τοῦ κατὰ τὴν KD
ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον καὶ ἡ ἐπιφάνεια
tοῦ περιγεγραμμένου σχῆματος περὶ τὴν σφαῖραν
diaírēthēsetai ὑπὸ τοῦ ἐπιπέδου. καὶ φανερὸν, ὅτι
tα αὐτὰ πέρατα ἔχουσιν ἐν ἐπιπέδῳ ἀμφιτέρων
γὰρ τῶν ἐπιπέδων πέρας ἐστὶν ἡ τοῦ κύκλου
περιφέρεια τοῦ περὶ διάμετρον τῆς KD ὀρθοῦ πρὸς
τὸν ΑΒΓΔ κύκλον· καὶ εἰς ἀμφότερα ἐπὶ τὰ
αὐτὰ κοίλαι, καὶ περιλαμβάνεται ἡ ἔτερα αὐτῶν
ὑπὸ τῆς ἐτέρας ἐπιφανείας καὶ τῆς ἐπιπέδου τῆς
tὰ αὐτὰ πέρατα ἐχούσις· ἐλάσσων οὖν ἔστιν ἡ
περιλαμβανομένη τοῦ τμῆματος τῆς σφαίρας ἐπι-
φάνεια τῆς ἐπιφανείας τοῦ σχῆματος τοῦ περι-

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sphere, having the same centre as the lesser sphere; the points of contact in which the sides touch [the smaller circle] will describe circles on the lesser sphere at right angles to the circle $\Delta \Gamma \Lambda$, and the angles of the polygon, except those at the points $E$, $H$ will traverse the circumferences of circles on the surface of the greater sphere at right angles to the circle $\varepsilon \zeta \pi \theta$, while the sides of the polygon will traverse surfaces of cones, as in the former case; there will therefore be a figure, bounded by surfaces of cones, described about the lesser sphere and inscribed in the greater. That the surface of the circumscribed figure is greater than the surface of the sphere will be proved thus.

Let $K\Delta$ be a diameter of one of the circles in the lesser sphere, $K$, $\Delta$ being points at which the sides of the circumscribed polygon touch the circle $\Delta \Gamma \Lambda$. Now, since the sphere is divided by the plane containing $K\Delta$ at right angles to the circle $\Delta \Gamma \Lambda$, the surface of the figure circumscribed about the sphere will be divided by the same plane. And it is manifest that they $^a$ have the same extremities in a plane; for the extremity of both surfaces $^b$ is the circumference of the circle about the diameter $K\Delta$ at right angles to the circle $\Delta \Gamma \Lambda$; and they are both concave in the same direction, and one of them is included by the other and the plane having the same extremities; therefore the included surface of the segment of the sphere is less than the surface of

\[ \text{i.e., the surface formed by the revolution of the circular segment } K\Delta \text{ and the surface formed by the revolution of the portion } K \ldots E \ldots \Delta \text{ of the polygon.} \]

$^a$ In the text $\varepsilon \pi \pi \varepsilon \delta \omega \nu$ should obviously be $\varepsilon \mu \phi \alpha \nu \varepsilon \omega \nu$.

$^b$ ai $\pi \lambda \varepsilon \rho \alpha i$ Heiberg; om. codd.
If the radius of the inner sphere is $a$ and that of the outer sphere $a'$, and the regular polygon has $4n$ sides, then
$$a' = a \sec \frac{\pi}{4n}.$$ 

This proposition shows that
\[
\text{Area of figure circumscribed in circle of radius } a = \text{Area of figure inscribed in circle of radius } a'.
\]
the figure circumscribed about it [Post. 4]. Similarly the surface of the remaining segment of the sphere is less than the surface of the figure circumscribed about it; it is clear therefore that the whole surface of the sphere is less than the surface of the figure circumscribed about it.

Prop. 29

The surface of the figure circumscribed about the sphere is equal to a circle, the square of whose radius is equal to the rectangle contained by one side of the polygon and a straight line equal to the sum of all the straight lines joining the angles of the polygon, being parallel to one of the straight lines subtended by two sides of the polygon.

For the figure circumscribed about the lesser sphere is inscribed in the greater sphere [Prop. 28]; and it has been proved that the surface of the figure inscribed in the sphere and formed by surfaces of cones is equal to a circle, the square of whose radius is equal to the rectangle contained by one side of the polygon and a straight line equal to the sum of all the straight lines joining the angles of the polygon, being parallel to one of the straight lines subtended by two sides [Prop. 24]; what was aforesaid is therefore obvious.\(^6\)

\[
= 4\pi a^2 \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left( \frac{2n-1}{2n} \right) \right],
\]

or \(4\pi a^2 \cos \frac{\pi}{4n}\) by p. 91 n. b

\[
= 4\pi a^2 \sec^2 \frac{\pi}{4n} \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left( \frac{2n-1}{2n} \right) \right],
\]

or \(4\pi a^2 \sec \frac{\pi}{4n}\).
Τοῦ σχήματος τοῦ περιγεγραμμένου περὶ τὴν σφαίραν ἡ ἐπιφάνεια μείζων ἐστὶν ἡ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

'Εστώ γὰρ ἡ τε σφαίρα καὶ ὁ κύκλος καὶ τὰ ἄλλα τὰ αὐτὰ τοὺς πρῶτον προκειμένους, καὶ ὁ Δ κύκλος ἵσος τῇ ἐπιφανείᾳ ἐστὼ τοῦ προκειμένου περιγεγραμμένου περὶ τὴν ἑλάσσονα σφαίραν.

'Επεὶ οὖν ἐν τῷ ΕΖΗΘ Κύκλῳ πολύγωνον ἰσόπλευρον ἐγγέγραπται καὶ ἀρτιογώνιον, αἱ ἐπιζευγνύουσαι τὰς τοῦ πολύγωνος πλευρὰς παράλληλοι οὖσαι τῇ ΖΘ πρὸς τὴν ΖΘ τοῦ αὐτοῦ λόγου ἕχουσιν, διὸ ἦ ΘΚ πρὸς ΚΖ: ἵσων ἀρα ἐστὶν τὸ περιεχόμενον σχῆμα ὑπὸ τε μιᾶς πλευρᾶς τοῦ πολύγωνον καὶ τῆς ἵσης πάσαις ταῖς ἐπιζευγνυόουσαι τὰς γωνίας τοῦ πολύγωνον τῷ περιεχόμενω ὑπὸ τῶν ΖΘΚ: ὡστε ἦ ἐκ τοῦ κέντρου τοῦ Δ κύκλου ἰσον δύναται τῷ ὑπὸ ΖΘΚ: μείζων ἀρα
Prop. 30

The surface of the figure circumscribed about the sphere is greater than four times the greatest of the circles in the sphere.

For let there be both the sphere and the circle and the other things the same as were posited before, and let the circle $\Lambda$ be equal to the surface of the given figure circumscribed about the lesser sphere.

Therefore since in the circle $EZH\Theta$ there has been inscribed an equilateral polygon with an even number of angles, the [sum of the straight lines] joining the sides of the polygon, being parallel to $Z\Theta$, have the same ratio to $Z\Theta$ as $\Theta K$ to $KZ$ [Prop. 21]; therefore the rectangle contained by one side of the polygon and the straight line equal to the sum of the straight lines joining the angles of the polygon is equal to the rectangle contained by $Z\Theta$, $\Theta K$ [Eucl. vi. 16]; so that the square of the radius of the circle $\Lambda$ is equal to the rectangle contained by $Z\Theta$, $\Theta K$.
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ἐστίν ἡ ἐκ τοῦ κέντρου τοῦ Δ κύκλου τῆς ΘΚ. ἡ δὲ ΘΚ ἐστὶ τῇ διαμέτρῳ τοῦ ABΓΔ κύκλου [διπλασία γὰρ ἐστὶν τῆς ΧΣ οὗσι ἐκ τοῦ κέντρου τοῦ ABΓΔ κύκλου].¹ δήλον οὖν, ὅτι μείζων ἐστίν ἡ τετραπλάσιος ὁ Δ κύκλος, τούτεστιν ἡ ἐπι-
φάνεια τοῦ περιγεγραμμένου σχήματος περὶ τὴν ἐλάσσονα σφαῖραν, τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

λγ'

Πάσης σφαίρας ἡ ἐπιφάνεια τετραπλάσια ἐστὶ τοῦ μεγίστου κύκλου τῶν ἐν αὐτῇ.

"Εστω γὰρ σφαῖρα τις, καὶ ἐστω τετραπλάσιος τοῦ μεγίστου κύκλου ὁ Α· λέγω, ὅτι ὁ Α ἴσος ἐστὶν τῇ ἐπιφάνεια τῆς σφαίρας.

Εἰ γὰρ μῆ, ἦτοι μείζων ἐστὶν ἡ ἐλάσσων. ἐστω πρῶτερον μείζων ἡ ἐπιφάνεια τῆς σφαίρας τοῦ κύκλου. ἐστὶ δὴ δύο μεγέθη ἄνισα ἡ τε ἐπιφάνεια τῆς σφαίρας καὶ ὁ Δ κύκλος· δυνατὸν ἃ ἐστὶ λαβεῖν δύο εὐθείας ἄνισους, ὥστε τὴν μείζονα πρὸς τὴν ἐλάσσονα λόγον ἔχειν ἐλάσσονα τοῦ, ὅν ἔχει ἡ

¹ διπλασία ... κύκλου om. Heiberg.

* Because ZΘ>ΘΚ [Eucl. iii. 15].

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[Prop. 29]. Therefore the radius of the circle $\Lambda$ is greater than $\Theta K$.

Now $\Theta K$ is equal to the diameter of the circle $\Lambda \Gamma \Delta$; it is therefore clear that the circle $\Lambda$, that is, the surface of the figure circumscribed about the lesser sphere, is greater than four times the greatest of the circles in the sphere.

Prop. 33

The surface of any sphere is four times the greatest of the circles in it.

For let there be a sphere, and let $A$ be four times the greatest circle; I say that $A$ is equal to the surface of the sphere.

For if not, either it is greater or less. First, let the surface of the sphere be greater than the circle.

Then there are two unequal magnitudes, the surface of the sphere and the circle $A$; it is therefore possible to take two unequal straight lines so that the greater bears to the less a ratio less than that which the sur-
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ἐπιφάνεια τῆς σφαίρας πρὸς τὸν κύκλον. εἰλήφθωσαν αἱ Β, Γ, καὶ τῶν Β, Γ μέση ἀνάλογον ἔστω ἡ Δ, νοείσθω δὲ καὶ ἡ σφαίρα ἐπιπέδῳ τετμημένη διὰ τοῦ κέντρου κατὰ τὸν ΕΖΗΘ κύκλον, νοείσθω δὲ καὶ εἰς τὸν κύκλον ἐγγεγραμμένον καὶ περιγεγραμμένον πολύγωνον, ὡστε ὁμοιον εἶναι τὸ περιγεγραμμένον τῷ ἐγγεγραμμένῳ πολύγωνῳ καὶ τὴν τοῦ περιγεγραμμένου πλευράν ἑλάσσονα λόγον ἔχειν τοῦ, ὃν ἔχει ἡ Β πρὸς Δ [καὶ ὁ διπλάσιος ἀρα λόγος τοῦ διπλασίου λόγου ἐστὶν ἑλάσσων. καὶ τοῦ μὲν τῆς Β πρὸς Δ διπλάσιος ἐστὶν ὁ τῆς Β πρὸς τὴν Γ, τῆς δὲ πλευρᾶς τοῦ περιγεγραμμένου πολύγωνου πρὸς τὴν πλευράν τοῦ ἐγγεγραμμένου διπλάσιος ὁ τῆς ἐπιφάνειας τοῦ περιγεγραμμένου στερεοῦ πρὸς τὴν ἐπιφάνειαν τοῦ ἐγγεγραμμένου]¹. ἡ ἐπιφάνεια ἀρα τοῦ περιγεγραμμένου σχήματος περὶ τῆς σφαίρας πρὸς τὴν ἐπιφάνειαν τοῦ ἐγγεγραμμένου σχήματος ἑλάσσονα λόγον ἔχει ἕπερ ἡ ἐπιφάνεια τῆς σφαίρας πρὸς τὸν Α κύκλον ὁπερ ἀτοπον ἡ μὲν γὰρ ἐπιφάνεια τοῦ περιγεγραμμένου τῆς ἐπιφάνειας τῆς σφαίρας μείζων ἐστὶν, ἡ δὲ ἐπιφάνεια τοῦ ἐγγεγραμμένου σχήματος τοῦ Α κύκλου ἑλάσσων ἐστὶ [δεδεικται γὰρ ἡ ἐπιφάνεια τοῦ ἐγγεγραμμένου ἑλάσσων τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ ἡ τετραπλασία, τοῦ δὲ ἐμεγίστου κύκλου τετραπλασίος ἐστὶν ὁ Α κύκλος].² οὐκ ἀρα ἡ ἐπιφάνεια τῆς σφαίρας μείζων ἐστὶ τοῦ Α κύκλου.

¹ καὶ ... ἐγγεγραμμένον om. Heiberg.
² δεδεικται ... κύκλος “repetitionem inutilem Prop. 25,” om. Heiberg.

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face of the sphere bears to the circle [Prop. 2]. Let B, Γ be so taken, and let Δ be a mean proportional between B, Γ, and let the sphere be imagined as cut through the centre along the [plane of the] circle EZHΘ, and let there be imagined a polygon inscribed in the circle and another circumscribed about it in such a manner that the circumscribed polygon is similar to the inscribed polygon and the side of the circumscribed polygon has [to the side of the inscribed polygon] a ratio less than that which B has to Δ [Prop. 3]. Therefore the surface of the figure circumscribed about the sphere has to the surface of the inscribed figure a ratio less than that which the surface of the sphere has to the circle A; which is absurd; for the surface of the circumscribed figure is greater than the surface of the sphere [Prop. 28], while the surface of the inscribed figure is less than the circle A [Prop. 25]. Therefore the surface of the sphere is not greater than the circle A.

Archimedes would not have omitted: πρὸς τὴν τοῦ ἐγγεγραμμένου.
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Δέγω δή, ὅτι οὐδὲ ἐλάσσον. εἰ γὰρ δυνατὸν, ἐστῳ· καὶ ὡμοίως εὐρήσθωσαν αἱ Β, Γ εὐθεῖαι, ὃστε τὴν Β πρὸς Γ ἐλάσσονα λόγον ἔχειν τοῦ, ὃν ἔχει ὁ Α κύκλος πρὸς τὴν ἐπιφάνειαν τῆς σφαίρας, καὶ τῶν Β, Γ μέσῃ ἀνάλογον ἡ Δ, καὶ ἐγγεγράφθω καὶ περιγεγράφθω πάλιν, ὃστε τὴν τοῦ περιγε- γραμμένου ἐλάσσονα λόγον ἔχειν τοῦ τῆς Β πρὸς Δ [καὶ τὰ διπλάσια ἀρὰ]1. ἡ ἐπιφάνεια ἄρα τοῦ περιγεγραμμένου πρὸς τὴν ἐπιφάνειαν τοῦ ἐγ- γεγραμμένου ἐλάσσονα λόγον ἔχει ἦπερ [ἡ Β πρὸς Γ. ἡ δὲ Β πρὸς Γ ἐλάσσονα λόγον ἔχει ἦπερ2 ὁ Α κύκλος πρὸς τὴν ἐπιφάνειαν τῆς σφαίρας. ὁπερ ἀτοπον. ἡ μὲν γὰρ τοῦ περιγεγραμμένου ἐπιφάνεια μειζὸν ἔστι τοῦ Α κύκλου, ἡ δὲ τοῦ ἐγγεγραμμένου ἐλάσσων τῆς ἐπιφάνειας τῆς σφαίρας.

Ω尼克 ἄρα οὐδὲ ἐλάσσον ἡ ἐπιφάνεια τῆς σφαίρας τοῦ Α κύκλου. ἐδείχθη δὲ, ὅτι οὐδὲ μειζὸν. ἡ ἄρα ἐπιφάνεια τῆς σφαίρας ἢση ἔστι τῶν Α κύκλων, τούτεστι τῷ τετραπλαοῦ τοῦ μεγίστου κύκλου.

1 καὶ . . . ἄρα om. Heiberg.
2 ἡ Β . . . ἦπερ om. Heiberg.

Archimedes would not have omitted these words.

On p. 100 n. a it was proved that the area of the inscribed figure is

\[4\pi a^2 \sin \frac{\pi}{n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left(\frac{2n - 1}{2n}\right) \frac{\pi}{2n} \right],\]

or \[4\pi a^2 \cos \frac{\pi}{4n}.

On p. 108 n. a it was proved that the area of the circumscribed figure is

\[4\pi a^2 \sec^2 \frac{\pi}{4n} \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left(\frac{2n - 1}{2n}\right) \frac{\pi}{2n} \right],\]

or \[4\pi a^2 \sec \frac{\pi}{4n}.

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I say now that neither is it less. For, if possible let it be; and let the straight lines $B, \Gamma$ be similarly found, so that $B$ has to $\Gamma$ a less ratio than that which the circle $A$ has to the surface of the sphere, and let $\Delta$ be a mean proportional between $B, \Gamma$, and let [polygons] be again inscribed and circumscribed, so that the [side] of the circumscribed polygon has [to the side of the inscribed polygon] a less ratio than that of $B$ to $\Delta$; then the surface of the circumscribed polygon has to the surface of the inscribed polygon a ratio less than that which the circle $A$ has to the surface of the sphere; which is absurd; for the surface of the circumscribed polygon is greater than the circle $A$, while that of the inscribed polygon is less than the surface of the sphere.

Therefore the surface of the sphere is not less than the circle $A$. And it was proved not to be greater; therefore the surface of the sphere is equal to the circle $A$, that is to four times the greatest circle.

When $n$ is indefinitely increased, the inscribed and circumscribed figures become identical with one another and with the circle, and, since $\cos \frac{\pi}{4n}$ and $\sec \frac{\pi}{4n}$ both become unity, the above expressions both give the area of the circle as $4\pi a^2$.

But the first expressions are, when $n$ is indefinitely increased, precisely what is meant by the integral

$$4\pi a^2 \cdot \frac{1}{2} \int_0^\pi \sin \phi \, d\phi,$$

which is familiar to every student of the calculus as the formula for the area of a sphere and has the value $4\pi a^2$.

Thus Archimedes’ procedure is equivalent to a genuine integration, but when it comes to the last stage, instead of saying, "Let the sides of the polygon be indefinitely
increased," he prefers to prove that the area of the sphere cannot be either greater or less than $4\pi a^2$. By this double *reductio ad absurdum* he avoids the logical difficulties of dealing with indefinitely small quantities, difficulties that were not fully overcome until recent times.

The procedure by which in this same book Archimedes...
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Prop. 34

Any sphere is four times as great as the cone having a base equal to the greatest of the circles in the sphere and height equal to the radius of the sphere.

For let there be a sphere in which $AB\Gamma\Delta$ is the greatest circle. If the sphere is not four times the aforesaid cone, let it be, if possible, greater than four times; let $\Xi$ be a cone having a base four times the circle $AB\Gamma\Delta$ and height equal to the radius of the sphere; then the sphere is greater than the cone $\Xi$. Accordingly there will be two unequal magnitudes, the sphere and the cone; it is therefore possible to take two unequal straight lines so that finds the surface of the segment of a sphere is equivalent to the integration

$$\pi a^2 \int^a 2 \sin \theta \, d\theta = 2\pi a^2 (1 - \cos a).$$

Concurrently Archimedes finds the volumes of a sphere and segment of a sphere. He uses the same inscribed and circumscribed figures, and the procedure is equivalent to multiplying the above formulae by $\frac{1}{3}a$ throughout. Other “integrations” effected by Archimedes are the volume of a segment of a paraboloid of revolution, the volume of a segment of a hyperboloid of revolution, the volume of a segment of a spheroid, the area of a spiral and the area of a segment of a parabola. He also finds the area of an ellipse, but not by a method equivalent to integration. The subject is fully treated by Heath, *The Works of Archimedes*, pp. cxlii-cliv, to whom I am much indebted in writing this note.
Eutocius supplies a proof on these lines. Let the lengths of $K, I, \Theta, H$ be $a, b, c, d$. Then $a - b = b - c = c - d$, and it is required to prove that $a : d > a^2 : b^2$.

Take $x$ such that $a : b = b : x$.

Then $a - b : a = b - x : b$,

and since $a > b$,

$\frac{a - b}{b} > \frac{b - x}{b}$.

But, by hypothesis,

$\frac{a - b}{b} = \frac{b - c}{b}$.

Therefore $\frac{b - c}{b} > \frac{b - x}{b}$,

and so $x > c$.
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the greater will have to the less a less ratio than that which the sphere has to the cone \( \mathcal{E} \). Therefore let the straight lines \( K, H, \) and the straight lines \( I, \Theta, \) be so taken that \( K \) exceeds \( I, \) and \( I \) exceeds \( \Theta \) and \( \Theta \) exceeds \( H \) by an equal quantity; let there be imagined inscribed in the circle \( AB\Gamma\Delta \) a polygon the number of whose sides is divisible by four; let another be circumscribed similar to that inscribed so that, as before, the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that \( K : I \); and let \( A\Gamma, B\Delta \) be diameters at right angles. Then if, while the diameter \( A\Gamma \) remains stationary, the surface in which the polygons lie be revolved, there will result two [solid] figures, one inscribed in the sphere and the other circumscribed, and the circumscribed figure will have to the inscribed the triplicate ratio of that which the side of the circumscribed figure has to the side of the figure inscribed in the circle \( AB\Gamma\Delta \) [Prop. 32]. But the ratio of the one side to the other is less than \( K : I \) \( \textit{ex hypothesi} \); and so the circumscribed figure has \( \text{to the inscribed} \) a ratio less than \( K^3 : I^3 \). But \( K : H > K^3 : I^3 \); by much more there-

Again, take \( y \) such that \( b : z = x : y \).
Then, as before \( b - x > x - y \).
Therefore, \( a \) \textit{fortiori}, \( b - c > x - y \).
But, by hypothesis, \( b - c = c - d \).
Therefore \( c - d > x - y \).
But \( x > c \), and so \( y > d \).
But, by hypothesis, \( a : b = b : x = x : y \),
\( a : y = a^3 : b^3 \) \hspace{1cm} [Eucl. v. Def. 10, also vol. i. p. 258 n. b.]
Therefore \( a : d > a^3 : b^3 \).
ἐχει δὲ καὶ ἡ Κ πρὸς Η μεῖζον λόγων ἡ τριπλάσιον τοῦ, ὅν ἐχει ἡ Κ πρὸς Ι [τούτο γὰρ φανερὸν διὰ λημμάτων]. πολλῷ ἄρα τὸ περιγραφέν πρὸς τὸ ἐγγραφέν ἐλάσσονα λόγων ἔχει τοῦ, ὅν ἐχει ἡ Κ πρὸς Η. ἡ δὲ Κ πρὸς Η ἐλάσσονα λόγων ἔχει ἦπερ ἡ σφαῖρα πρὸς τὸν Χ κώνου. καὶ ἐναλλάξῃ ὁπερ ἀδύνατον. τὸ γὰρ σχῆμα τὸ περιγεγραμμένον μεῖζον ἐστὶ τῆς σφαίρας, τὸ δὲ ἐγγεγραμμένον ἐλάσσον τοῦ Χ κώνου [διὸτι ὁ μὲν Χ κώνος τετραπλάσιος ἐστὶ τοῦ κώνου τοῦ βάσιν μὲν ἤχοντος ἵσῃ τῷ ΑΒΓΔ κύκλω, ὑψὸς δὲ ἰσον τῇ ἐκ τοῦ κέντρου τῆς σφαίρας, τὸ δὲ ἐγγεγραμμένον σχῆμα ἐλάσσον τοῦ εἰρημένου κώνου ἡ τετραπλάσιον]. οὐκ ἄρα μεῖζων ἡ τετραπλασία ἡ σφαῖρα τοῦ εἰρημένου.

Έστω, εἰ δυνατὸν, ἐλάσσων ἡ τετραπλασία· ὅστε ἐλάσσων ἐστὶ τῆς σφαίρας τοῦ Χ κώνου. εἰλήφθωσαν δὴ αἱ Κ, Η εὐθεῖαι, ὅστε τὴν Κ μεῖζον εἶναι τῆς Η καὶ ἐλάσσονα λόγων ἔχειν πρὸς αὐτὴν τοῦ, ὅν ἐχει ὁ Χ κώνος πρὸς τὴν σφαῖραν, καὶ αἱ Θ, Ι ἐκκεῖσθωσαν, καθὼς πρότερον, καὶ εἰς τὸν ΑΒΓΔ κύκλον νοείσθω πολύγωνον ἐγγεγραμμένον καὶ ἀλλο περιγεγραμμένον, ὡστε τὴν πλευράν τοῦ περιγεγραμμένον πρὸς τὴν πλευράν τοῦ ἐγγεγραμμένου ἐλάσσονα λόγων ἔχειν ἦπερ ἡ Κ πρὸς Ι, καὶ τα ἀλλα κατεσκευασμένα τον αὐτὸν τρόπον τοὺς πρότερον. ἔχει ἄρα καὶ τὸ περιγεγραμμένον στερεὸν σχῆμα πρὸς τὸ ἐγγεγραμμένο τριπλασίων πλούτου λόγων ἦπερ ἡ πλευρά τοῦ περιγεγραμμένου πεῖρᾷ τοῦ ΑΒΓΔ κύκλου πρὸς τὴν τοῦ ἐγγεγραμμένου. ἢ δὲ πλευρά πρὸς τὴν πλευράν ἐλάσσονα λόγων ἔχει
fore the circumscribed figure has to the inscribed a ratio less than $K : H$. But $K : H$ is a ratio less than that which the sphere has to the cone $E$ \textit{[ex hypothesi]}; therefore the circumscribed figure has to the inscribed a ratio less than that which the sphere has to the cone $E$; and \textit{permutando}, [the circumscribed figure has to the sphere a ratio less than that which the inscribed figure has to the cone] \(^a\); which is impossible; for the circumscribed figure is greater than the sphere [Prop. 28], but the inscribed figure is less than the cone $E$ [Prop. 27]. Therefore the sphere is not greater than four times the aforesaid cone.

Let it be, if possible, less than four times, so that the sphere is less than the cone $E$. Let the straight lines $K$, $H$ be so taken that $K$ is greater than $H$ and $K : H$ is a ratio less than that which the cone $E$ has to the sphere [Prop. 2]; let the straight lines $\Theta$, $I$ be placed as before; let there be imagined in the circle $AB\Gamma\Delta$ one polygon inscribed and another circumscribed, so that the side of the circumscribed figure has to the side of the inscribed a ratio less than $K : I$; and let the other details in the construction be done as before. Then the circumscribed solid figure will have to the inscribed the triplicate ratio of that which the side of the figure circumscribed about the circle $AB\Gamma\Delta$ has to the side of the inscribed figure [Prop. 32]. But the ratio of the sides

\footnote{A marginal note in one ms. gives these words, which Archimedes would not have omitted.}

\footnote{\(\tau\omicron\upsilon\omicron\) \ldots \(\lambda\iota\mu\mu\dot{\acute{a}}\tau\omicron\) \textit{om. Heiberg}.}

\footnote{\(\delta\iota\omicron\omicron\) \ldots \(\tau\epsilon\tau\rho\alpha\nu\pi\lambda\alpha\iota\omicron\) \textit{om. Heiberg}.}
ηπερ ή Κ πρὸς Ι. ἔξει οὖν τὸ σχῆμα τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ή τριπλάσιον τού, δν ἔχει ή Κ πρὸς τὴν Ι. ή δὲ Κ πρὸς τὴν Η μείζονα λόγον ἔχει ή τριπλάσιον τού, δν ἔχει ή Κ πρὸς τὴν Ι. ὡστε ἐλάσσονα λόγον ἔχει τὸ σχῆμα τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ή ή Κ πρὸς τὴν Η. ή δὲ Κ πρὸς τὴν ἐλάσσονα λόγον ἔχει ή ὁ Ξ κώνος πρὸς τὴν σφαῖραν· ὅπερ ἀδύνατον· τὸ μὲν γὰρ ἐγγεγραμμένον ἐλασσόν έστι τῆς σφαῖρας, τὸ δὲ περιγεγραμμένον μείζον τοῦ Ξ κώνου. οὐκ ἢ ἀρα οὐδὲ ἐλάσσον ἐστὶν ή τετραπλασία ή σφαῖρα τοῦ κώνου τοῦ βάσιν μὲν ἔχοντος ἵσθιν τῷ ΑΒΓΔ κύκλῳ, ύψος δὲ τὴν ἵσθιν τῇ ἕκ τοῦ κέντρου τῆς σφαῖρας. ἐδείχθη δὲ, ὅτι οὐδὲ μείζων· τετραπλασία ἢ ἀρα.

[Πόρισμα] 1

Προδεδειγμένων δὲ τούτων φανερῶν, ὅτι πᾶσ κύλινδρος βάσιν μὲν ἔχων τὸν μέγιστον κύκλου τῶν ἐν τῇ σφαῖρα, ύψος δὲ ἵσον τῇ διαμέτρῳ τῆς σφαῖρας, ἡμιόλιος ἐστὶ τῆς σφαῖρας καὶ ἡ ἐπιφάνεια αὐτοῦ μετὰ τῶν βάσεων ἡμιολία τῆς ἐπιφάνειας τῆς σφαῖρας.

Ὀ μὲν γὰρ κύλινδρος ὁ προειρημένος ἕξαπλάσιος ἐστὶ τοῦ κώνου τοῦ βάσιν μὲν ἔχοντος τὴν αὐτήν, ύψος δὲ ἵσον τῇ ἕκ τοῦ κέντρου, ἡ δὲ σφαιρα δἐδεικται τοῦ αὐτοῦ κώνου τετραπλασία οὔσα· δῆλον οὖν, ὅτι ὁ κύλινδρος ἡμιόλιος ἐστὶ τῆς σφαῖρας. πάλιν, ἐπεὶ ἡ ἐπιφάνεια τοῦ κύλινδρου χωρίς τῶν βάσεων ἵσθι δἐδεικται κύκλῳ, οὐ ή ἕκ

1 πόρισμα. The title is not found in some mss.
is less than $K : I$ \emph{[ex hypothesi]}; therefore the circumscribed figure has to the inscribed a ratio less than $K^3 : I^3$. But $K : H > K^3 : I^3$; and so the circumscribed figure has to the inscribed a ratio less than $K : H$. But $K : H$ is a ratio less than that which the cone $\Xi$ has to the sphere \emph{[ex hypothesi]}; [therefore the circumscribed figure has to the inscribed a ratio less than that which the cone $\Xi$ has to the sphere]\footnote{These words, which Archimedes would not have omitted, are given in a marginal note to one ms.}; which is impossible; for the inscribed figure is less than the sphere [Prop. 28], but the circumscribed figure is greater than the cone $\Xi$ [Prop. 31, coroll.]. Therefore the sphere is not less than four times the cone having its base equal to the circle $A\beta\gamma\Delta$, and height equal to the radius of the sphere. But it was proved that it cannot be greater; therefore it is four times as great.

\footnote{These words, which Archimedes would not have omitted, are given in a marginal note to one ms.}

\[\text{[corollary]}\]

From what has been proved above it is clear that any cylinder having for its base the greatest of the circles in the sphere, and having its height equal to the diameter of the sphere, is one-and-a-half times the sphere, and its surface including the bases is one-and-a-half times the surface of the sphere.

For the aforesaid cylinder is six times the cone having the same basis and height equal to the radius [from Eucl. xii. 10], while the sphere was proved to be four times the same cone [Prop. 34]. It is obvious therefore that the cylinder is one-and-a-half times the sphere. Again, since the surface of the cylinder excluding the bases has been proved equal to a circle
(c) Solution of a Cubic Equation

Archim. De Sphaera et Cyl. ii., Prop. 4, Archim. ed. Heiberg i. 186. 15–192. 6

As the geometrical form of proof is rather diffuse, and may conceal from the casual reader the underlying nature of the operation, it may be as well to state at the outset the various stages of the proof. The problem is to cut a given sphere by a plane so that the segments shall have a given ratio, and the stages are:

(a) Analysis of this main problem in which it is reduced to a particular case of the general problem, “so to cut a given straight line ΔZ at X that \( ZX \) bears to the given...
whose radius is a mean proportional between the side of the cylinder and the diameter of the base [Prop. 13], and the side of the aforementioned cylinder circumscribing the sphere is equal to the diameter of the base, while the circle having its radius equal to the diameter of the base is four times the base [Eucl. xii. 2], that is to say, four times the greatest of the circles in the sphere, therefore the surface of the cylinder excluding the bases is four times the greatest circle; therefore the whole surface of the cylinder, including the bases, is six times the greatest circle. But the surface of the sphere is four times the greatest circle. Therefore the whole surface of the cylinder is one-and-a-half times the surface of the sphere.

(c) Solution of a Cubic Equation

Archimedes, On the Sphere and Cylinder ii., Prop. 4, Archim. ed. Heiberg i. 186. 15-192. 6

To cut a given sphere, so that the segments of the sphere shall have, one towards the other, a given ratio.  

straight line the same ratio as a given area bears to the square on \( \Delta X \); in algebraical notation, to solve the equation \( \frac{a-x}{b} = \frac{c^2}{x^2} \) or \( x^2(a-x) = bc^2 \).

(b) Analysis of this general problem, in which it is shown that the required point can be found as the intersection of a parabola \( ax^2 = c^2y \) and a hyperbola \((a-x)y = ab\). It is stated, for the time being without proof, that \( x^2(a-x) \) is greatest when \( x = \frac{3}{4}a \); in other words, that for a real solution \( bc^2 \geq \frac{1}{3}a^3 \).

(c) Synthesis of this general problem, according as \( bc^2 \) is greater than, equal to, or less than \( \frac{1}{3}a^3 \). If it be greater, there is no real solution; if equal, there is one real solution; if less, there are two real solutions.

(d) Proof that \( x^2(a-x) \) is greatest when \( x = \frac{3}{4}a \), deferred
in (b). This is done in two parts, by showing that (1) if \( x \) has any value less than \( \frac{2}{3}a \), (2) if \( x \) has any value greater than \( \frac{2}{3}a \), then \( x^2(a - x) \) has a smaller value than when \( x = \frac{2}{3}a \).

(e) Proof that, if \( b^2c^2 < \frac{4}{27}a^4 \), there are always two real solutions.

(f) Proof that, in the particular case of the general problem to which Archimedes has reduced his original problem, there is always a real solution.

(g) Synthesis of the original problem.

Of these stages, (a) and (g) alone are found in our texts of Archimedes; but Eutocius found stages (b)-(d) in an old book, which he took to be the work of Archimedes; and he added stages (e) and (f) himself. When it is considered that all these stages are traversed by rigorous geometrical
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Let $AB\Gamma \Delta$ be the given sphere; it is required so to cut it by a plane that the segments of the sphere shall have, one towards the other, the given ratio.

Let it be cut by the plane $\Delta \Gamma$; then the ratio of the segment $A\Delta \Gamma$ of the sphere to the segment $A\beta \Gamma$ of the sphere is given. Now let the sphere be cut through the centre [by a plane perpendicular to the plane through $\Delta \Gamma$], and let the section be the great circle $AB\Gamma \Delta$ of centre $K$ and diameter $\Delta B$, and let [$\Delta, \beta$ be taken on $B\Delta$ produced in either direction so that]

$$KA + \Delta X : \Delta X = PX : XB,$$
$$KB + \beta X : \beta X = \Delta X : X\Delta,$$
and let $\Delta \beta$, $\Delta \Gamma$, $\beta \Gamma$, $\beta \Gamma$ be joined; then the cone $\Delta \Delta \Gamma$ is equal to the segment $A\Delta \Gamma$ of the sphere, and

the cone $A\beta \Gamma$ to the segment $A\beta \Gamma$ [Prop. 2]; therefore the ratio of the cone $\Delta \Delta \Gamma$ to the cone $A\beta \Gamma$ is given. But cone $\Delta \Delta \Gamma :$ cone $A\beta \Gamma = \Delta X : \beta \Gamma.$\footnote{Since they have the same base.} Therefore the ratio $\Delta X : \beta \Gamma$ is given. And in the methods, the solution must be admitted a veritable tour de force. It is strictly analogous to the modern method of solving a cubic equation, but the concept of a cubic equation did not, of course, come within the purview of the ancient mathematicians.
The words καὶ... ἀπὸ ΔΧ are shown by Eutocius's comment to be an interpolation. The words πάλιν... πρὸς ΒΧ and καὶ... πρὸς ΖΧ must also be interpolated, as, in order to prove that ΔΛ : ΔΧ is given, Eutocius first proves that ΒΖ : ΖΧ = ΛΔ : ΛΧ, which he would hardly have done if Archimedes had himself provided the proof.

* This is proved by Eutocius thus:

Since

\[ \text{ΔΛ + ΔΧ} : \text{ΔΧ} = \text{ΡΧ} : \text{ΧΒ}, \]
\[ \text{ΚΔ} : \text{ΔΧ} = \text{ΡΒ} : \text{ΒΧ}, \]
\[ \text{ΚΔ} : \text{ΒΡ} = \text{ΔΧ} : \text{ΧΒ}, \]
\[ \text{ΚΒ} : \text{ΒΡ} = \text{ΔΧ} : \text{ΧΒ}. \]

Again, since

\[ \text{ΚΒ + ΒΧ} : \text{ΧΒ} = \text{ΔΧ} : \text{ΧΔ}, \]

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same way as in a previous proposition [Prop. 2], by construction,
\[ \Delta \Delta : K\Delta = KB : BP = \Delta X : XB. \]

And since \( PB : BK = K\Delta : \Delta \), \([\text{Eucl. v. 7}, \text{coroll.}]\)

\( \text{componendo,} \)
\( PK : KB = K\Lambda : \Delta \Delta, \) \([\text{Eucl. v. 18}]\)

\( \therefore PK : K\Delta = K\Lambda : \Delta \Delta. \)

\( \therefore PA : K\Lambda = K\Lambda : \Delta \Delta. \) \([\text{Eucl. v. 12}]\)

\( \therefore PA : \Delta \Delta = \Delta K^2. \) \([\text{Eucl. vi. 17}]\)

\( \therefore PA : \Delta \Delta = K\Lambda^2 : \Delta \Delta^2. \)

And since \( \Delta \Delta : \Delta K = \Delta X : XB, \)

\( \text{invertendo et} \) \( \text{componendo,} \)
\( K\Lambda : \Delta \Delta = B\Delta : \Delta X. \) \([\text{Eucl. v. 7}, \text{coroll. and v. 18}]\)

Let \( BZ \) be placed equal to \( KB. \) It is plain that \([Z]\) will fall beyond \( P. \)

Since the ratio \( \Delta \Delta : \Delta X \) is given, therefore the ratio \( PA : \Delta X \) is given. \( \therefore \) Then,

\( \text{dirimendo et permutando} \)
\( \Delta X : XB = \Delta \Delta : \Delta K. \)

Now \( \Delta X : XB = KB : BP. \)

Therefore \( \Delta \Delta : \Delta K = \Delta X : XB = KB : BP. \)

\( \therefore \) Since \( X\Delta : XB = KB : BP, \) and \( \Delta X > XB, \) \( \therefore KB > BP. \)

\( \therefore BZ > BP. \)

\( \therefore \) As Eutocius’s note shows, what Archimedes wrote was:

“Since the ratio \( \Delta \Delta : \Delta X \) is given, and the ratio \( PA : \Delta X \),
therefore the ratio \( PA : \Delta \Delta \) is also given.”

Eutocius’s proof is:

Since \( KB + BX : BX = \Delta X : X\Delta, \)

\( \therefore ZX : XB = \Delta X : X\Delta; \)

\( \therefore XZ : ZB = X\Lambda : \Delta \Delta; \)

\( \therefore BZ : ZX = \Delta \Delta : \Delta X. \)

But the ratio \( BZ : ZX \) is given because \( ZB \) is equal to the radius of the given sphere and \( BX \) is given. Therefore \( \Delta \Delta : \Delta X \) is given.

Again, since the ratio of the segments is given, the ratio of
dotheis. ἐπεῖ οὖν ὁ τῆς ΡΑ πρὸς ΛΧ λόγος συνήπται ἐκ τε τοῦ, ὅπερ ἔχει ἡ ΡΑ πρὸς ΛΔ, καὶ ἡ ΔΔ πρὸς ΛΧ, ἀλλ' ὡς μὲν ἡ ΡΑ πρὸς ΛΔ, τὸ ἀπὸ ΔΒ πρὸς τὸ ἀπὸ ΔΧ, ὡς δὲ ἡ ΔΔ πρὸς ΛΧ, οὔτως ἡ ΒΖ πρὸς ΖΧ, ὁ ἀρα τῆς ΡΑ πρὸς ΛΧ λόγος συνήπται ἐκ τε τοῦ, ὅπερ ἔχει τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ, καὶ ἡ ΒΖ πρὸς ΖΧ. πεποιήσων δὲ, ὡς ἡ ΡΑ πρὸς ΛΧ, ἡ ΒΖ πρὸς ΖΘ· λόγος δὲ τῆς ΡΑ πρὸς ΛΧ δοθεὶς· λόγος ἀρα καὶ τῆς ΒΖ πρὸς ΖΘ δοθεῖς. δοθεῖσα δὲ ἡ ΒΖ—ίση γάρ ἐστι τῇ ἐκ τοῦ κέντρου· δοθεῖσα ἀρα καὶ ἡ ΖΘ. καὶ ὁ τῆς ΒΖ ἁρα λόγος πρὸς ΖΘ συνήπται ἐκ τε τοῦ, ὅπερ ἔχει τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ, καὶ ἡ ΒΖ πρὸς ΖΧ. ἀλλ' ὁ ΒΖ πρὸς ΖΘ λόγος συνήπται ἐκ τε τοῦ τῆς ΒΖ πρὸς ΖΧ καὶ τοῦ τῆς ΖΧ πρὸς ΖΘ [κοινὸς ἄφθισθω ὁ τῆς ΒΖ πρὸς ΖΧ]. λοιπῶν ἁρα ἐστίν, ὡς τὸ ἀπὸ ΒΔ, τούτεστι δοθεῖν, πρὸς τὸ ἀπὸ ΔΧ, οὔτως ἡ ΧΖ πρὸς ΖΘ, τούτεστι πρὸς δοθεῖν. καὶ ἐστιν δοθεῖσα ἡ ΖΔ εὐθεία· εὐθείαν ἁρα δοθείσαν τὴν ΔΖ τεμεῖν δεῖ κατὰ τὸ Χ καὶ ποιεῖν, ὡς τῆν ΧΖ πρὸς δοθείσαν [τὴν ΖΘ], οὔτως τὸ δοθέν [τὸ ἀπὸ ΒΔ]² πρὸς τὸ ἀπὸ ΔΧ. τούτῳ οὔτως ἀπλῶς μὲν λεγόμενον ἔχει διορισμὸν,

¹ κοινὸς ... πρὸς ΖΧ. Eutocius's comment shows that these words are interpolated.
² τὴν ΖΘ, τὸ ἀπὸ ΒΔ. Eutocius's comments show these words to be glosses.

the cones ΛΛΓ, ΑΠΓ is also given, and therefore the ratio ΔΧ:XP. Therefore the ratio ΡΑ:ΛΧ is given. Since the ratios ΡΑ:ΛΧ and ΛΔ:ΛΧ are given, it follows that the ratio ΡΑ:ΛΔ is given.

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since the ratio $PA:AX$ is composed of the ratios $PA:DA$ and $DA:AX$.

and since \[ PA : DA = DB^2 : DX^2, \]

\[ DA : DX = BZ : ZX, \]

therefore the ratio $PA : AX$ is composed of the ratios $BDA^2 : DX^2$ and $BZ : ZX$. Let $[\Theta$ be chosen so that]

\[ PA : AX = BZ : ZX. \]

Now the ratio $PA : AX$ is given; therefore the ratio $ZB : Z\Theta$ is given. Now $BZ$ is given—for it is equal to the radius; therefore $Z\Theta$ is also given. Therefore $b$ the ratio $BZ : Z\Theta$ is composed of the ratios $BDA^2 : DX^2$ and $BZ : ZX$. But the ratio $BZ : Z\Theta$ is composed of the ratios $BZ : ZX$ and $ZX : Z\Theta$. Therefore, the remainder $c$ $BDA^2 : DX^2 = XZ : Z\Theta$, in which $BDA^2$ and $Z\Theta$ are given. And the straight line $IZ$ is given; therefore it is required so to cut the given straight line $IZ$ at $X$ that $XZ$ bears to a given straight line the same ratio as a given area bears to the square on $AX$.

When the problem is stated in this general form,$d$ it is necessary to investigate the limits of possibility,

$^a$ For \[ PA : DA = DK^2 : DA^2 = BDA^2 : DX. \]

$^b$ "Therefore" refers to the last equation.

$^c$ $i.e.$ the remainder in the process given fully by Eutocius as follows:

\[ (BDA^2 : DX^2) \cdot (BZ : ZX) = BZ : Z\Theta = (BZ : ZX) \cdot (XZ : Z\Theta). \]

Removing the common element $BZ : ZX$ from the extreme terms, we find that the remainder $BDA^2 : DX^2 = XZ : Z\Theta$.

$^d$ In algebraic notation, if $AX = x$ and $IZ = a$, while the given straight line is $b$ and the given area is $c^2$, then

\[ \frac{a-x}{b} = \frac{c^2}{x^2}, \]

or

\[ x^2(a-x) = bc^2. \]
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προστιθεμένων δὲ τῶν προβλημάτων τῶν ἐνθάδε ὑπαρχόντων [τούτεστι τοῦ τε διπλασίαν εἶναι τὴν ΔΒ τῆς ΒΖ καὶ τοῦ μείζονα τῆς ΖΘ τῆν ΖΒ, ωσ κατὰ τὴν ἀνάλυσιν]¹ οὐκ ἔχει διορισμόν· καὶ ἔσται τὸ πρόβλημα τοιοῦτον· δυὸ δοθεισῶν εὐθειῶν τῶν ΒΔ, ΒΖ καὶ διπλασίας οὕσης τῆς ΒΔ τῆς ΒΖ καὶ σημεῖον ἐπὶ τῆς ΒΖ τοῦ Θ τεμεῦν τὴν ΔΒ κατὰ τὸ Χ καὶ ποιεῖν, ώσ τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ, τὴν ΧΖ πρὸς ΖΘ· ἐκάτερα δὲ ταῦτα ἐπὶ τέλει ἀναλυθῆσεται τε καὶ συντεθῆσεται.


'Επὶ τέλει μὲν τὸ προρηθὲν ἐπηγγείλατο δεῖξαι, ἐν οὐδενὶ δὲ τῶν ἀντιγράφων εὐρεῖν ἔνεστι τὸ ἐπάγγελμα. οἴθεν καὶ Διονυσόδωρον μὲν εὐρίσκομεν μὴ τῶν αὐτῶν ἐπιτυχόντα, ἀδυνατήσαντα δὲ ἐπιβαλεῖν τῷ καταλειφθέντι λήμματι, ἐφ᾽ ἑτέραν ὁδὸν τοῦ ὅλου προβλήματος ἐλθεῖν, ἣντων ἔξής γράψομεν. Διοκλῆς μέντοι καὶ αὐτὸς ἐν τῷ Περὶ πυρῶν αὐτῶ συγγεγραμμένω βιβλίῳ ἐπηγγέλθαι νομίζων τὸν 'Αρχιμήδη, μὴ πεποιηκέναι δὲ τὸ ἐπάγγελμα, αὐτὸς ἀναπληροῦν ἐπεχείρησεν, καὶ τὸ ἐπιχείρημα ἔξης γράψομεν· ἐστὶν γὰρ καὶ αὐτὸ οὐδένα μὲν ἐχον πρὸς τὰ παραλειμμένα λόγον, ομοίως δὲ τῷ Διονυσόδωρῳ δι᾽ ἑτέρας ἀποδείξεως κατασκευάζον τὸ πρόβλημα. ἐν τινι μέντοι παλαιῷ

¹ τούτεστι . . . ἀνάλυσιν. Eutocius’s notes make it seem likely that these words are interpolated.

* In the technical language of Greek mathematics, the 184
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but under the conditions of the present case no such investigation is necessary.\(^a\) In the present case the problem will be of this nature: \textit{Given two straight lines }B\Delta, BZ, \textit{in which }B\Delta=2BZ, \textit{and a point }\Theta \textit{upon }BZ, \textit{so to cut }\Delta B \textit{at }X \textit{that}

\[B\Delta^2 : \Delta X^2 = XZ : Z\Theta;\]

and the analysis and synthesis of both problems will be given at the end.\(^b\)

\[\text{Eutocius, Commentary on Archimedes' Sphere and Cylinder ii., Archim. ed. Heiberg iii. 130. 17-150. 22}\]

He promised that he would give at the end a proof of what is stated, but the fulfilment of the promise cannot be found in any of his extant writings. Dionysodorus also failed to light on it, and, being unable to tackle the omitted lemma, he approached the whole problem in an altogether different way, which I shall describe in due course. Diocles, indeed, in his work \textit{On Burning Mirrors} maintained that Archimedes made the promise but had not fulfilled it, and he undertook to supply the omission himself, which attempt I shall also describe in its turn; it bears, however, no relation to the missing discussion, but, like that of Dionysodorus, it solves the problem by a construction reached by a different proof. But general problem requires a \textit{diorismos}, for which \textit{v. vol. i. p. 151 n. }h \textit{and p. 396 n. }a. In algebraic notation, there must be limiting conditions if the equation

\[x^2(a-x)=bc^2\]

is to have a real root lying between \(0\) and \(a.\)

\(^b\) Having made this promise, Archimedes proceeded to give the formal synthesis of the problem which he had thus reduced.
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in a certain ancient book—for I pursued the inquiry thoroughly—I came upon some theorems which, though far from clear—owing to errors and to manifold faults in the diagrams, nevertheless gave the substance of what I sought, and furthermore preserved in part the Doric dialect beloved by Archimedes, while they kept the names favoured by ancient custom, the parabola being called a section of a right-angled cone and the hyperbola a section of an obtuse-angled cone; in short, I felt bound to consider whether these theorems might not be what he had promised to give at the end. For this reason I applied myself with closer attention, and, although it was difficult to get at the true text owing to the multitude of the mistakes already mentioned, gradually I routed out the meaning and now set it out, so far as I can, in more familiar and clearer language. In the first place the theorem will be treated generally, in order to make clear what he says about the limits of possibility; then will follow the special form it takes under the conditions of his analysis of the problem.

"Given a straight line AB and another straight line AG and an area Δ, let it be required to find a point E on AB such that $AE : AG = Δ : EB^2$.

"Suppose it found, and let AG be at right angles to AB, and let GE be joined and produced to Z, and through Γ let ΓH be drawn parallel to AB, and through B let ZBH be drawn parallel to AG, meeting
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ἐκατέρα τῶν ΓΕ, ΓΗ, καὶ συμπεπληρώσθω τὸ ΗΘ παράλληλόγραμμον, καὶ διὰ τοῦ Ε ὁποτέρα τῶν ΓΘ, ΗΖ παράλληλος ἥχθω ἡ ΚΕΛ, καὶ τῷ Δ ἴσον ἔστω τὸ ύπὸ ΓΗΜ.

"Ἐπεὶ οὖν ἔστω, ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως τὸ Δ πρὸς τὸ ἀπὸ ΕΒ, ὡς δὲ ἡ ΕΑ πρὸς ΑΓ; οὔτως ἡ ΓΗ πρὸς ΗΖ, ὡς δὲ ἡ ΓΗ πρὸς ΗΖ, οὔτως τὸ ἀπὸ ΓΗ πρὸς τὸ ύπὸ ΓΗΖ, ὡς ἄρα τὸ ἀπὸ ΓΗ πρὸς τὸ ύπὸ ΓΗΖ, οὔτως τὸ Δ πρὸς τὸ ἀπὸ ΕΒ, τοὐτέστι πρὸς τὸ ἀπὸ ΚΖ· καὶ ἐναλλάξ, ὡς τὸ ἀπὸ ΓΗ πρὸς τὸ Δ, τοὐτέστι πρὸς τὸ ύπὸ ΓΗΜ, οὔτως τὸ ύπὸ ΓΗΖ πρὸς τὸ ἀπὸ ΖΚ. ἀλλ' ὡς τὸ ἀπὸ ΓΗ πρὸς τὸ ύπὸ ΓΗΜ, οὔτως ἡ ΓΗ πρὸς ΗΜ· καὶ ὡς ἄρα ἡ ΓΗ πρὸς ΗΜ, οὔτως τὸ ύπὸ ΓΗΖ πρὸς τὸ ἀπὸ ΖΚ. ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΜ, τῆς ΗΖ κοινοῦ ὑψος λαμβανο-μένης οὔτως τὸ ύπὸ ΓΗΖ πρὸς τὸ ύπὸ ΜΗΖ· ὡς ἄρα τὸ ύπὸ ΓΗΖ πρὸς τὸ ύπὸ ΜΗΖ, οὔτως τὸ ύπὸ ΓΗΖ πρὸς τὸ ἀπὸ ΖΚ· ἴσον ἄρα τὸ ύπὸ ΜΗΖ τῷ ἀπὸ ΖΚ. ἐὰν ἄρα περὶ ἄξονα τῆς ΖΗ

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both $\Gamma E$ and $\Gamma H$, and let the parallelogram $H\Theta$ be completed, and through $E$ let $KE\Lambda$ be drawn parallel to either $\Gamma \Theta$ or $HZ$, and let $[M$ be taken so that $]

\Gamma H \cdot HM = \Delta.

" Then, since $EA : AG = \Delta : EB^2$ [ex hyp.
and $EA : AG = \Gamma H : HZ$,
and $\Gamma H : HZ = \Gamma H^2 : \Gamma H \cdot HZ$,
$\therefore \Gamma H^2 : \Gamma H \cdot HZ = \Delta : EB^2$
$= \Delta : KZ^2$;
and, permutando, $\Gamma H^2 : \Delta$ $\left(= \Gamma H \cdot HZ : ZK^2, \right)$
i.e., $\Gamma H^2 : \Gamma H \cdot HM = \Gamma H \cdot HZ : ZK^2$.
But $\Gamma H^2 : \Gamma H \cdot HM = \Gamma H : HM$;
$\therefore \Gamma H : HM = \Gamma H \cdot HZ : ZK^2$.

But, by taking a common altitude $HZ$,
$\Gamma H : HM = \Gamma H \cdot HZ : MH \cdot HZ$;
$\therefore \Gamma H \cdot HZ : MH \cdot HZ = \Gamma H \cdot HZ : ZK^2$;
$\therefore MH \cdot HZ = ZK^2$. 
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Let $AB = a$, $\Gamma \Sigma = b$, and $\Delta = \Gamma \Pi$, $\Pi \Pi = c$, so that $HM = \frac{c^2}{a}$. Then if $\Pi \Gamma$ be taken as the axis of $x$ and $HZ$ as the axis of $y$, the equation of the parabola is

$$x^2 = \frac{c^2}{a}y,$$

and the equation of the hyperbola is

$$(a-x)y = ab.$$ 

Their points of intersection give solutions of the equation

$$x^2(a-x) = bc^2.$$
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If, therefore, a parabola be drawn through H about the axis ZH with the parameter HM, it will pass through K [Apoll. Con. i. 11, converse], and it will be given in position because HM is given in magnitude [Eucl. Data 57], comprehending with the given straight line HI the given area \( \Delta \); therefore K lies on a parabola given in position. Let it then be drawn, as described, and let it be HK.

"Again, since the area \( \Theta \Delta = \Gamma B \) [Eucl. i. 43 i.e., \( \Theta K \cdot K \Lambda = AB \cdot BH \),

if a hyperbola be drawn through B having \( \Theta \Gamma, \Gamma H \) for asymptotes, it will pass through K by the converse to the eighth theorem of the second book of Apollonius's Elements of Conics, and it will be given in position because both the straight lines \( \Theta \Gamma, \Gamma H \), and also the point B, are given in position. Let it be drawn, as described, and let it be KB; therefore K lies on a hyperbola given in position. But it lies also on a parabola given in position; therefore K is given. And KE is the perpendicular drawn from it to the straight line AB given in position; therefore E is given. Now since the ratio of EA to the given straight line \( \Lambda \Gamma \) is equal to the ratio of the given area \( \Delta \) to the square on EB, we have two solids, whose bases are the square on EB and \( \Delta \) and whose altitudes are EA, \( \Lambda \Gamma \), and the bases are inversely pro-

to which, as already noted, Archimedes had reduced his problem. (N.B.—The axis of \( x \) is for convenience taken in a direction contrary to that which is usual; with the usual conventions, we should get slightly different equations.)
In our algebraical notation, \( x^2(a - x) \) is a maximum when \( x = \frac{1}{2}a \). We can easily prove this by the calculus. For, by differentiating and equating to zero, we see that \( x^2(a - x) \) has a maximum at \( x = \frac{1}{2}a \).
portional to the altitudes; therefore the solids are equal [Eucl. xi. 34]; therefore

\[ EB^2 \cdot EA = \Delta \cdot \Gamma A, \]

in which both \( \Delta \) and \( \Gamma A \) are given. But, of all the figures similarly taken upon \( BA \), \( BE^2 \cdot BA \) is greatest when \( BE = 2EA \), as will be proved; it is therefore necessary that the product of the given area and the given straight line should not be greater than

\[ BE^2 \cdot EA. \]

"The synthesis is as follows: Let \( AB \) be the given straight line, let \( \Delta \Gamma \) be any other given straight line, let \( \Delta \) be the given area, and let it be required to cut \( AB \) so that the ratio of one segment to the given straight line \( \Delta \Gamma \) shall be equal to the ratio of the given area \( \Delta \) to the square on the remaining segment.

"Let \( AE \) be taken, the third part of \( AB \); then \( \Delta \cdot \Delta \Gamma \) is greater than, equal to or less than \( BE^2 \cdot EA \).

"If it is greater, no synthesis is possible, as was shown in the analysis; if it is equal, the point \( E \) satisfies the conditions of the problem. For in equal solids the bases are inversely proportional to the altitudes, and \( EA : \Delta \Gamma = \Delta : BE^2 \).

"If \( \Delta \cdot \Delta \Gamma \) is less than \( BE^2 \cdot EA \), the synthesis is thus accomplished: let \( \Delta \Gamma \) be placed at right angles to \( AB \), and through \( \Gamma \) let \( \Gamma Z \) be drawn parallel to a stationary value when \( 2ax - 3x^2 = 0 \), i.e., when \( x = 0 \) (which gives a minimum value) or \( x = \frac{3}{2}a \) (which gives a maximum). No such easy course was open to Archimedes.

* Sc. "not greater than \( BE^2 \cdot EA \) when \( BE = 2EA."

* Figure on p. 146.
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άλληλος ἡχθων ἡ ΓΖ, διὰ δὲ τοῦ Β τη ἌΓ παράλληλος ἡχθων ἡ ΒΖ καὶ συμπεπτέτω τη ΓΕ ἐκβληθείση κατὰ τὸ Η, καὶ συμπεπληρώσθω τὸ ΖΘ παράλληλογράμμων, καὶ διὰ τοῦ Ε τη ΖΗ παράλληλος ἡχθων ἡ ΚΕΛ. ἐπεὶ οὖν τὸ Δ ἐπὶ τὴν ἌΓ ἔλασσὸν ἐστὶ τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ, ἐστὶν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως τὸ Δ πρὸς ἔλασσὸν τι τοῦ ἀπὸ τῆς ΒΕ, τουτέστι τοῦ ἀπὸ τῆς ΗΚ. ἐστὶν οὖν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως τὸ Δ πρὸς τὸ ἀπὸ ΗΜ, καὶ τῷ Δ ἵππον ἐστὶν τὸ ὑπὸ ΓΖΝ. ἐπεὶ οὖν ἐστὶν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως τὸ Δ, τουτέστι τὸ ὑπὸ ΓΖΝ, πρὸς τὸ ἀπὸ ΗΜ, ἂλλ' ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως ἡ ΓΖ πρὸς ΖΗ, ὡς δὲ ἡ ΓΖ πρὸς ΖΗ, οὔτως τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΗ, καὶ ὡς ἀρα τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΗ, οὔτως τὸ ὑπὸ ΓΖΝ πρὸς τὸ ἀπὸ ΗΜ· καὶ ἐναλλάξ, ὡς τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΝ, οὔτως τὸ ὑπὸ ΓΖΗ πρὸς τὸ ἀπὸ ΗΜ. ἂλλ' ὡς τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΝ, ἡ ΓΖ πρὸς ΖΝ, ὡς δὲ ἡ ΓΖ πρὸς ΖΝ, τῆς ΖΗ κοινοῦ ψυχος λαμβανομένης οὔτως τὸ ὑπὸ ΓΖΗ πρὸς τὸ ὑπὸ ΝΖΗ· καὶ ὡς ἀρα τὸ ὑπὸ ΓΖΗ πρὸς τὸ ὑπὸ ΝΖΗ, οὔτως τὸ ὑπὸ ΓΖΗ πρὸς τὸ ἀπὸ ΗΜ· ἵππον ἂρα ἐστὶ τὸ ἀπὸ ΗΜ τῷ ὑπὸ ΗΖΝ.

"...Εὰν ἂρα διὰ τοῦ Ζ περὶ ἄξονα τῆς ΖΗ γράψω-μεν παραβολήν, ὡστε τὰς καταγομένας δύνασθαι παρὰ τῆν ΖΝ, ἦξει διὰ τοῦ Μ. γεγράφθω, καὶ ἐστὶν ὡς ἡ ΜΕΖ. καὶ ἐπεὶ ἵππον ἐστὶ τὸ ΘΑ τῷ ΑΖ, τουτέστι τὸ ὑπὸ ΘΚΛ τῷ ὑπὸ ΑΒΖ, εὰν διὰ 144
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AB, and through B let BZ be drawn parallel to AG, and let it meet ΕΕ produced at H, and let the parallelogram ZΘ be completed, and through Ε let KΕΑ be drawn parallel to ZH. Now

since \( Δ \cdot AG \leq BE^2 \cdot EA, \)

\[ Δ : AG = Δ : (\text{the square of some quantity less than } BE) \]

\[ Δ : (\text{the square of some quantity less than } HK). \]

Let \( EA : AG = Δ : HM^2, \)

and let \( Δ = ΖZ \cdot ZN. \)

Then \( EA : AG = Δ : HM^2 = ΖZ \cdot ZN : HM^2. \)

But \( EA : AG = ΖZ : ZH, \)

and \( ΖZ : ZH = ΖZ^2 : ΖZ \cdot ZH; \)

\[ ΖZ^2 : ΖZ \cdot ZH = ΖZ \cdot ZN : HM^2; \]

and permutando, \( ΖZ^2 : ΖZ \cdot ZN = ΖZ \cdot ZH : HM^2. \)

But \( ΖZ^2 : ΖZ \cdot ZN = ΖZ : ZN, \)

and \( ΖZ : ZN = ΖZ \cdot ZH : NZ \cdot ZH, \)

by taking a common altitude ZH;

and \( ΖZ \cdot ZH : NZ \cdot ZH = ΖZ \cdot ZH : HM^2; \)

\[ \therefore \quad HM^2 = HZ \cdot ZN. \]

"Therefore if we describe through Z a parabola about the axis ZH and with parameter ZN, it will pass through M. Let it be described, and let it be as ΜΕZ. Then since

\( ΘΛ = ΑZ, \) [Eucl. i. 43]

i.e. \( ΘΚ \cdot ΚΛ = AB \cdot BZ, \)
τοῦ Β περὶ ἀσυμμπτῶτους τὰς ΘΓ, ΓΖ γράψωμεν ὑπερβολήν, ἥξει διὰ τοῦ Κ διὰ τὴν ἀντιστροφὴν τοῦ η' θεωρήματος τῶν Ἀπολλωνίου Κωνικῶν στοιχείων. γεγράφθω, καὶ ἔστω ὡς ἡ ΒΚ τέμνουσα τὴν παραβολὴν κατὰ τὸ Ε, καὶ ἀπὸ τοῦ Ε ἐπὶ τὴν ΑΒ κάθετος ἥχθως ἡ ΞΟΠ, καὶ διὰ τοῦ Ε τῇ ΑΒ παράλληλος ἥχθω ἡ ΡΞΣ. ἔπει οὖν ὑπερβολὴ ἐστὶν ἡ ΒΕΚ, ἀσύμπτωτοι δὲ αἱ ΘΓ, ΓΖ, καὶ παράλληλοι ἡγεῖται εἰς ἑαυτὸν αἱ ΡΞΠ ταῖς ΑΒΖ, ἵσον ἐστὶ τὸ ὑπὸ ΡΞΠ τῷ ὑπὸ ΑΒΖ. ὅπερ καὶ τὸ ΡΟ τῷ ΟΖ. ἔαν ἄρα ἀπὸ τοῦ Γ ἐπὶ τὸ Σ ἐπιζευγθῇ εὐθεία, ἥξει διὰ τοῦ Ο. ἐρχέσθω, καὶ ἔστω ὡς ἡ ΓΟΣ. ἔπει οὖν ἐστὶν, ὡς ἡ ΟΑ πρὸς ΑΓ, οὕτως ἡ ΟΒ πρὸς ΒΣ, τοῦτεστών ἡ ΓΖ πρὸς ΖΣ, ὡς δὲ ἡ ΓΖ πρὸς ΖΣ, τῆς ΖΝ κοινῶν ὑψοὺς λαμβανομένης οὕτως τὸ ὑπὸ ΓΖΝ πρὸς τὸ ὑπὸ ΖΝ, καὶ ὡς ἄρα ἡ ΟΑ πρὸς ΑΓ, οὕτως τὸ ὑπὸ ΓΖΝ πρὸς τὸ ὑπὸ ΖΝ. καὶ ἔστι τῶν μὲν ὑπὸ ΓΖΝ ἵσον τὸ Δ χωρίον, τῷ δὲ ὑπὸ ΖΝ ἵσον τὸ ἀπὸ ΣΞ, τοῦτεστὶ τὸ ἀπὸ ΒΟ, διὰ τὴν παραβολὴν ὡς ἄρα ἡ ΟΑ πρὸς ΑΓ, οὕτως τὸ Δ

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if we describe through $B$ a hyperbola in the asymptotes $\Theta \Gamma, \Gamma Z$, it will pass through $K$ by the converse of the eighth theorem [of the second book] of Apollonius's *Elements of Conics*. Let it be described, and let it be as $BK$ cutting the parabola in $\Xi$, and from $\Xi$ let $\Xi \Omega \Pi$ be drawn perpendicular to $AB$, and through $\Xi$ let $P \Xi \Sigma$ be drawn parallel to $AB$. Then since $B \Xi K$ is a hyperbola and $\Theta \Gamma, \Gamma Z$ are its asymptotes, while $P \Xi, \Xi \Pi$ are parallel to $AB, BZ$,

$$P \Xi . \Xi \Pi = AB . BZ; \quad [\text{Apoll. ii. 12}]$$

$$\therefore \quad PO = OZ.$$ 

Therefore if a straight line be drawn from $\Gamma$ to $\Sigma$ it will pass through $O$ [Eucl. i. 43, converse]. Let it be drawn, and let it be as $\Gamma O \Sigma$. Then since

$$OA : A \Gamma = OB : B \Sigma \quad [\text{Eucl. vi. 4}]

= \Gamma Z : Z \Sigma,$$

and

$$\Gamma Z : Z \Sigma = \Gamma Z . ZN : \Sigma Z . ZN,$$

by taking a common altitude $ZN$,

$$\therefore \quad OA : A \Gamma = \Gamma Z . ZN : \Sigma Z . ZN.$$ 

And $\Gamma Z . ZN = \Delta, \Sigma Z . ZN = \Sigma \Xi^2 = BO^2$, by the property of the parabola [Apoll. i. 11].

$$\therefore \quad OA : A \Gamma = \Delta : BO^2; \quad 147$$
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Ωτὶ Ἀριστοτέλης ἀπὸ τῆς EU ἡ ἄλλη ἀπὸ τῆς BE ἐν ἡ ἀνάλυσιν τῶν ὀρθών ὑποβάσεων ἐπὶ τοῦ ἈΒ. Αἀριστοτέλης ἀπὸ τῆς EU ἡ ἄλλη ἀπὸ τῆς BE ἐν ἡ ἀνάλυσιν τῶν ὀρθών ὑποβάσεων ἐπὶ τοῦ ἈΒ. Αἀριστοτέλης ἀπὸ τῆς EU ἡ ἄλλη ἀπὸ τῆς BE ἐν ἡ ἀνάλυσιν τῶν ὀρθών ὑποβάσεων ἐπὶ τοῦ ἈΒ. Αἀριστοτέλης ἀπὸ τῆς EU ἡ ἄλλη ἀπὸ τῆς BE ἐν ἡ ἀνάλυσιν τῶν ὀρθών ὑποβάσεων ἐπὶ τοῦ ἈΒ. 

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Χαριστοτέλης ἀπὸ τῆς EU ἡ ἄλλη ἀπὸ τῆς BE ἐν ἡ ἀνάλυσιν τῶν ὀρθών ὑποβάσεων ἐπὶ τοῦ ἈΒ. Αἀριστοτέλης ἀπὸ τῆς EU ἡ ἄλλη ἀπὸ τῆς BE ἐν ἡ ἀνάλυσιν τῶν ὀρθών ὑποβάσεων ἐπὶ τοῦ ἈΒ. Αἀριστοτέλης ἀπὸ τῆς EU ἡ ἄλλη ἀπὸ τῆς BE ἐν ἡ ἀνάλυσιν τῶν ὀρθών ὑποβάσεων ἐπὶ τοῦ ἈΒ. 

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therefore the point O has been found satisfying the conditions of the problem.

"That \( BE^2 \cdot EA \) is the greatest of all the figures similarly taken upon BA when \( BE = 2EA \) will be thus proved. Let there again be, as in the analysis, a given straight line \( A\Gamma \) at right angles to \( AB \), and let \( \Gamma E \) be joined and let it, when produced, meet at \( Z \) the line through \( B \) drawn parallel to \( A\Gamma \), and through \( \Gamma, Z \) let \( \Theta Z, \Gamma H \) be drawn parallel to \( AB \), and let \( \Gamma A \) be produced to \( \Theta \), and through \( E \) let \( K\Theta A \) be drawn parallel to it, and let

\[
EA : A\Gamma = \Gamma H \cdot HM : EB^2 ;
\]

then

\[
BE^2 \cdot EA = (\Gamma H \cdot HM) \cdot A\Gamma,
\]

owing to the fact that in two [equal] solids the bases are inversely proportional to the altitudes. I assert, then, that \( (\Gamma H \cdot HM) \cdot A\Gamma \) is the greatest of all the figures similarly taken upon BA.

"For let there be described through \( H \) a parabola about the axis \( ZH \) and with parameter \( HM \); it will pass through \( K \), as was proved in the analysis, and, if produced, it will meet \( \Theta \Gamma \), being parallel to the axis \( b \) of the parabola, by the twenty-seventh theorem of the first book of Apollonius's Elements of Conics.\( ^{c} \)

Let it be produced, and let it meet at \( N \), and through \( B \) let a hyperbola be drawn in the asymptotes \( N\Gamma, \Gamma H \); it will pass through \( K \), as was shown in the analysis. Let it be described as \( BK \), and let \( ZH \) be produced to \( \Xi \) so that \( ZH = H\Xi \), and let \( \Xi K \) be joined

\( ^a \) Figure on p. 151.

\( ^b \) Lit. "diameter," in accordance with Archimedes' usage.

\( ^c \) Apoll. i. 26 in our texts.
καὶ ἐκβεβλήσθω ἐπὶ τὸ Ὀ. φανερὸν ἀρα, ὅτι ἐφάπτεται τῆς παραβολῆς διὰ τὴν ἀντιστροφὴν τοῦ τετάρτου καὶ τριακοστοῦ θεωρήματος τοῦ πρῶτου βιβλίου τῶν Ἄπολλωνίου Κωνικῶν στοιχείων. ἔπει διπλὴ ἐστιν ἡ ΒΕ τῆς ΕΑ—οὕτως γὰρ ύπόκειται—τούτεστιν ἡ ΖΚ τῆς ΚΘ,

- Apoll. i. 33 in our texts.
and produced to O; it is clear that it will touch the parabola by the converse of the thirty-fourth

Theorem of the first book of Apollonius's *Elements of Conics*. Then since $BE = 2EA$—for this hypothesis has been made—therefore $ZK = 2K\Theta$, and the triangle
In the same notation as before, the condition $\text{BE}^2 \cdot \text{EA} = (\Theta H \cdot \text{HM}) \cdot \Delta \Gamma$ is $\frac{4}{27}a^3 = b^2$; and Archimedes has proved that, when this condition holds, the parabola $x^2 = \frac{c^2}{a}y$ touches the hyperbola $(a - x)y = ab$ at the point $\left(\frac{2}{3}a, 3b\right)$ because they both touch at this point the same straight line, that is the
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OθK is similar to the triangle ΕΖΚ, so that ΕΚ = 2ΚΟ. But ΕΚ = 2ΚΠ because ΕΖ = 2ΕΗ and ΠΗ is parallel to ΚΖ; therefore OK = KΠ. Therefore OKΠ, which meets the hyperbola and lies between the asymptotes, is bisected; therefore, by the converse of the third theorem of the second book of Apollonius’s Elements of Conics, it is a tangent to the hyperbola. But it touches the parabola at the same point K. Therefore the parabola touches the hyperbola at K.

Let the hyperbola be therefore conceived as produced to P, and upon AB let any point Σ be taken, and through Σ let ΣΥ be drawn parallel to KA and let it meet the hyperbola at T, and through T let ΦTX be drawn parallel to ΗΗ. Now by virtue of the property of the hyperbola and its asymptotes, ΦΥ = ΤΒ, and, the common element ΤΣ being subtracted, ΦΣ = ΣΗ, and therefore the straight line drawn from Σ to X will pass through Υ [Eucl. i. 43, conv.]. Let it be drawn, and let it be as ΣΧ. Then since, in virtue of the property of the parabola, ΨΧ = ΧΗ. HM, [Apoll. i. 11 line 9bx - ay - 3ab = 0, as may easily be shown. We may prove this fact in the following simple manner. Their points of intersection are given by the equation

\[ x^2(a - x) = bc^2, \]

which may be written

\[ x^2 - ax^2 + \frac{4}{27}a^3 = \frac{4}{27}a^3 - bc^3, \]

or

\[ (x - \frac{2}{3}a)^2(x + \frac{a}{3}) = \frac{4}{27}a^3 - bc^2. \]

Therefore, when \( bc^2 = \frac{4}{27}a^3 \) there are two coincident solutions, \( x = \frac{2}{3}a \), lying between 0 and \( a \), and a third solution \( x = -\frac{a}{3} \), outside that range.
ἐστι τῷ ὑπὸ ΧΗΜ. γεγονέτω οὖν τῷ ἀπὸ ΤΧ ἱσον τῷ ὑπὸ ΧΗΩ. ἐπεὶ οὖν ἔστιν, ὡς ἡ ΣΑ πρὸς ΑΓ, οὕτως ἡ ΓΗ πρὸς ΗΧ, ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΧ, τῆς ΗΩ κοινοῦ ύψους λαμβανομένης οὖτως τῷ ὑπὸ ΓΗΩ πρὸς τῷ ὑπὸ ΧΗΩ καὶ πρὸς τῷ ὑσον αὐτῷ τῷ ἀπὸ ΧΤ, τουτέστι τό ἀπὸ ΒΣ, τό ἀρὰ ἀπὸ ΒΣ ἐπὶ τήν ΣΑ ἱσον ἔστι τῷ ὑπὸ ΓΗΩ ἐπὶ τήν ΓΑ. τῷ δὲ ὑπὸ ΓΗΩ ἐπὶ τήν ΓΑ ἔλασσον ἔστι τοῦ ὑπὸ ΓΗΜ ἐπὶ τήν ΓΑ· τῷ ἀρὰ ἀπὸ ΒΣ ἐπὶ τήν ΣΑ ἔλασσον ἔστι τοῦ ἀπὸ ΒΕ ἐπὶ τήν ΕΑ. ὅμοιως δὴ δειχθήσεται καὶ ἐπὶ πάντων τῶν σημείων τῶν μεταξὺ λαμβανομένων τῶν Ε, Β.

"Αλλà δὴ εἰληφθῶ μεταξὺ τῶν Ε, Α σημείων τῷ 5. λέγω, ὅτι καὶ οὕτως τῷ ἀπὸ τῆς ΒΕ ἐπὶ τῆς ΕΑ μείζον ἔστι τοῦ ἀπὸ ΒΣ ἐπὶ τῆς 5Α.

"Τῶν γὰρ αὐτών κατεσκευασμένων ήχθω διὰ τοῦ 5 τῇ ΚΛ παράλληλος ή ζΣΡ καὶ συμβαλλέτω τῇ ὑπερβολὴ κατὰ τό Ρ· συμβαλεῖ γὰρ αὐτῆς διὰ το παράλληλος εἰναι τῇ ἀσυμπτώτῳ· καὶ διὰ τοῦ Ρ παράλληλος ἀχθείσα τῇ ΑΒ ή Α'ΡΒ' συμβαλλέτω τῇ ΗΖ ἐκβαλλομένη κατὰ τῷ Β'. καὶ ἐπεὶ πάλιν διὰ τὴν ὑπερβολὴν ἱσον ἔστι τῷ Γ' τῷ ΑΗ, ἡ ἀπὸ τοῦ Γ ἐπὶ τῷ Β' ἐπιζευγνυμένη εὐθεία Ἦξει διὰ τοῦ Ε. ἐρχέσθω καὶ ἔστω ὡς ἡ ΓζΒ'. καὶ ἐπεὶ πάλιν διὰ τὴν παραβολὴν ἱσον ἔστι τῷ ἀπὸ Α'Β' τῷ ὑπὸ Β'ΗΜ, τῷ ἀρα ἀπὸ ΡΒ' ἔλασσον ἔστι τοῦ ὑπὸ Β'ΗΜ. γεγονέτω τῷ ἀπὸ ΡΒ' ἱσον τῷ ὑπὸ Β'ΗΩ. ἐπεὶ οὖν ἔστω, ὡς ἡ 5Α πρὸς ΑΓ, οὕτως ἡ ΓΗ πρὸς ΗΒ', ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΒ', τῆς

* Figure on p. 156.
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:. \( TX^2 < XH \cdot HM \).

Let \( TX^2 = XH \cdot H\Omega \).

Then since \( \Sigma A : \Delta = \Gamma H : HX \),
while \( \Gamma H : HX = \Gamma H : H\Omega : XH \cdot H\Omega \),
by taking a common altitude \( H\Omega \),

\[
= \Gamma H \cdot H\Omega : XT^2
= \Gamma H \cdot H\Omega : BS^2,
\]

:. \( BS^2 \cdot \Sigma A = (\Gamma H \cdot H\Omega) \cdot GA \).

But \( (\Gamma H \cdot H\Omega) \cdot GA < (\Gamma H \cdot HM) \cdot GA \);

:. \( BS^2 \cdot \Sigma A < BE^2 \cdot EA \).

This can be proved similarly for all points taken between \( E, B \).

"Now let there be taken a point \( \gamma \) between \( E, A \).
I assert that in this case also \( BE^2 \cdot EA > H \gamma \cdot \gamma A \).

"With the same construction, let \( \gamma \gamma P \) be drawn through \( \gamma \) parallel to \( KA \) and let it meet the hyperbola at \( P \); it will meet the hyperbola because it is parallel to an asymptote [Apoll. ii. 13]; and through \( P \) let \( A'PB' \) be drawn parallel to \( AB \) and let it meet \( HZ \) produced in \( B' \). Since, in virtue of the property of the hyperbola, \( \Gamma' \gamma = AH \), the straight line drawn from \( \Gamma' \) to \( B' \) will pass through \( \gamma \) [Eucl. i. 43, conv.].

Let it be drawn and let it be as \( \Gamma' B' \). Again, since, in virtue of the property of the parabola,

\[
A'B'^2 = B'H \cdot HM,
\]

:. \( PB'^2 < B'H \cdot HM \).

Let \( PB'^2 = B'H \cdot H\Omega \).

Then since \( \gamma A : \Delta = \Gamma H : HB' \),
while \( \Gamma H : HB' = \Gamma H : H\Omega : B'H \cdot H\Omega \),
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Ἡ ὦ κοινοὶ ὑψους λαμβανομένης οὕτως τὸ ύπὸ ΓΗΩ πρὸς τὸ ύπὸ Β'ΗΩ, τουτέστι πρὸς τὸ ἀπὸ

PB', τουτέστι πρὸς τὸ ἀπὸ Bς, τὸ ἀρα ἀπὸ Bς ἐπὶ τὴν SΑ ἴσων ἐστὶ τῷ ύπὸ ΓΗΩ ἐπὶ τὴν ΤΑ. καὶ μείζον τὸ ύπὸ ΤΗΜ τοῦ ύπὸ ΓΗΩ μείζον ἀρα καὶ τὸ ἀπὸ BE ἐπὶ τὴν EA τοῦ ἀπὸ Bς ἐπὶ

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by taking a common altitude $\mathbb{H}\Omega$,

$$= \Gamma\mathbb{H}.\mathbb{H}\Omega : PB'^2$$

$$= \Gamma\mathbb{H}.\mathbb{H}\Omega : B\mathbb{S}^2,$$

$$\therefore \quad B\mathbb{S}^2.\mathbb{S}\mathbb{A} = (\Gamma\mathbb{H}.\mathbb{H}\Omega).\Gamma\mathbb{A}.$$

And

$$\Gamma\mathbb{H}.\mathbb{H}\mathbb{M} > \Gamma\mathbb{H}.\mathbb{H}\mathbb{\Omega};$$

$$\therefore \quad B\mathbb{E}^2.\mathbb{E}\mathbb{A} > B\mathbb{S}^2.\mathbb{S}\mathbb{A}.$$
GREEK MATHEMATICS

tήν 5Α. ὅμως ἰδὴ δειχθῆσεται καὶ ἐπὶ πάντων τῶν σημείων τῶν μεταξὺ τῶν Ε, Α λαμβανομένων. ἐδείχθη δὲ καὶ ἐπὶ πάντων τῶν μεταξὺ τῶν Ε, Β· πάντων ἀρα τῶν ἐπὶ τῆς AB ὅμως λαμβανομένων μέγιστον ἔστω τὸ ἀπὸ τῆς BE ἐπὶ τὴν EA, ὅταν ἡ διπλασία ἡ BE τῆς EA.'''

Ἐπιστῆσαι δὲ χρὴ καὶ τοῖς ἀκολουθοῦσιν κατὰ τὴν εἰρημένην καταγραφήν. ἔπει γὰρ δεδεικται τὸ ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ καὶ τὸ ἀπὸ Βς ἐπὶ τὴν 5Α ἐλασσον τοῦ ἀπὸ BE ἐπὶ τὴν EA, δυνατὸν ἐστὶ καὶ τοῦ δοθέντος χωρίου ἐπὶ τὴν δοθεῖσαν ἑλάσσονος ὄντος τοῦ ἀπὸ τῆς BE ἐπὶ τὴν EA κατὰ δύο σημεία τὴν AB τεμνομένην ποιεῖν τὸ ἐξ ἀρχῆς πρόβλημα. τούτο δὲ γίνεται, εἰ νοθησαμεν περὶ διάμετρον τὴν ΧΘ γραφόμενην παραβολήν, ὥστε τὰς καταγομένας δύνασθαι παρὰ τὴν ΗΩ· ἢ γὰρ τουαύτῃ παραβολῇ πάντως ἔρχεται διὰ τοῦ Τ. καὶ ἐπειδὴ ἀνάγκη αὐτὴν συμπίπτειν τῇ ΓΝ παραλλήλῳ οὖσῃ τῇ διαμέτρῳ, δῆλον, ὅτι τέμνει τὴν ύπερβολήν καὶ κατ’ ἄλλο σημείον ἀνωτέρω τοῦ Κ, ὡς ἐνταύθα κατὰ τὸ Ρ, καὶ ἀπὸ τοῦ Ρ ἐπὶ τὴν AB κάθετος ἀγομένη, ὡς ἐνταύθα ἡ Ρς, τέμνει τὴν AB κατὰ τὸ 5, ὥστε τὸ 5 σημείον ποιεῖν τὸ πρό-

a There is some uncertainty where the quotation from Archimedes ends and Eutocius's comments are resumed. Heiberg, with some reason, makes Eutocius resume his comments at this point.

b In the mss. the figures on pp. 150 and 156 are com-
This can be proved similarly for all points taken between E, A. And it was proved for all points between E, B; therefore for all figures similarly taken upon AB, BE^2. EA is greatest when BE = 2EA."

The following consequences a should also be noticed in the aforementioned figure. Inasmuch as it has been proved that

\[ B\Sigma^2 \cdot \Sigma A < BE^2 \]

and

\[ B\Sigma^2 \cdot \Sigma A < BE^2 \cdot EA, \]

if the product of the given space and the given straight line is less than BE^2 . EA, it is possible to cut AB in two points satisfying the conditions of the original problem.c This comes about if we conceive a parabola described about the axis XH with parameter \( H\Omega \); for such a parabola will necessarily pass through T.d And since it must necessarily meet \( TN \), being parallel to a diameter [Apoll. Con. i. 26], it is clear that it cuts the hyperbola in another point above K, as at P in this case, and a perpendicular drawn from P to AB, as \( P\Sigma \) in this case, will cut AB in \( \Sigma \), so that the point \( \Sigma \) satisfies the conditions of the

bined; in this edition it is convenient, for the sake of clarity, to give separate figures.

With the same notation as before this may be stated: when \( bc^2 < \frac{4}{27}a^3 \), there are always two real solutions of the cubic equation \( x^3(a-x) = bc^2 \) lying between 0 and \( a \). If the cubic has two real roots it must, of course, have a third real root as well, but the Greeks did not recognize negative solutions.

By Apoll. i. 11, since \( TX^2 = XH \cdot H\Omega \).
βλήμα, καὶ ἵσον γίνεσθαι τὸ ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ τῷ ἀπὸ Βς ἐπὶ τὴν ΣΑ, ὡς ἐστὶ διὰ τῶν προερημένων ἀποδείξεων ἐμφανές. ὥστε δυνατοῦ ὄντος ἐπὶ τῆς BA δύο σημεία λαμβάνειν ποιοῦντα τὸ ζητούμενον, ἔξεστιν, ὅποτερον τις βούλιοτο, λαμβάνειν ἡ τὸ μεταξὺ τῶν Ε, Β ἡ τὸ μεταξὺ τῶν Ε, Α. εἰ μὲν γὰρ τὸ μεταξὺ τῶν Ε, Β, ὡς εἰρηται, τῆς διὰ τῶν Η, Τ σημείων γραφομένης παραβολῆς κατὰ δύο σημεία τεμνούσης τὴν ὑπερβολήν τὸ μὲν ἐγνύτερον τοῦ Η, τούτεστι τοῦ αξόνος τῆς παραβολῆς, εὑρῆσει τὸ μεταξὺ τῶν Ε, Β, ὡς ἐνταῦθα τὸ Τ εὑρίσκει τὸ Σ, τὸ δὲ ἀπωτέρῳ τὸ μεταξὺ τῶν Ε, Α, ὡς ἐνταῦθα τὸ Ρ εὑρίσκει τὸ τ':

Καθόλου μὲν οὖν οὕτως ἀναλέλυται καὶ συντεθεῖται τὸ πρόβλημα: ἵνα δὲ καὶ τοῖς 'Αρχιμηδείοις ῥήμασιν ἐφαρμοσθῇ, λείποςθω ὡς ἐν αὐτῇ τῇ τοῦ ῥήτου καταγραφῆ διάμετρος μὲν τῆς σφαίρας ἡ ΔΒ, ἢ δὲ ἐκ τοῦ κέντρου ἡ ΒΖ, καὶ ἡ δεδομένη ἡ ΖΘ. κατηντήσαμεν ἀρά, φησίν, εἰς τὸ 'τὴν ΔΖ τεμεῖν κατὰ τὸ Χ, ωστε εἶναι, ὡς τὴν ΧΖ πρὸς τὴν δοθείσαν, οὕτως τὸ δὸθὲν πρὸς τὸ ἀπὸ τῆς ΔΧ. τούτῳ δὲ ἀπλῶς μὲν λεγόμενον ἕχει διαφορισμόν:' εἰ γὰρ τὸ δὸθὲν ἐπὶ τὴν δοθείσαν μεῖζον ἔτυγχανεν τοῦ ἀπὸ τῆς ΔΒ ἐπὶ τὴν ΒΖ, ἀδύνατον ἢν τὸ πρόβλημα, ως δεδεικταί, εἰ δὲ ἵσον, τὸ Β σημεῖον ἐποίει τὸ πρόβλημα, καὶ οὕτως δὲ οὐδὲν ἢν πρὸς τὴν ἐξ ἀρχῆς 'Αρχιμηδους προθεσιν. ἡ γὰρ σφαίρα οὐκ ἔτεινετο εἰς τὸν δοθέντα

a Archimedes' figure is re-drawn (v. page 162) so that Β, Ζ come on the left of the figure and Δ on the right, instead of B, Z on the right and Δ on the left.

b v. supra, p. 133.
problem, and $B\Sigma^2 \cdot \Sigma A = B\xi^2 \cdot \xi A$, as is clear from the above proof. Inasmuch as it is possible to take on BA two points satisfying what is sought, it is permissible to take whichever one wills, either the point between E, B or that between E, A. If we choose the point between E, B, the parabola described through the points H, T will, as stated, cut the hyperbola in two points; of these the one nearer to H, that is to the axis of the parabola, will determine the point between E, B, as in this case T determines $\Sigma$, while the point farther away will determine the point between E, A, as in this case P determines $\xi$.

The analysis and synthesis of the general problem have thus been completed; but in order that it may be harmonized with Archimedes' words, let there be conceived, as in Archimedes' own figure, a diameter $\Delta B$ of the sphere, with radius [equal to] $BZ$, and a given straight line $Z\Theta$. We are therefore faced with the problem, he says, "so to cut $\Delta Z$ at $X$ that $XZ$ bears to the given straight line the same ratio as the given area bears to the square on $\Delta X$. When the problem is stated in this general form, it is necessary to investigate the limits of possibility." If therefore the product of the given area and the given straight line chanced to be greater than $\Delta B^2 \cdot BZ$, the problem would not admit a solution, as was proved, and if it were equal the point B would satisfy the conditions of the problem, which also would be of no avail for the purpose Archimedes set himself at the outset; for the sphere would not be

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*For $\Delta B = \frac{\Delta Z}{3}$ [ex hyp.], and so $\Delta B$ in the figure on p. 162 corresponds with $BE$ in the figure on p. 146, while $BZ$ in the figure on p. 162 corresponds with $EA$ in the figure on p. 146.*
λόγον. ἀπλῶς ἄρα λεγόμενον εἶχεν προσδιορισμόν: "προστιθεμένων δὲ τῶν προβλημάτων τῶν ἐνθάδε ὑπαρχόντων," τοιτέστι τοῦ τε διπλασίαν εἶναι τὴν ΔΒ τῆς ΖΒ καὶ τοῦ μείζονα εἶναι τὴν ΒΖ τῆς ΖΘ, "οὐκ ἔχει διορισμόν." τὸ γὰρ ἀπὸ ΔΒ τὸ δοθὲν ἐπὶ τὴν ΖΘ τὴν δοθεῖσαν ἐλαττῶν ἐστὶ τοῦ ἀπὸ τῆς ΔΒ ἐπὶ τὴν ΒΖ διὰ τὸ τὴν ΒΖ τῆς ΖΘ μείζονα εἶναι, οὕπερ ὑπάρχοντος ἐδείξαμεν δυνατὸν, καὶ ὀπωσ προβαίνει τὸ πρόβλημα.

* Eutocius proceeds to give solutions of the problem by Dionysodorus and Diocles, by whose time, as he has explained, Archimedes' own solution had already disappeared. Dionysodorus solves the less general equation by means of the intersection of a parabola and a rectangular hyperbola; Diocles solves the general problem by the intersection of an ellipse with a rectangular hyperbola, and his proof is both ingenious and intricate. Details may be consulted in Heath, *H.G.M.* ii. 46-49 and more fully in Heath, 162
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cut in the given ratio. Therefore when the problem was stated generally, an investigation of the limits of possibility was necessary as well; "but under the conditions of the present case," that is, if \( \Delta B = 2ZB \) and \( BZ > Z\Theta \), "no such investigation is necessary." For the product of the given area \( \Delta B^2 \) into the given straight line \( Z\Theta \) is less than the product of \( \Delta B^2 \) into \( BZ \) by reason of the fact that \( BZ \) is greater than \( Z\Theta \), and we have shown that in this case there is a solution, and how it can be effected.\(^a\)

\(^a\) The Works of Archimedes, pp. cxxiii-cxli, which deals with the whole subject of cubic equations in Greek mathematical history. It is there pointed out that the problem of finding mean proportionals is equivalent to the solution of a pure cubic equation, \( a^3 = \frac{a}{b} \), and that Menaechmus's solution, by the intersection of two conic sections (v. vol. i. pp. 278-283), is the precursor of the method adopted by Archimedes, Dionysodorus and Diocles. The solution of cubic equations by means of conics was, no doubt, found easier than a solution by the manipulation of parallelepipeds, which would have been analogous to the solution of quadratic equations by the application of areas (v. vol. i. pp. 186-215). No other examples of the solution of cubic equations have survived, but in his preface to the book On Conoids and Spheroids Archimedes says the results there obtained can be used to solve other problems, including the following, "from a given spheroidal figure or conoid to cut off, by a plane drawn parallel to a given plane, a segment which shall be equal to a given cone or cylinder, or to a given sphere" (Archim. ed. Heiberg i. 258. 11-15); the case of the paraboloid of revolution does not lead to a cubic equation, but those of the spheroid and hyperboloid of revolution do lead to cubics, which Archimedes may be presumed to have solved. The conclusion reached by Heath is that Archimedes solved completely, so far as the real roots are concerned, a cubic equation in which the term in \( x \) is absent; and as all cubic equations can be reduced to this form, he may be regarded as having solved geometrically the general cubic.
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(d) CONOIDS AND SPHEROIDS

(i.) Preface

i. 246. 1-14

Ἀρχιμήδης Δοσιθέω εὑρίσκεις.
Ἀποστέλλω τοι γράφας ἐν τῷ δε τῷ βιβλίῳ τῶν τε λοιπῶν θεωρημάτων τὰς ἀποδείξεις, ὡς οὐκ εἴσε ἐν τοῖς πρότερον ἀπεσταλμένοις, καί ἅλλων υστερον ποτεξευρημένων, ἀ πρότερον μὲν ἢ δη πολλάκις ἐγχειρήσας ἐπισκέπτεσθαι δύσκολον ἔχειν τι φανείσας μοι τὰς εὑρέσιος αὐτῶν ἀπόρρησα. διόπερ οὐδὲ συνεξεδόθην τοῖς ἅλλοις αὐτὰ τὰ προ- βεβλημένα. υστερον δὲ ἐπιμελέστερον ποτ' αὐτοῖς γενόμενος ἐξεύρον τὰ ἀπορρηθέντα. ἢν δὲ τὰ μὲν λοιπὰ τῶν προτέρων θεωρημάτων περὶ τοῦ ὁρθο- γωνίου κωνοειδεός προβεβλημένα, τὰ δὲ νῦν ἐντι ποτεξευρημένα περὶ τε ἀμβλυγωνίου κωνοειδεός καὶ περὶ σφαιροειδεών σχημάτων, ὡς τὰ μὲν παραμάκεα, τὰ δὲ ἐπιπλατέα καλέω.

(ii.) Two Lemmas

Ibid., Lemma ad Prop. 1, Archim. ed. Heiberg
i. 260. 17-24

Εἰ κα ἔως τι μεγέθεα ὑποσαοῦν τῷ ἵσῳ ἄλλαλων

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a In the books On the Sphere and Cylinder, On Spirals and on the Quadrature of a Parabola.
b i.e., the paraboloid of revolution.
c i.e., the hyperboloid of revolution.
d An oblong spheroid is formed by the revolution of an
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(d) CONOIDS AND SPHEROIDS

(i.) Preface

Archimedes, On Conoids and Spheroids, Preface, Archim. ed. Heiberg i. 246. 1-14

Archimedes to Dositheus greeting.

I have written out and now send you in this book the proofs of the remaining theorems, which you did not have among those sent to you before,* and also of some others discovered later, which I had often tried to investigate previously but whose discovery was attended with some difficulty and I was at a loss over them; and for this reason not even the propositions themselves were forwarded with the rest. But later, when I had studied them more carefully, I discovered what had left me at a loss before. Now the remainder of the earlier theorems were propositions about the right-angled conoid †; but the discoveries now added relate to an obtuse-angled conoid * and to spheroidal figures, of which I call some oblong and others flat. ‡

(ii.) Two Lemmas

Ibid., Lemma to Prop. 1, Archim. ed. Heiberg i. 260. 17-24

If there be a series of magnitudes, as many as you please, in which each term exceeds the previous term by an ellipse about its major axis, a flat spheroid by the revolution of an ellipse about its minor axis.

In the remainder of our preface Archimedes gives a number of definitions connected with those solids. They are of importance in studying the development of Greek mathematical terminology.
If \( h \) is the common difference, the first series is \( h, 2h, 3h \ldots \ n h \), and the second series is \( nh, nh \ldots \) to \( n \) terms, its sum obviously being \( n^2h \). The lemma asserts that

\[
2(h + 2h + 3h + \ldots + nh) < n^2h < 2(h + 2h + 3h + \ldots + nh).
\]

It is proved in the book *On Spirals*, Prop. 11. The proof is geometrical, lines being placed side by side to represent the

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ὅπερέχοντα, ἢ δὲ ὁ ὑπεροχὰ ἵσα τῷ ἐλαχίστῳ, καὶ ἄλλα μεγέθεα τῷ μὲν πλήθει ἵσα τούτου, τῷ δὲ μεγέθει ἐκαστὸν ἵσον τῷ μεγίστῳ, πάντα τὰ μεγέθεα, ὡς ἐστιν ἐκαστὸν ἵσον τῷ μεγίστῳ, πάντων μὲν τῶν τῷ ἵσῳ ὑπερεχόντων ἐλάσσονα ἐσσοῦνται ἡ διπλάσια, τῶν δὲ λοιπῶν χωρὶς τοῦ μεγίστου μείζονα ἡ διπλάσια. ἃ δὲ ἀπόδειξις τούτου φανερά.

Ibid., Prop. 1, Archim. ed. Heiberg 1. 260, 26-261. 22

Εἰ καὶ μεγέθεα ὁποσαοῦν τῷ πλήθει ἄλλοις μεγέθεσιν ἵσοις τῷ πλήθει κατὰ δύο τὸν αὐτὸν λόγον ἔχοντι τὰ ὁμοίως τεταγμένα, λέγηται δὲ τὰ τε πρῶτα μεγέθεα ποτ' ἄλλα μεγέθεα ἡ πάντα ἡ τίνα αὐτῶν ἐν λόγοις ὁποιοισοῦν, καὶ τὰ ύστερον ποτ' ἄλλα μεγέθεα τὰ ὁμόλογα ἐν τοῖς αὐτοῖς λόγοις, πάντα τὰ πρῶτα μεγέθεα ποτὶ πάντα, ἃ λέγονται, τὸν αὐτὸν ἐξοῦντι λόγον, ὃν ἔχοντι πάντα τὰ ύστερον μεγέθεα ποτὶ πάντα, ἃ λέγονται.

"Εστω τινὰ μεγέθεα τὰ Α, Β, Γ, Δ, Ε, Ζ ἄλλοις μεγέθεσιν ἵσοις τῷ πλήθει τοῖς, Η, Θ, Ι, Κ, Λ, Μ
equal quantity, which common difference is equal to the least term, and if there be a second series of magnitudes, equal to the first in number, in which each term is equal to the greatest term [in the first series], the sum of the magnitudes in the series in which each term is equal to the greatest term is less than double of the sum of the magnitudes differing by an equal quantity, but greater than double of the sum of those magnitudes less the greatest term. The proof of this is clear.¹

Ibid., Prop. 1, Archim. ed. Heiberg i. 260. 26–261. 22

If there be a series of magnitudes, as many as you please, and it be equal in number to another series of magnitudes, and the terms have the same ratio two by two, and if any or all of the first series of magnitudes form any proportion with another series of magnitudes, and if the second series of magnitudes form the same proportion with the corresponding terms of another series of magnitudes, the sum of the first series of magnitudes bears to the sum of those with which they are in proportion the same ratio as the sum of the second series of magnitudes bears to the sums of the terms with which they are in proportion.

Let the series of magnitudes A, B, Г, Δ, Е, Z be equal in number to the series of magnitudes H, Θ, И, terms of the arithmetical progression and produced until each is equal to the greatest term. It is equivalent to this algebraical proof:

Let \[ S_n = h + 2h + 3h + \ldots + nh. \]

Then \[ S_n = nh + (n - 1)h + (n - 2)h + \ldots + h. \]

Adding, \[ 2S_n = n(n + 1)h, \]

and so \[ 2S_{n-1} = (n - 1)nh. \]

Therefore \[ 2S_{n-1} < n^2h < 2Sn. \]
Since \( N : A = T : H, A : B = H : \Theta \), [ex hyp.]

\[ \therefore \text{ex aequo} \quad N : B = T : \Theta. \] [Eucl. v. 22]

But \( B : \Xi = \Theta : \Upsilon \); [ex hyp.]

\[ \therefore \text{ex aequo} \quad N : \Xi = T : \Upsilon. \] [Eucl. v. 22]

Similarly

\[ \Xi : O = \Upsilon : \Phi, O : \Pi = \Phi : X, \Pi : P = X : \Psi, P : \Sigma = \Psi : \Omega. \]

Now since \( A : B = H : \Theta \), [ex hyp.]

\[ \therefore \text{componendo} \quad A + B : A = H + \Theta : H, \] [Eucl. v. 18]

i.e., permutando \( A + B : H + \Theta = A : H. \) [Eucl. v. 16]

But since \( N : A = T : H, \) [ex hyp.]

\[ \therefore A : H = N : T \] [Eucl. v. 16]

\[ = \Xi : \Upsilon \] [ibid.]

\[ = O : \Phi \] [ibid.]

\[ = \Gamma : I. \] [ibid.]

\[ \therefore A + B : H + \Theta = \Gamma : I. \] [Eucl. v. 18]

\[ = O : \Phi \] [Eucl. v. 16]
K, Λ, M, and let them have the same ratio two by two, so that

\[ A : B = H : \Theta, \quad B : \Gamma = \Theta : I, \]

and so on, and let the series of magnitudes A, B, Σ, Δ, E, Z form any proportion with another series of magnitudes N, Ξ, O, Π, P, Σ, and let H, Θ, I, K, Λ, M form the same proportion with the corresponding terms of another series, T, Υ; Φ, X, Ψ, Ω so that

\[ A : N = H : T, \quad B : \Xi = \Theta : \Upsilon, \]

and so on; it is required to prove that

\[ \frac{A + B + \Gamma + \Delta + E + Z}{N + \Xi + O + \Pi + P + \Sigma} = \frac{H + \Theta + I + K + \Lambda + M}{T + \Upsilon + \Phi + X + \Psi + \Omega}, \]

\[ = \Pi : X \quad \text{[ibid.]} \]
\[ = \Delta : K. \quad \text{[ibid.]} \]

By pursuing this method it may be proved that

\[ A + B + \Gamma + \Delta + E + Z : H + \Theta + I + K + \Lambda + M = A : H, \]

or, permutando,

\[ A + B + \Gamma + \Delta + E + Z : A = H + \Theta + I + K + \Lambda + M : H. \quad (1) \]

Now

\[ N : \Xi = T : \Upsilon; \]

\[ \therefore \text{componendo et permutando,} \]

\[ N + \Xi : T + \Upsilon = \Xi : \Upsilon \quad \text{[Eucl. v. 18, v. 16]} \]
\[ = O : \Phi; \quad \text{[Eucl. v. 16]} \]

whence

\[ N + \Xi + O : T + \Upsilon + \Phi = O : \Phi, \quad \text{[Eucl. v. 18]} \]

and so on until we obtain

\[ N + \Xi + O + \Pi + P + \Sigma : T + \Upsilon + \Phi + X + \Psi + \Omega = N : T. \quad \text{[2]} \]

But

\[ A : N = H : T; \quad \text{[ex hyp.]} \]

\[ \therefore \text{by (1) and (2),} \]

\[ \frac{A + B + \Gamma + \Delta + E + Z}{N + \Xi + O + \Pi + P + \Sigma} = \frac{H + \Theta + I + K + \Lambda + M}{T + \Upsilon + \Phi + X + \Psi + \Omega}. \quad \text{Q.E.D.} \]

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(iii.) *Volume of a Segment of a Paraboloid of Revolution*

*Ibid.,* Prop. 21, Archim. ed. Heiberg i. 344. 21–354. 20

Πᾶν τμᾶμα ὀρθογωνίου κωνοειδεὸς ἀποτετμαμένον ἐπιπέδῳ ὀρθῶ ποτὶ τὸν ἄξονα ἡμιολίων ἦστι τοῦ κώνου τοῦ βάσιν ἔχοντος τὰς αὐτὰν τῷ τμᾶματι καὶ ἄξονα.

"Εστώ γὰρ τμᾶμα ὀρθογωνίου κωνοειδεὸς ἀποτετμαμένον ὀρθῶ ἐπιπέδῳ ποτὶ τὸν ἄξονα, καὶ τμαθόντος αὐτοῦ ἐπιπέδῳ ἄλλω διὰ τοῦ ἄξονος τὰς μὲν ἐπιφανείας τομὰ ἦστω ἄ ΑΒΓ ὀρθογωνίου κώνου τομά, τοῦ δὲ ἐπιπέδου τοῦ ἀποτέμνοντος τὸ τμᾶμα ἄ ΓΑ εὐθεία, ἄξων δὲ ἦστω τοῦ τμᾶματος ἄ ΒΔ, ἦστω δὲ καὶ κώνος τὰς αὐτὰς βάσιν ἔχων τῷ τμᾶματι καὶ ἄξονα τῶν αὐτῶν, οὗ κορυφὰ τὸ Β. δεικτέον, ὅτι τὸ τμᾶμα τοῦ κωνοειδεὸς ἡμιολίων ἦστι τοῦ κώνου τούτου.

'Εκκείσθω γὰρ κώνος ὁ Ψ ἡμιολίος ἐὼν τοῦ κώνου, οὗ βάσις ὁ περὶ διάμετρον τῶν ΑΓ, ἄξων δὲ ἄ ΒΔ, ἦστω δὲ καὶ κύλινδρος βάσιν μὲν ἔχων 170
(iii.) Volume of a Segment of a Paraboloid of Revolution


Any segment of a right-angled conoid cut off by a plane perpendicular to the axis is one-and-a-half times the cone having the same base as the segment and the same axis.

For let there be a segment of a right-angled conoid cut off by a plane perpendicular to the axis, and let it be cut by another plane through the axis, and let the section be the section of a right-angled cone $\Delta B\Gamma$, and let $\Gamma A$ be a straight line in the plane cutting off the segment, and let $B\Delta$ be the axis of the segment, and let there be a cone, with vertex $B$, having the same base and the same axis as the segment. It is required to prove that the segment of the conoid is one-and-a-half times this cone.

For let there be set out a cone $\Psi$ one-and-a-half times as great as the cone with base about the diameter $A\Gamma$ and with axis $B\Delta$, and let there be a

* It is proved in Prop. 11 that the section will be a parabola.

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τὸν κύκλον τὸν περὶ διάμετρον τὰν ἈΓ, ἄξονα δὲ τὰν ΒΔ· ἐσσεῖται οὖν ὁ Ψ κώνος ἡμίσεος τοῦ κυλίνδρου [ἐπείπερ ἡμιόλιος ἤστιν ὁ Ψ κώνος τοῦ αὐτοῦ κώνου].

λέγω, διὸ τὸ τμῆμα τοῦ κωνοειδεὸς ίσον ἐστὶ τῷ Ψ κώνῳ.

Εἰ γάρ μὴ ἦστιν ἴσον, ήτοι μείζον ἐντὸ ἢ ἐλασσον. ἦστω δὴ πρότερον, εἰ δυνατόν, μείζον.

ἐγγεγράφθη δὴ σχῆμα στερεὸν εἰς τὸ τμῆμα, καὶ ἀλλο περιγεγράφθη ἐκ κυλίνδρων ὑψος ἴσον ἐχόντων συγκείμενον, ὥστε τὸ περιγραφέν σχῆμα τοῦ ἐγγραφήστος ὑπερέχει ἐλάσσον, ἡ ἀλίκῳ ὑπερέχει τὸ τοῦ κωνοειδεὸς τμῆμα τοῦ Ψ κώνου, καὶ ἦστω τῶν κυλίνδρων, ἐξ ὧν σύγκειται τὸ περιγραφέν σχῆμα, μέγιστος μὲν ὁ βάσιν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ἈΓ, ἄξονα δὲ τὰν ΕΔ, ἐλάχιστος δὲ ὁ βάσιν μὲν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΣΤ, ἄξονα δὲ τὰν ΒΙ, τῶν δὲ κυλίνδρων, ἐξ ὧν σύγκειται τὸ ἐγγραφὲν σχῆμα, μέγιστος μὲν ἦστω ὁ βάσιν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΚΛ, ἄξονα δὲ τὰν ΔΕ, ἐλάχιστος δὲ ὁ βάσιν μὲν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΣΤ, ἄξονα δὲ τὰν ΘΙ, ἐκβεβλήσθω δὲ τὰ ἐπίπεδα πάντων τῶν κυλίνδρων ποτὶ τὰν

1 ἐπείπερ . . . κώνου om. Heiberg.

* For the cylinder is three times, and the cone Ψ one-and-a-

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cylinder having for its base the circle about the diameter \( \Delta \Gamma \) and for its axis \( \beta \Delta \); then the cone \( \Psi \) is one-half of the cylinder; I say that the segment of the conoid is equal to the cone \( \Psi \).

If it be not equal, it is either greater or less. Let it first be, if possible, greater. Then let there be inscribed in the segment a solid figure and let there be circumscribed another solid figure made up of cylinders having an equal altitude, in such a way that the circumscribed figure exceeds the inscribed figure by a quantity less than that by which the segment of the conoid exceeds the cone \( \Psi \) [Prop. 19]; and let the greatest of the cylinders composing the circumscribed figure be that having for its base the circle about the diameter \( \Delta \Gamma \) and for axis \( \varepsilon \Delta \), and let the least be that having for its base the circle about the diameter \( \Sigma \Theta \) and for axis \( \beta \Theta \); and let the greatest of the cylinders composing the inscribed figure be that having for its base the circle about the diameter \( \varepsilon \Delta \) and for axis \( \delta \Delta \), and let the least be that having for its base the circle about the diameter \( \Sigma \Theta \) and for axis \( \theta \Theta \); and let the planes of all the cylinders be half times, as great as the same cone; but because \( \tau \omega \) \( \alpha \nu \tau \omega \) \( \kappa \omega \nu \nu \) is obscure and \( \epsilon \pi \epsilon \tau \pi \epsilon \) often introduces an interpolation, Heiberg rejects the explanation to this effect in the text.

Archimedes has used those inscribed and circumscribed figures in previous propositions. The paraboloid is generated by the revolution of the parabola \( \Lambda \beta \Gamma \) about its axis \( \beta \Delta \). Chords \( \kappa \Lambda \ldots \Sigma \Theta \) are drawn in the parabola at right angles to the axis and at equal intervals from each other. From the points where they meet the parabola, perpendiculars are drawn to the next chords. In this way there are built up inside and outside the parabola "staggered" figures consisting of decreasing rectangles. When the parabola revolves, the rectangles become cylinders, and the segment of the paraboloid lies between the inscribed set of cylinders and the circumscribed set of cylinders.
ἐπιφάνειαν τοῦ κυλινδροῦ τοῦ βάσιν ἔχοντος τὸν κύκλον τὸν περὶ διάμετρον τὰν ΔΓ, ἄξονα δὲ τὰν ΒΔ· ἐσσεῖται δὴ ὁ ὄλος κυλινδρος διηρημένος εἰς κυλινδροὺς τῷ μὲν πλήθει ἵσους τοῖς κυλινδροῖς τοῖς ἐν τῷ περιγεγραμμένω σχήματι, τῷ δὲ με-γέθει ἵσους τῷ μεγίστῳ αὐτῶν. καὶ ἐπεὶ τὸ περιγεγραμμένον σχήμα περὶ τὸ τμῆμα ἐλάσσον ὑπερέχει τοῦ ἐγγεγραμμένου σχήματος ἢ τὸ τμῆμα τοῦ κώνου, δῆλον, ὅτι καὶ τὸ ἐγγεγραμμένον σχήμα ἐν τῷ τμῆματι μειζόν ἔστι τοῦ Ψ κώνου. οὗ δὴ πρῶτος κυλινδρος τῶν ἐν τῷ ὀλῳ κυλινδρῷ ὁ ἔχων ἄξονα τὰν ΔΕ ποτὶ τὸν πρῶτον κυλινδρον τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι τὸν ἔχοντα ἄξονα τὰν ΔΕ τὸν αὐτὸν ἔχει λόγον, ὅπως δὲ ἐστὶν ὁ αὐτὸς τῷ, ὅπως ἔχει ἄ ποτὶ τὰν ΚΕ δυνάμει· οὕτος δὲ ἐστὶν ὁ αὐτὸς τῷ, ὅπως ἔχει ἄ ΒΔ ποτὶ τὰν ΒΕ, καὶ τῷ, ὅπως ἔχει ἄ ΔΑ ποτὶ τὰν ΕΣ. ὅμως δὲ δείχθηκε ποτὶ ὁ δεύτερος κυλινδρος τῶν ἐν τῷ ὀλῳ κυλινδρῷ ὁ ἔχων ἄξονα τὸν ΕΖ ποτὶ τὸν δεύτερον κυλινδρον τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι τὸν αὐτὸν ἔχειν λόγον, ὅπως ἄ ΠΕ, τοιτέστιν ἄ ΔΑ, ποτὶ τὰν ΖΟ, καὶ τῶν ἀλλών κυλινδρων ἐκαστός τῶν ἐν τῷ ὀλῳ κυλινδρῷ ἄξονα ἐχόντων ἵσον τὰ ΔΕ ποτὶ ἐκαστόν τῶν κυλινδρῶν τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι ἄξονα ἐχόντων τοῦ αὐτοῦ ἔχειν τούτον τὸν λόγον, ὅπως ἡ ἡμίσεια τὰς διαμέτρου τὰς βάσις αὐτοῦ ποτὶ τῶν ἀπολελαμμέναν ἀπ’ αὐτῶς μεταξὺ τῶν ΑΒ, ΒΔ εὐθείαν· καὶ πάντες οὖν οἱ κυλινδροι οἱ ἐν τῷ κυλινδρῷ, οἱ βάσις μὲν ἐστὶν ὁ κύκλος ὁ περὶ διάμετρον τὰν ΔΓ, ἄξων δὲ [ἔστιν]¹ ἄ ΔΙ εὐθεία, ποτὶ πάντας τοὺς κυλινδροὺς τοὺς ἐν τῷ ἐγ-γεγραμμένῳ σχήματι τὸν αὐτὸν ἔξοντι λόγον, ὅπως 174
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produced to the surface of the cylinder having for its base the circle about the diameter $\Gamma \Delta$ and for axis $B\Delta$; then the whole cylinder is divided into cylinders equal in number to the cylinders in the circumscribed figure and in magnitude equal to the greatest of them. And since the figure circumscribed about the segment exceeds the inscribed figure by a quantity less than that by which the segment exceeds the cone, it is clear that the figure inscribed in the segment is greater than the cone $\Psi$.

Now the first cylinder of those in the whole cylinder, that having $\Delta E$ for its axis, bears to the first cylinder in the inscribed figure, which also has $\Delta E$ for its axis, the ratio $\Delta A^2 : KE^2$ [Eucl. xii. 11 and xii. 2]; but $\Delta A^2 : KE^2 = BA : BE$ = $\Delta A : E\Xi$. Similarly it may be proved that the second cylinder of those in the whole cylinder, that having $EZ$ for its axis, bears to the second cylinder in the inscribed figure the ratio $PE : ZO$, that is, $\Delta A : ZO$, and each of the other cylinders in the whole cylinder, having its axis equal to $\Delta E$, bears to each of the cylinders in the inscribed figure, having the same axis in order, the same ratio as half the diameter of the base bears to the part cut off between the straight lines $AB, B\Delta$; and therefore the sum of the cylinders in the cylinder having for its base the circle about the diameter $\Gamma \Delta$ and for axis the straight line $\Delta I$ bears to the sum of the cylinders in the inscribed figure the same ratio as the sum of

---

*a Because the circumscribed figure is greater than the segment.

*b By the property of the parabola; v. Quadr. parab. 3.

1 έστιν om. Heiberg.
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πάσαι αἱ εὐθεῖαι αἱ ἐκ τῶν κέντρων τῶν κύκλων, οἱ ἐν τί βάσις τῶν εἰρημένων κυλινδρῶν, ποτὶ πάσας τὰς εὐθείας τὰς ἀπολελαμμένας ἀπ' αὐτῶν μεταξὺ τῶν AB, BD. αἱ δὲ εἰρημέναι εὐθεῖαι τῶν εἰρημένων χωρίς τὰς ΔΔ μεῖζονες ἐντὶ ἦ διπλάσια: ὡστε καὶ οἱ κυλινδροὶ πάντες οἱ ἐν τῷ κυλινδρῷ, οὐ ἄξωνς ὁ ΔΙ, μεῖζονες ἐντὶ ἦ διπλάσιοι τοῦ ἐγγεγραμμένου σχήματος· πολλῷ ἀρα καὶ ὁ ὁλος κυλινδρὸς, οὐ ἄξωνς ὁ ΔΒ, μεῖζων ἐντὶ ἦ διπλασίων τοῦ ἐγγεγραμμένου σχήματος. τοῦ δὲ Ψ κώνου ἦν διπλασίων: ἐλάσσον ἀρα τὸ ἐγγεγραμ-
μένον σχήμα τοῦ Ψ κώνου· ὁπερ ἀδύνατον· ἐδείχθη γὰρ μεῖζον. οὐκ ἀρα ἐστὶν μεῖζον τὸ κωνοειδὲς τοῦ Ψ κώνου.

'Ομοίως δὲ οὐδὲ ἐλάσσον· παλιν γὰρ ἐγγε-
γράφθω τὸ σχήμα καὶ περιγεγράφθω, ὡστε ὑπερέχειν [ἐκαστὸν]¹ ἐλάσσον, ἦ ἀλίκω ὑπερέχει ὁ Ψ κώνος τοῦ κωνοειδέος, καὶ τὰ ἄλλα τὰ αὐτὰ τοῖς πρότερον κατεσκευάσθω. ἐπεὶ οὖν ἐλασσόν ἐστι τὸ ἐγγεγραμμένον σχήμα τοῦ τμάματος, καὶ τὸ ἐγγραφέν τοῦ περιγραφέντος ἐλάσσον λείπεται ἦ τὸ τμάμα τοῦ Ψ κώνου, δήλον, ὡς ἐλασσόν ἐστι τὸ περιγραφέν σχήμα τοῦ Ψ κώνου. παλιν δὲ ὁ

¹ ἐκαστὸν om. Heiberg, ἐκαστὸν ἐκάστοι Torelli (for ἐκάτερον ἐκατέρου).

<table>
<thead>
<tr>
<th>i.e.</th>
<th>First cylinder in whole cylinder</th>
<th>ΔΛ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>First cylinder in inscribed figure</td>
<td>EΞ'</td>
</tr>
<tr>
<td></td>
<td>Second cylinder in whole cylinder</td>
<td>EΠ</td>
</tr>
<tr>
<td></td>
<td>Second cylinder in inscribed figure</td>
<td>ZΟ'</td>
</tr>
</tbody>
</table>

and so on.

| Whole cylinder | ΔΛ + EΠ + ⋯ |
| Inscribed figure | EΞ + ZΟ + ⋯ |
the radii of the circles, which are the bases of the aforesaid cylinders, bears to the sum of the straight lines cut off from them between $AB$, $B\Delta$. But the sum of the aforesaid straight lines is greater than double of the aforesaid straight lines without $A\Delta$; so that the sum of the cylinders in the cylinder whose axis is $A\Omega$ is greater than double of the inscribed figure; therefore the whole cylinder, whose axis is $A\Omega$, is greater by far than double of the inscribed figure. But it was double of the cone $\Psi$; therefore the inscribed figure is less than the cone $\Psi$; which is impossible, for it was proved to be greater. Therefore the conoid is not greater than the cone $\Psi$.

Similarly [it can be shown] not to be less; for let the figure be again inscribed and another circumscribed so that the excess is less than that by which the cone $\Psi$ exceeds the conoid, and let the rest of the construction be as before. Then because the inscribed figure is less than the segment, and the inscribed figure is less than the circumscribed by some quantity less than the difference between the segment and the cone $\Psi$, it is clear that the circumscribed figure is less than the cone $\Psi$. Again, the first

This follows from Prop. 1, for

\[
\frac{\text{First cylinder in whole cylinder}}{\text{Second cylinder in whole cylinder}} = 1 = \frac{\Delta A}{\Pi}
\]

and so on, and thus the other condition of the theorem is satisfied.

\[\text{For } \Delta A, \varepsilon, \zeta, \ldots \text{ is a series diminishing in arithmetical progression, and } \Delta A, \Pi, \ldots \text{ is a series, equal in number, in which each term is equal to the greatest in the arithmetical progression. Therefore, by the Lemma to Prop. 1,}
\]

\[
\Delta A + \Pi + \ldots > 2(\varepsilon + \zeta + \ldots).
\]
πρώτος κύλινδρος τῶν ἐν τῷ ὅλῳ κυλίνδρῳ ὁ ἐξων ἄξονα τὰν ΔΕ ποτὶ τὸν πρώτον κύλινδρον τῶν ἐν τῷ περιγεγραμμένῳ σχῆματι τὸν τὸν αὐτὸν ἔχοντα ἄξονα τὰν ΕΔ τὸν αὐτὸν ἔχει λόγον, ὅν τὸ ἀπὸ τᾶς ΑΔ τετράγωνον ποτὶ τὸ αὐτὸ, ὁ δὲ δεύτερος κύλινδρος τῶν ἐν τῷ ὅλῳ κυλίνδρῳ ὁ ἐξων ἄξονα τὰν ΕΖ ποτὶ τὸν δεύτερον κύλινδρον τῶν ἐν τῷ περιγεγραμμένῳ σχῆματι τὸν ἔχοντα ἄξονα τὰν ΕΖ τὸν αὐτὸν ἔχει λόγον, ὅν ᾧ ΔΑ ποτὶ τὰν ΚΕ δυνάμει. οὕτως δὲ ἐστὶν ὁ αὐτὸς τῷ, ὅτι ἔχει ᾧ ΒΔ ποτὶ τὰν ΒΕ, καὶ τῷ, ὅτι ἔχει ᾧ ΔΑ ποτὶ τὰν ΕΞ· καὶ τῶν ἀλλων κυλίνδρων ἐκαστὸς τῶν ἐν τῷ ὅλῳ κυλίνδρῳ ἄξονα ἔχοντων ἴσον τὰ ΔΕ ποτὶ ἐκαστὸν τῶν κυλίνδρων τῶν ἐν τῷ περιγεγραμμένῳ σχῆματι ἄξονα ἔχοντων τὸν αὐτὸν, ἔχει τούτον τὸν λόγον, ὅν ᾧ ἡμίσεια τὰς διαμέτρου τὰς βάσις αὐτοῦ ποτὶ τὰν ἀπολελαμμέναν ἀπ’ αὐτῶς μεταξὺ τὰν ΑΒ, ΒΔ εὐθεῖαν· καὶ πάντες οὕτως οἱ κυλίνδροι οἱ ἐν τῷ ὅλῳ κυλίνδρῳ, ὃς ἄξων ἐστὶν ᾧ ΒΔ εὐθεία, ποτὶ πάντας τοὺς κυλίνδρους τοὺς ἐν τῷ περιγεγραμμένῳ σχῆματι τὸν αὐτὸν ἔξοντι λόγον, ὅτι πάσης αἰ εὐθείαι ποτὶ πάσας τὰς εὐθείας. αἰ δὲ εὐθείαι πάσαι αἰ ἐκ τῶν κέντρων τῶν κύκλων, οἱ βάσις ἔντι τῶν κυλίνδρων, τὰν εὐθείαν πασάν τὰν ἀπολελαμμέναν ἀπ’ αὐτῶν σὺν τὰ ἈΔ ἐλάσσονες ἐντὸς

- As before,

First cylinder in whole cylinder \[=\Delta \Delta\]

First cylinder in circumscribed figure \[=\Delta \Delta\]

Second cylinder in whole cylinder \[=\Delta \Delta \text{ EΠ}\]

Second cylinder in circumscribed figure \[=\text{ΕΞ} = \text{ΕΞ}\]

and so on.

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cylinder of those in the whole cylinder, having ΔE for its axis, bears to the first cylinder of those in the circumscribed figure, having the same axis EA, the ratio $\Delta A^2 : \Delta A^2$; the second cylinder in the whole cylinder, having EZ for its axis, bears to the second cylinder in the circumscribed figure, having EZ also for its axis, the ratio $\Delta A^2 : KE^2$; this is the same as $B\Delta : BE$, and this is the same as $\Delta A : E\Xi$; and each of the other cylinders in the whole cylinder, having its axis equal to ΔE, will bear to the corresponding cylinder in the circumscribed figure, having the same axis, the same ratio as half the diameter of the base bears to the portion cut off from it between the straight lines AB, BΔ; and therefore the sum of the cylinders in the whole cylinder, whose axis is the straight line BΔ, bears to the sum of the cylinders in the circumscribed figure the same ratio as the sum of the one set of straight lines bears to the sum of the other set of straight lines.\textsuperscript{a} But the sum of the radii of the circles which are the bases of the cylinders is less than double of the sum of the straight lines cut off from them together with $\Delta A$; it is therefore clear

And

\[
\frac{\text{First cylinder in whole cylinder}}{\text{Second cylinder in whole cylinder}} = 1 = \frac{\Delta A}{EI},
\]

and so on.

Therefore the conditions of Prop. 1 are satisfied and

\[
\frac{\text{Whole cylinder}}{\text{Circumscribed figure}} = \frac{\Delta A + EI + \ldots}{\Delta A + E\Xi + \ldots}.
\]

\textsuperscript{a} As before, $\Delta A$, $E\Xi$ \ldots is a series diminishing in arithmetical progression, and $\Delta A$, $EI$ \ldots is a series, equal in number, in which each term is equal to the greatest in the arithmetical progression.

Therefore, by the Lemma to Prop. 1,

\[
\Delta A + EI + \ldots < 2(\Delta A + E\Xi + \ldots).
\]
GREEK MATHEMATICS

Archimedes' proof may be shown to be equivalent to an integration, as Heath has done (The Works of Archimedes, cxlvii-cxlviii).

For, if \( n \) be the number of cylinders in the whole cylinder, and \( \Delta = nh \), Archimedes has shown that

\[
\begin{align*}
\text{Whole cylinder} & = \frac{n^2h}{h + 2h + 3h + \ldots + (n-1)h} > 2, \\
\text{Inscribed figure} & = \frac{n^2h}{n + 2h + 3h + \ldots + nh} < 2.
\end{align*}
\]

[Lemma to Prop. 1]

[ibid.]

In Props. 19 and 20 he has meanwhile shown that, by increasing \( n \) sufficiently, the inscribed and circumscribed figures can be made to differ by less than any assigned volume.

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that the sum of all the cylinders in the whole cylinder is less than double of the cylinders in the circumscribed figure; therefore the cylinder having for its base the circle about the diameter $A \Gamma$ and for axis $B \Delta$ is less than double of the circumscribed figure. But it is not, for it is greater than double; for it is double of the cone $\Psi$, and the circumscribed figure was proved to be less than the cone $\Psi$. Therefore the segment of the conoid is not less than the cone $\Psi$. But it was proved not to be greater; therefore it is one-and-a-half times the cone having the same base as the segment and the same axis.$^4$

When $n$ is increased, $h$ is diminished, but their product remains constant; let $nh=c$.

Then the proof is equivalent to an assertion that, when $a$ is indefinitely increased,

$$\text{limit of } h[h + 2h + 3h + \ldots + (n - 1)h] = \frac{1}{3}c^2,$$

which, in the notation of the integral calculus reads,

$$\int_0^c x dx = \frac{1}{3}c^2.$$

If the paraboloid is formed by the revolution of the parabola $y^2 = ax$ about its axis, we should express the volume of a segment as

$$\int_0^c \pi y^2 dx,$$

or

$$\pi a \int_0^c x dx.$$

The constant does not appear in Archimedes' proof because he merely compares the volume of the segment with the cone, and does not give its absolute value. But his method is seen to be equivalent to a genuine integration.

As in other cases, Archimedes refrains from the final step of making the divisions in his circumscribed and inscribed figures indefinitely large; he proceeds by the orthodox method of reductio ad absurdum.
(e) The Spiral of Archimedes

Definitions

Archim. De Lin. Spir., Deff., Archim. ed. Heiberg II. 44. 17–46. 21

α'. Εἴ κα εὐθεία ἐπιζευξθῇ γραμμά ἐν ἐπιπέδῳ καὶ μένοντος τοῦ ἑτέρου πέρατος αὐτᾶς ἴσοταχέως περιενεχθεῖσα ὀσακισθοῦν ἀποκατασταθῇ πάλιν, οὔθεν ἦρμασαν, ἀμα δὲ τὰ γραμμὰ περιαγομένα φέρηται τι σαμείον ἴσοταχέως αὐτὸ ἑαυτῷ κατὰ τᾶς εὐθείας ἀρξάμενον ἀπὸ τοῦ μένοντος πέρατος, τὸ σαμείον ἑλικὰ γράφει ἐν τῷ ἐπιπέδῳ.

β'. Καλείσθω ὁδὸν τὸ μὲν πέρας τὰς εὐθείας τὸ μένον περιαγομένας αὐτᾶς ἁρχὰ τὰς ἑλικὸς.

γ'. 'Α δὲ θέσις τὰς γραμμὰς, ἁφ' ἃς ἀρξάτο αἱ εὐθεία περιφέρεσθαι, ἁρχὰ τῆς περιφορᾶς.

δ'. Εὐθεία, ὁν μὲν ἐν τὰ πρῶτα περιφορᾷ διαπορευθῆ τὸ σαμείον τὸ κατὰ τὰς εὐθείας φερόμενον, πρῶτα καλείσθω, ἀν δ' ἐν τὰ δευτέρᾳ περιφορᾷ τὸ αὐτὸ σαμείον διανύσῃ, δευτέρα, καὶ αἳ ἄλλαι ὁμοίως ταῦται ὁμονύμως ταῖς περιφοραῖς καλεῖσθωσαν.

ε'. Τὸ δὲ χωρίον τὸ περιλαθὲν ὑπὸ τε τὰς ἑλικὸς τὰς ἐν τὰ πρῶτα περιφορᾶ γραφείσας καὶ τὰς εὐθείας, ἃ ἐστὶν πρῶτα, πρῶτον καλείσθω, τὸ δὲ περιλαθὲν ὑπὸ τε τὰς ἑλικὸς τὰς ἐν τὰ δευτέρᾳ περιφορᾶ γραφείσας καὶ τὰς εὐθείας τὰς δευτέρας δεύτερον καλείσθω, καὶ τὰ ἄλλα ἐξῆς οὕτω καλείσθω.

ς'. Καὶ εἴ κα ἀπὸ τοῦ σαμείου, ὃ ἐστὶν ἁρχὰ τὰς ἑλικῶν, ἀχθῇ τις εὐθεία γραμμά, τὰς εὐθείας ταῦτας 182
(e) The Spiral of Archimedes

(i.) Definitions

Archimedes, On Spirals, Definitions, Archim. ed.
Heiberg ii. 44. 17-46. 21

1. If a straight line drawn in a plane revolve uniformly any number of times about a fixed extremity until it return to its original position, and if, at the same time as the line revolves, a point move uniformly along the straight line, beginning at the fixed extremity, the point will describe a spiral in the plane.

2. Let the extremity of the straight line which remains fixed while the straight line revolves be called the origin of the spiral.

3. Let the position of the line, from which the straight line began to revolve, be called the initial line of the revolution.

4. Let the distance along the straight line which the point moving along the straight line traverses in the first turn be called the first distance, let the distance which the same point traverses in the second turn be called the second distance, and in the same way let the other distances be called according to the number of turns.

5. Let the area comprised between the first turn of the spiral and the first distance be called the first area, let the area comprised between the second turn of the spiral and the second distance be called the second area, and let the remaining areas be so called in order.

6. And if any straight line be drawn from the origin, let [points] on the side of this straight line in
\[
\text{GREEK MATHEMATICS}
\]

\[
\text{τά ἐπὶ τὰ αὐτά, ἐφ' ἃ κα ἀ περιφορὰ γένηται, προαγούμενα καλείσθω, τὰ δὲ ἐπὶ θάτερα ἐπόμενα.}
\]

\[
\text{ξ'. Ὁ τε γραφεῖς κύκλος κέντρῳ μὲν τῷ σαμείῳ, ὁ ἐστὶν ἀρχὰ τὰς ἔλικος, διαστῆματί δὲ τὰ εὐθεία, ἀ ἐστὶν πρῶτα, πρῶτος καλείσθω, ὁ δὲ γραφεῖς κέντρῳ μὲν τῷ αὐτῷ, διαστήματι δὲ τὰ διπλασία εὐθείᾳ δεύτερος καλείσθω, καὶ οἱ άλλοι δὲ ἐξεστο τούτως τὸν αὐτὸν τρόπον.}
\]

(ii.) Fundamental Property

\[
\text{Ibid., Prop. 14, Archim. έδ. Heiberg ii. 50. 9–52. 15}
\]

\[
\text{Εἰ κα ποτὶ τὰν ἔλικα τὰν ἐν τὰ πρῶτα περιφορά γεγραμμέναν ποτιπεσώντι δύο εὐθείαν ἀπὸ τοῦ σαμείου, ὁ ἐστὶν ἀρχὰ τὰς ἔλικος, καὶ ἐκβληθέωντι ποτὶ τὰν τοῦ πρῶτον κύκλου περιφέρειαν, τὸν αὐτὸν ἐξοντὶ λόγον αἱ ποτὶ τὰν ἔλικα ποτιπεσώντους ποτ' ἀλλάζον, δὲν ἃ περιφέρειαν τοῦ κύκλου αἱ μεταξὺ τοῦ πέρατος τὰς ἔλικος καὶ τῶν περάτων τὰν ἐκβληθεισάν εὐθείαν τῶν ἐπὶ τὰς περιφέρειας γυμνομένων, ἐπὶ τὰ προαγούμενα λαμβανομέναν τὰν περιφέρειαν ἀπὸ τοῦ πέρατος τὰς ἔλικος.}
\]

\[
\text{"Εστώ ἔλιξ ἃ ΑΒΓΔΕΘ ἐν τὰ πρῶτα περιφορά γεγραμμένα, ἀρχὰ δὲ τὰς μὲν ἔλικος ἐστὼ τὸ Α σαμείον, ἃ δὲ ΘΑ εὐθεία ἀρχὰ τὰς περιφορὰς ἐστὼ, καὶ κύκλος ὁ ΘΚΗ ἐστὼ ὁ πρῶτος, ποτιπεσώντων δὲ ἀπὸ τοῦ Α σαμείου ποτὶ τὰν ἔλικα αἱ ΑΕ, ΑΔ καὶ ἐκπεσώντων ποτὶ τὰν τοῦ κύκλου περιφέρειαν ἐπὶ τὰ Ζ, Η. δεικτέον, ὅτι τὸν αὐτὸν ἐχοντὶ λόγον ἃ ΑΕ ποτὶ τὰν ΑΔ, ὅν ἃ ΘΚΖ περιφέρεια ποτὶ τὰν ΘΚΗ περιφέρειαν.}
\]

\[
\text{Περιαγομένας γὰρ τὰς ΑΘ γραμμὰς δήλον, ως 184}
\]
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the direction of the revolution be called *forward*, and let those on the other side be called *rearward*.

7. Let the circle described with the origin as centre and the first distance as radius be called the *first circle*, let the circle described with the same centre and double of the radius of the first circle be called the *second circle*, and let the remaining circles in order be called after the same manner.

(ii.) *Fundamental Property*


If, from the origin of the spiral, two straight lines be drawn to meet the first turn of the spiral and produced to meet the circumference of the first circle, the lines drawn to the spiral will have the same ratio one to the other as the arcs of the circle between the extremity of the spiral and the extremities of the straight lines produced to meet the circumference, the arcs being measured in a forward direction from the extremity of the spiral.

Let $AB\Gamma\Delta E\Theta$ be the first turn of a spiral, let the point $A$ be the origin of the spiral, let $\Theta A$ be the initial line, let $\Theta K H$ be the first circle, and from the point $A$ let $AE$, $A\Delta$ be drawn to meet the spiral and be produced to meet the circumference of the circle at $Z$, $H$. It is required to prove that $AE : A\Delta = \text{arc } \Theta K Z : \text{arc } \Theta K H$.

When the line $A\Theta$ revolves it is clear that the point

*i.e.*, with radius equal to the sum of the radii of the first and second circles.
A κατὰ τὰς εὐθείας φερόμενον τὰν ΑΘ γραμμὰν πορεύεται, καὶ τὸ Θ σαμεῖον κατὰ τὰς τοῦ κύκλου περιφερείας φερόμενον τὰν ΘΚΖ περιφέρειαν, τὸ δὲ Α τὰν ΑΕ εὐθείαν, καὶ πάλιν τὸ τε Α σαμεῖον τὰν ΑΔ γραμμὰν καὶ τὸ Θ τὰν ΘΚΗ περιφέρειαν, ἐκάτερον ἰσοταχέως αὐτὸ ἐαυτῷ φερόμενον· δὴ λογὶς ὁδ' ὃτι τὸν αὐτὸν ἔχοντι λόγον ἂ ΑΕ ποτὶ τὰν ΑΔ, ὃν ἄ ΘΚΖ περιφέρεια ποτὶ τὰν ΘΚΗ περιφέρειαν [δέδεικται γὰρ τοῦτο ἐξ ὑπὸ τοὺς πρῶτους].

Ὅμως δὲ δεικθῆσθαι, καὶ εἴ καὶ ἀ άτέρα τὰν ποτιπποτοῦσαν ἐπὶ τὸ πέρας τὰς ἕλικους ποτιππίτη, ὃτι τὸ αὐτὸ συμβαίνει.

(iii.) A Verging

Ibid., Prop. 7, Archim. ed. Heiberg ii. 22. 14–24. 7

Τῶν αὐτῶν δεδομένων καὶ τὰς ἐν τῷ κύκλῳ εὐθείας ἐκβεβλημένας δυνάτων ἐστιν ἀπὸ τοῦ
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\[ \Theta \text{ moves uniformly round the circumference } \Theta KH \text{ of the circle while the point } A, \text{ which moves along the straight line, traverses the line } A\Theta; \text{ the point } \Theta \text{ which moves round the circumference of the circle traverses the arc } \Theta KZ \text{ while } A \text{ traverses the straight line } AE; \text{ and furthermore the point } A \text{ traverses the line } \Delta \Delta \text{ in the same time as } \Theta \text{ traverses the arc } \Theta KH, \text{ each moving uniformly;} \text{ it is clear, therefore, that } AE : A\Delta = \text{arc } \Theta KZ : \text{arc } \Theta KH \text{ [Prop. 2].}

Similarly it may be shown that if one of the straight lines be drawn to the extremity of the spiral the same conclusion follows.\(^a\)

(iii.) \textit{A Verging} \(^b\)

Ibid., Prop. 7, Archim. ed. Heiberg ii. 22. 14–24. 7

With the same data and the chord in the circle produced,\(^c\) it is possible to draw a line from the centre to meet

\* In Prop. 15 Archimedes shows (using different letters, however) that if $AE, A\Delta$ are drawn to meet the second turn of the spiral, while $AZ, AH$ are drawn, as before, to meet the circumference of the first circle, then

\[ AE : A\Delta = \text{arc } \Theta KZ + \text{circumference of first circle} : \text{arc } \Theta KH + \text{circumference of first circle}, \]

and so on for higher turns.

In general, if $E, \Delta$ lie on the $n$th turn of the spiral, and the circumference of the first circle is $c$, then

\[ AE : A\Delta = \text{arc } \Theta KZ + n - 1c : \text{arc } \Theta KH + n - 1c. \]

These theorems correspond to the equation of the curve $r = a\theta$ in polar co-ordinates.

\* This theorem is essential to the one that follows.

\* See n. a on this page.

\(^1\) δέδεικται...πρώτοις om. Heiberg.
κέντρου ποτιβαλεῖν ποτὶ τὰν ἐκβεβλημέναν, ὥστε τὰν μεταξὺ τᾶς περιφερείας καὶ τὰς ἐκβεβλημένας ποτὶ τὰν ἐπιζευγθείσαν ἀπὸ τοῦ πέρατος τὰς ἐναπολαθείσας ποτὶ τὸ πέρας τὰς ἐκβεβλημένας τὸν ταχθέντα λόγου ἔχευν, εἰ καὶ ὃ δοθεὶς λόγος μεῖζων ἢ τοῦ, ὃν ἔχει ἀ ἡμίσεια τᾶς ἑν τῷ κύκλῳ δεδομένας ποτὶ τὰν ἀπὸ τοῦ κέντρου καθετον ἐπ᾽ αὐτῶν ἀγμέναν.

Δεδόσθω τὰ αὐτά, καὶ ἔστω ἄ ἐν τῷ κύκλῳ γραμμά ἐκβεβλημένα, ὃ δὲ δοθεὶς λόγος ἔστω, ὃν ἔχει ἄ Ζ ποτὶ τὰν Ἁ, μεῖζων τοῦ, ὃν ἔχει ἄ ΓΘ ποτὶ τὰν ΘΚ. μεῖζων οὖν ἐσσεῖται καὶ τοῦ, ὃν ἔχει ἄ ΚΓ ποτὶ ΓΛ. ὃν ἡ λόγου ἔχει ἄ Ζ ποτὶ Ἁ τοῦτον ἐξεί ἄ ΚΓ ποτὶ ἐλάσσονα τὰς ΓΛ. ἐχέσων ποτὶ IN. νεύοσαν ἐπὶ τὸ Γ—δυνατόν δὲ ἐστιν οὖτως τέμνει—καὶ πεσεῖται ἐντὸς τὰς ΓΛ, ἐπειδὴ ἐλάσσον ἐστὶ τὰς ΓΛ. ἐπεὶ οὖν τὸν αὐτὸν ἔχει λόγον ἄ ΚΓ ποτὶ IN, ὃν ἄ Ζ ποτὶ Ἁ, καὶ ἄ ΕΙ ποτὶ ΙΓ τὸν αὐτὸν ἐξεὶ λόγον, ὃν ἄ Ζ ποτὶ τὰν Ἁ.

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* ΑΓ is a chord in a circle of centre K, and BN is the diameter drawn parallel to AG and produced. From K, KΘ is drawn perpendicular to AG, and ΓΛ is drawn perpendicular to KG so as to meet the diameter in Λ. Archimedes asserts that it is possible to draw KE to meet the circle in I and AG produced in E so that EI : IG = Z : H, an assigned ratio, provided that Z : H > ΓΘ : ΘΚ. The straight line IG meets BA in N. In Prop. 5 Archimedes has proved a similar proposition when AG is a tangent, and in Prop. 6 he has proved the proposition for the case where the positions of I, Γ are reversed.

* For triangle ΓΕΙ is similar to triangle KIN, and therefore ΚΙ : IN = EI : IG [Eucl. vi. 4]; and ΚΙ = KG.

* The type of problem known as νεόσεις, vergings, has already been encountered (vol. i. p. 244 n. a). In this proposition, as in Props. 5 and 6, Archimedes gives no hint how
the produced chord so that the distance between the circumference and the produced chord shall bear to the distance between the extremity of the line intercepted [by the circle] and the extremity of the produced chord an assigned ratio, provided that the given ratio is greater than that which half of the given chord in the circle bears to the perpendicular drawn to it from the centre.

Let the same things be given, and let the chord in the circle be produced, and let the given ratio be \( Z : H \), and let it be greater than \( \Gamma \Theta : \Theta \Gamma \); therefore it will be greater than \( \Gamma \Gamma : \Gamma \Lambda \) [Eucl. vi. 4]. Then \( Z : H \) is equal to the ratio of \( \Gamma \Gamma \) to some line less than \( \Gamma \Lambda \) [Eucl. v. 10]. Let it be to \( IN \) verging upon \( \Gamma \)—for it is possible to make such an intercept—and \( IN \) will fall within \( \Gamma \Lambda \), since it is less than \( \Gamma \Lambda \). Then since \( \Gamma \Gamma : IN = Z : H \), therefore \( EI : II = Z : H \).

the construction is to be accomplished, though he was presumably familiar with a solution.

In the figure of the text, let \( T \) be the foot of the perpen-
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(iv.) Property of the Subtangent

Ibid., Prop. 20, Archim. ed. Heiberg ii. 72. 4–74. 26

Εἰ κα τάς ἐλικος τάς ἐν τά πρώτα περιφορά γεγραμμένας εὐθεία γραμμά ἐπισφαύη μὴ κατὰ τό πέρας τάς ἐλικος, ἀπὸ δὲ τάς ἀφᾶς ἐπὶ τάν ἀρχὰν τάς ἐλικος εὐθεία ἐπιζευγθῆ, καὶ κέντρῳ μὲν τά ἀρχὰ τάς ἐλικος, διαστήματι δὲ τά ἐπιζευγθείσα κύκλος γραφῆ, ἀπὸ δὲ τάς ἀρχὰς τάς ἐλικος ἀχθῇ τις ποτ' ὀρθὰς τά ἀπὸ τάς ἀφᾶς ἐπὶ τάν ἀρχὰν τάς ἐλικος ἐπιζευγθείσα, συμπεσείται αὕτα ποτὶ τάν ἐπισφαύοντασ, καὶ ἐσσεῖται ἀ μεταξύ εὐθεία τᾶς τε συμπτώσιος καὶ τάς ἀρχὰς τάς ἐλικος ἵσα τά περιφερεία τοῦ γραφέντος κύκλου τά μεταξύ τᾶς ἀφᾶς καὶ τᾶς τομᾶς, καθ' ἀν τέμνει ὁ γραφεὶς κύκλος τάν ἀρχὰν τάς περιφοράς, ἐπὶ τά προαγούμενα λαμβανομένας τάς περιφερείας ἀπὸ τοῦ σαμείου τοῦ ἐν τά ἀρχὰ τάς περιφορᾶς.

"Εστω ἐλιξ, ἐφ' ἂς ἄ ΑΒΓΔ, ἐν τά πρώτα περιφορά γεγραμμένα, καὶ ἐπισφαύετω τις αὐτάς εὐθεία ἄ ΕΖ κατὰ τό Δ, ἀπὸ δὲ τοῦ 'Δ ποτὶ τάν
dicular from Γ to ΒΛ, and let Δ be the other extremity of the diameter through B. Let the unknown length KN=a, let ΠΤ=a, KT=b, ΒΔ=2c, and let IN=k, a given length.

Then

\[ \text{NI} \cdot \text{NI} = \text{NA} \cdot \text{NB}, \]

i.e.,

\[ k\sqrt{a^2+(x-b)^2}=(x-c)(x+c), \]

which, after rationalization, is an equation of the fourth degree in \( x \).

Alternatively, if we denote \( \text{NI} \) by \( y \), we can determine \( x \) and \( y \) by the two equations

\[ y^2=a^2+(x-b)^2, \]

\[ ky=x^2-c^2, \]
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(iv.) Property of the Subtangent

Ibid., Prop. 20, Archim. ed. Heiberg ii. 72. 4-74. 26

If a straight line touch the first turn of the spiral other than at the extremity of the spiral, and from the point of contact a straight line be drawn to the origin, and with the origin as centre and this connecting line as radius a circle be drawn, and from the origin a straight line be drawn at right angles to the straight line joining the point of contact to the origin, it will meet the tangent, and the straight line between the point of meeting and the origin will be equal to the arc of the circle between the point of contact and the point in which the circle cuts the initial line, the arc being measured in the forward direction from the point on the initial line.

Let $ABFG$ lie on the first turn of a spiral, and let

the straight line $EZ$ touch it at $\Delta$, and from $\Delta$ let $\Delta \Delta$

so that values of $x$ and $y$ satisfying the conditions of the problem are given by the points of intersection of a certain parabola and a certain hyperbola.

The whole question of vergings, including this problem, is admirably discussed by Heath, *The Works of Archimedes*, c-cxxii.
For in Prop. 16 the angle $\Delta Z$ was shown to be acute.

For $\Delta N$ touches the spiral and so can have no part within the spiral, and therefore cannot pass through $A$; therefore it is a chord of the circle and less than the diameter.

For, if a perpendicular be drawn from $A$ to $\Delta N$, it bisects
be drawn to the origin, and with centre A and radius \(A\Delta\) let the circle \(\Delta MN\) be described, and let this circle cut the initial line at \(K\), and let \(ZA\) be drawn at right angles to \(A\Delta\). That it will meet \([Z\Delta]\) is clear; it is required to prove that the straight line \(ZA\) is equal to the arc \(KM\Delta\).

If not, it is either greater or less. Let it first be, if possible, greater, and let \(\Delta A\) be taken less than the straight line \(ZA\), but greater than the arc \(KM\Delta\) [Prop. 4]. Again, \(KMN\) is a circle, and in this circle \(\Delta N\) is a line less than the diameter, and the ratio \(\Delta A : AA\) is greater than the ratio of half \(\Delta N\) to the perpendicular drawn to it from \(A\); it is therefore possible to draw from \(A\) a straight line \(AE\) meeting \(\Delta \Delta\) produced in such a way that

\[
EP : \Delta P = \Delta A : AA;
\]

for this has been proved possible [Prop. 7]; therefore

\[
EP : AP = \Delta P : AA.\]

But \(\Delta P : AA < \text{arc } \Delta P : \text{arc } KM\Delta\),

since \(\Delta P\) is less than the arc \(\Delta P\), and \(AA\) is greater than the arc \(KM\Delta\);

\[
\therefore \quad EP : PA < \text{arc } \Delta P : \text{arc } KM\Delta;
\]

\[
\therefore \quad AE : AP < \text{arc } KMP : \text{arc } KM\Delta.
\]

[Eucl. v. 18]

\(\Delta N\) [Eucl. iii. 3] and divides triangle \(\Delta AZ\) into two triangles of which one is similar to triangle \(\Delta AZ\) [Eucl. vi. 8]; therefore

\[\Delta A : AZ = \frac{1}{2}N\Delta : (\text{perpendicular from } A \text{ to } N\Delta).\]

[Eucl. vi. 4]

But \(AZ > AA\);

\[\therefore \quad \Delta A : AA > \frac{1}{2}N\Delta : (\text{perpendicular from } A \text{ to } N\Delta).\]

For \(\Delta A = AP\), being a radius of the same circle; and the proportion follows permutando.
This part of the proof involves a *verging* assumed in Prop. 8, just as the earlier part assumed the *verging* of Prop. 7. The *verging* of Prop. 8 has already been described (vol. i. p. 350 n. b) in connexion with Pappus’s comments on it.

Archimedes goes on to show that the theorem is true even if the tangent touches the spiral in its second or some higher turn, not at the extremity of the turn; and in Props. 18 and 19 he has shown that the theorem is true if the tangent should touch at an extremity of a turn.

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Now arc KMP : arc KMA = AX : AΔ; [Prop. 14]

∴ EA : AP < AX : AΔ;

which is impossible. Therefore ZA is not greater than the arc KMA. In the same way as above it may be shown to be not less⁹; therefore it is equal.⁹

(f) SEMI-REGULAR SOLIDS

Pappus, Collection v. 19, ed. Hultsch i. 352. 7-354. 10

Although many solid figures having all kinds of surfaces can be conceived, those which appear to be regularly formed are most deserving of attention. Those include not only the five figures found in the godlike Plato, that is, the tetrahedron and the cube, the octahedron and the dodecahedron, and fifthly the icosaahedron,⁹ but also the solids, thirteen in number, which were discovered by Archimedes and are contained by equilateral and equiangular, but not similar, polygons.

As Pappus (ed. Hultsch 302. 14-18) notes, the theorem can be established without recourse to propositions involving solid loci (for the meaning of which see vol. i. pp. 348-349), and proofs involving only "plane" methods have been developed by Tannery, Mémoires scientifiques, i., 1912, pp. 300-316 and Heath, H.G.M. ii. 556-561. It must remain a puzzle why Archimedes chose his particular method of proof, especially as Heath’s proof is suggested by the figures of Props. 6 and 9; Heath (loc. cit., p. 557) says "it is scarcely possible to assign any reason except his definite predilection for the form of proof by reductio ad absurdum based ultimately on his famous ‘Lemma’ or Axiom."

⁹ For the five regular solids, see vol. i. pp. 216-225.

⁹ Heron (Definitions 104, ed. Heiberg 66. 1-9) asserts that two were known to Plato. One is that described as P₂ below, but the other, said to be bounded by eight squares and six triangles, is wrongly given.
For the purposes of n. b, the thirteen polyhedra will be designated as $P_1, P_2 \ldots P_{13}$.

Kepler, in his Harmonice mundi (Opera, 1864, v. 123-126), appears to have been the first to examine these figures systematically, though a method of obtaining some is given in a scholium to the Vatican ms. of Pappus. If a solid angle of a regular solid be cut by a plane so that the same length is cut off from each of the edges meeting at the solid angle, 196
The first is a figure of eight bases, being contained by four triangles and four hexagons \([P_1]\).^{a}

After this come three figures of fourteen bases, the first contained by eight triangles and six squares \([P_2]\), the second by six squares and eight hexagons \([P_3]\), and the third by eight triangles and six octagons \([P_4]\).

After these come two figures of twenty-six bases, the first contained by eight triangles and eighteen squares \([P_5]\), the second by twelve squares, eight hexagons and six octagons \([P_6]\).

After these come three figures of thirty-two bases, the first contained by twenty triangles and twelve pentagons \([P_7]\), the second by twelve pentagons and twenty hexagons \([P_8]\), and the third by twenty triangles and twelve decagons \([P_9]\).

After these comes one figure of thirty-eight bases, being contained by thirty-two triangles and six squares \([P_{10}]\).

After this come two figures of sixty-two bases, the first contained by twenty triangles, thirty squares and twelve pentagons \([P_{11}]\), the second by thirty squares, twenty hexagons and twelve decagons \([P_{12}]\).

After these there comes lastly a figure of ninety-two bases, which is contained by eighty triangles and twelve pentagons \([P_{13}]\).^{b}

the section is a regular polygon which is a triangle, square or pentagon according as the solid angle is composed of three, four or five plane angles. If certain equal lengths be cut off in this way from all the solid angles, regular polygons will also be left in the faces of the solid. This happens (i) obviously when the cutting planes bisect the edges of the solid, and (ii) when the cutting planes cut off a smaller length from each edge in such a way that a regular polygon is left in each face with double the number of sides. This method gives (1) from the tetrahedron, \(P_1\); (2) from the
(g) **System of expressing Large Numbers**

Archim. *Aren. 3*, Archim. ed. Heiberg ii. 236. 17–240. 1

"Α μὲν οὖν υποτιθεμαι, ταῦτα· χρήσιμον δὲ εἰμεν ὑπολαμβάνω τὰν κατονόμαξιν τῶν ἀριθμῶν ῥήθημεν, ὅπως καὶ τῶν ἄλλων οἱ τῷ βιβλίῳ μὴ περιτετευχότες τῷ ποτὶ Ζεύξιππον γεγραμμένῳ μὴ πλανώνται διὰ τὸ μηδὲν εἰμεν ὑπὲρ αὐτὸς ἐν τῷ τῷ βιβλίῳ προειρημένον. συμβαίνει δὴ τὰ ὀνόματα τῶν ἀριθμῶν ἐς τὸ μὲν τῶν μυρίων ὑπάρχειν ἀμῖν παραδεδομένα, καὶ ὑπὲρ τὸ τῶν μυρίων [μὲν] ἀποχρεότως γιγνώσκομεν μυριάδων ἀριθμῶν λέγοντες ἐστε ποτὶ τὰς μυρίας μυριάδας. ἐστων οὖν ἀμῖν οἱ μὲν νῦν εἰρημένοι ἀριθμοὶ ἐς τὰς μυρίας μυριάδας πρῶτοι καλουμένοι, τῶν δὲ πρώτων ἀριθμῶν αἱ μύριαι μυριάδες μονὰς καλείσθω δευτέρων ἀριθμῶν, καὶ ἀριθμεῖσθων τῶν δευτέρων μονάδες καὶ ἐκ τῶν μονάδων δεκάδες καὶ ἐκατοντάδες καὶ χιλιάδες καὶ μυριάδες ἐς τὰς μυρίας μυριάδας. πάλιν δὲ καὶ αἱ μύριαι μυριάδες τῶν δευτέρων ἀριθμῶν μονὰς καλείσθω τρίτων ἀριθμῶν, καὶ ἀριθμεῖσθων τῶν τρίτων ἀριθμῶν μονάδες καὶ ἀπὸ τῶν μονάδων δεκάδες καὶ ἐκατοντάδες καὶ χιλιάδες καὶ μυριάδες ἐς τὰς μυρίας μυριάδας.

Τῶν αὐτῶν δὲ τρόπον καὶ τῶν τρίτων ἀριθμῶν μύριαι μυριάδες μονὰς καλείσθω τετάρτων ἀριθμῶν, 1 μὲν om. Heiberg.

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cube, \( P_2 \) and \( P_4 \); (3) from the octahedron, \( P_2 \) and \( P_5 \); (4) from the icosahedron, \( P_7 \) and \( P_8 \); (5) from the dodecahedron, \( P_7 \) and \( P_9 \). It was probably the method used by Plato.

Four more of the semi-regular solids are obtained by first cutting all the edges symmetrically and equally by planes parallel to the edges, and then cutting off angles. This
Such are then the assumptions I make; but I think it would be useful to explain the naming of the numbers, in order that, as in other matters, those who have not come across the book sent to Zeuxippus may not find themselves in difficulty through the fact that there had been no preliminary discussion of it in this book. Now we already have names for the numbers up to a myriad \([10^4]\), and beyond a myriad we can count in myriads up to a myriad myriads \([10^8]\). Therefore, let the aforesaid numbers up to a myriad myriads be called *numbers of the first order* [numbers from 1 to \(10^8\)], and let a myriad myriads of numbers of the first order be called a unit of *numbers of the second order* [numbers from \(10^8\) to \(10^{16}\)], and let units of the numbers of the second order be enumerable, and out of the units let there be formed tens and hundreds and thousands and myriads up to a myriad myriads. Again, let a myriad myriads of numbers of the second order be called a unit of *numbers of the third order* [numbers from \(10^{16}\) to \(10^{24}\)], and let units of numbers of the third order be enumerable, and from the units let there be formed tens and hundreds and thousands and myriads up to a myriad myriads. In the same manner, let a myriad myriads of numbers of the third order be gives (1) from the cube, \(P_5\) and \(P_6\); (2) from the icosahedron, \(P_{11}\); (3) from the dodecahedron, \(P_{12}\).

The two remaining solids are more difficult to obtain; \(P_{10}\) is the *snub cube* in which each solid angle is formed by the angles of four equilateral triangles and one square; \(P_{13}\) is the *snub dodecahedron* in which each solid angle is formed by the angles of four equilateral triangles and one regular pentagon.
Expressed in full, the last number would be 1 followed by 80,000 million millions of ciphers. Archimedes uses this system to show that it is more than sufficient to express the number of grains of sand which it would take to fill the universe, basing his argument on estimates by astronomers of the sizes and distances of the sun and moon and their relation to the size of the universe and allowing a wide margin for safety. Assuming that a poppy-head (for so μῆκων is here to be understood, not “poppy-seed,” v. D’Arcy W. Thompson, The Classical Review, lvi. (1942), p. 75) would contain not more than 10,000 grains of sand, and that its diameter is not less than a finger’s breadth, and having proved that the
called a unit of numbers of the fourth order [numbers from $10^{24}$ to $10^{32}$], and let a myriad myriads of numbers of the fourth order be called a unit of numbers of the fifth order [numbers from $10^{32}$ to $10^{40}$], and let the process continue in this way until the designations reach a myriad myriads taken a myriad myriad times $[10^8 \cdot 10^8]$.

It is sufficient to know the numbers up to this point, but we may go beyond it. For let the numbers now mentioned be called numbers of the first period [$1$ to $10^8 \cdot 10^8$], and let the last number of the first period be called a unit of numbers of the first order of the second period $[10^8 \cdot 10^8$ to $10^8 \cdot 10^8 \cdot 10^8]$. And again, let a myriad myriads of numbers of the first order of the second period be called a unit of numbers of the second order of the second period $[10^8 \cdot 10^8 \cdot 10^8$ to $10^8 \cdot 10^8 \cdot 10^8 \cdot 10^8]$. Similarly let the last of these numbers be called a unit of numbers of the third order of the second period $[10^8 \cdot 10^8 \cdot 10^{16}$ to $10^8 \cdot 10^8 \cdot 10^8 \cdot 10^{24}]$, and let the process continue in this way until the designations of numbers in the second period reach a myriad myriads taken a myriad myriad times $[10^8 \cdot 10^8 \cdot 10^8 \cdot 10^8$, or $10^8 \cdot 10^8]^2]$. Again, let the last number of the second period be called a unit of numbers of the first order of the third period $[(10^3 \cdot 10^8)^2$ to $(10^8 \cdot 10^8)^3 \cdot 10^8]$, and let the process continue in this way up to a myriad myriad units of numbers of the myriad myriadth order of the myriad myriadth period $[(10^8 \cdot 10^8)^{10^8}$ or $10^8 \cdot 10^{16}]$.

A sphere of the fixed stars is less than $10^7$ times the sphere in which the sun’s orbit is a great circle, Archimedes shows that the number of grains of sand which would fill the universe is less than “10,000,000 units of the eighth order of numbers,” or $10^{68}$. The work contains several references important for the history of astronomy.
GREEK MATHEMATICS

(h) Indeterminate Analysis: The Cattle Problem

ii. 528. 1–532. 9

Πρόβλημα

ὅπερ Ἀρχιμήδης ἐν ἐπιγράμμασιν εὑρὼν τοῖς ἐν Ἀλεξάνδρείᾳ περὶ ταῦτα πραγματευομένους ζητεῖν ἀπέστειλεν ἐν τῇ πρὸς Ἑρατοσθένην τὸν Κυρηναίου ἐπιστολῇ.

Πληθὺν Ἦλειον βοῶν, ὃ ξεῖνε, μέτρησον φροντίδ' ἐπιστήσας, εἰ μετέχεις σοφίας, πόση ἀρ' ἐν πεδίοις Σικελίης ποτ' ἐβόσκετο νήσου Θριγκίης τετραχῇ στίφεα δασσαμένη χροῆν ἀλλάσσοντα: τὸ μὲν λευκοῖο γάλακτος,

κυανέως δ' ἔτερον χρώματι λαμψόμενον, ἀλλὰ γε μὲν ξανθόν, τὸ δὲ πουκίλον. ἐν δὲ ἐκάστῳ στίφει ἔσσαν ταῦροι πλήθει βριθόμενοι συμμετρίς τοιησδέ τετευχότες: ἀργότριχας μὲν κυανέων ταύρων ἡμίσει ἢδὲ τρίτῳ καὶ ξανθοῖς σύμπασιν ἱσσου, ὃ ξεῖνε, νόσσουν,

αὐτὰρ κυανέους τῷ τετράτῳ τε μέρει μικτοχρόνων καὶ πέμπτῳ, ἕτη ξανθοῖς τε πάσιν.

τοὺς δ' ὑπολειπομένους πουκιλόχρωτας ἀθρεὶ ἀργεννων ταύρων ἐκτὸς μέρει ἐβδομάτω τε καὶ ξανθοῖς αὐτοὺς πάσιν ἵσαζομένους. θηλείαςι δὲ βοουὶ τάδ' ἐπλετο: λευκότριχες μὲν ἥσαν συμπάσις κυανής ἀγέλης τῷ τριτάτῳ τε μέρει καὶ τετράτω ἀτρεκές ἵσαι· αὐτὰρ κυάνεαι τῷ τετράτῳ τε πάλιν μικτοχρώνων καὶ πέμπτῳ ὁμοὶ μέρει ἵσαζοντο σὺν ταύροις πάσαις εἰς νομὸν ἐρχομέναις.

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(h) Indeterminate Analysis: The Cattle Problem

Archimedes (?), Cattle Problem,* Archim. ed. Heiberg ii. 528. 1-532. 9

A Problem

which Archimedes solved in epigrams, and which he communicated to students of such matters at Alexandria in a letter to Eratosthenes of Cyrene.

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, the third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now

* It is unlikely that the epigram itself, first edited by G. E. Lessing in 1773, is the work of Archimedes, but there is ample evidence from antiquity that he studied the actual problem. The most important papers bearing on the subject have already been mentioned (vol. i. p. 16 n. c), and further references to the literature are given by Heiberg ad loc.
GREEK MATHEMATICS

ξανθοτρίχων δ' ἁγέλης πέμπτω μέρει ἢδε καὶ ἐκτω πουκίλαι ἵσαριθμον πλήθος ἔχον τετραχῆ. 
ξανθαι δ' ἠριθμεύντο μέρους τρίτου ἡμίσει ίσαι ἀργενὴς ἁγέλης ἐβδομάτω τε μέρει.
ζείνε, συ δ', 'Ἡλιοίο βοῖς πόσαι, ἀτρεκές εἰπὼν, 
χωρίς μὲν ταύρων ζατρεφέων ἀριθμόν, 
χωρίς δ' αὖ, θῆλεια ὅσαι κατὰ χροῖν ἐκασται, 
οὐκ ἁίδρει κε λέγοι οὖδ' ἀριθμῶν ἀδαίσ, 
οὐ μὴν πὼ γε σοφοῖς ἐναρίθμοι. ἀλλ' ἰθι φράζευ καὶ τάδε πάντα βοῶν 'Ηνλίοιο πάθη.
ἀργότριχες ταύροι μὲν ἐπεὶ μιξαίατο πληθὺν 
κυανεός, ἵσταντ' ἐμπεδὸν ἴσημετροι 
eἰς βάθος εἰς εὐρός τε, τὰ δ' αὖ περιμείκεια πάντη 
πύμπλαντο πλήθος.1 Θρινακίης πεδία.
ξανθοὶ δ' αὖτ' εἰς ἐν καὶ πουκίλοι ἀθροισθέντες 
ἵσταντ' ἀμβολάδην ἐξ ἐνὸς ἀρχόμενοι 
σχήμα τελεοῦντες τὸ τρικάσπεδον οὔτε προσόντων ἀλλοχρῶν ταύρων οὔτ' ἐπιλειπομένων.
ταῦτα συνεξευρόν καὶ ἐνὶ πρατίδεσσιν ἄθροίσα 
καὶ πληθέων ἀποδοὺς, ζείνε, τὰ πάντα μέτρα 
ἐρχεο κυδιῶν νυκτὸρος ἵσθι τε πάντως 
κεκριμένος ταύτη γ' ὀμπνίος ἐν σοφίῃ.

1 πλήθος Krumbiegel, πλύθον cod.

* i.e. a fifth and a sixth both of the males and of the females.
* At a first glance this would appear to mean that the sum of the number of white and black bulls is a square, but this makes the solution of the problem intolerably difficult. There is, however, an easier interpretation. If the bulls are packed together so as to form a square figure, their number need not be a square, since each bull is longer than it is broad. The simplified condition is that the sum of the number of white and black bulls shall be a rectangle.
the dappled in four parts a were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise. But come, understand also all these conditions regarding the cows of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, b and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom. c

* If

\[ X, x \] are the numbers of white bulls and cows respectively, \\
\[ Y, y \] " " " black \\
\[ Z, z \] " " " yellow \\
\[ W, w \] " " " dappled

the first part of the epigram states that

(a) \[ X = (\frac{1}{4} + \frac{1}{4}) Y + Z \] . . . . (1) \\
\[ Y = (\frac{1}{4} + \frac{1}{4}) W + Z \] . . . . (2) \\
\[ W = (\frac{1}{4} + \frac{1}{4}) X + Z \] . . . . (3)
GREEK MATHEMATICS

(i) Mechanics: Centres of Gravity

(i.) Postulates

Heiberg ii. 124. 3–126. 3

α'. Αιτούμεθα τά ἵσα βάρεα ἀπὸ ἰσων μακέων ἱσορροπεῖν, τά δὲ ἰσα βάρεα ἀπὸ τῶν ἀνίσων μακέων μὴ ἱσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος τὸ ἀπὸ τοῦ μείζονος μάκεος.

(b)

\[ x = (\frac{1}{2} + \frac{1}{2})(Y + y) \]  
\[ y = (\frac{1}{2} + \frac{1}{2})(W + w) \]  
\[ w = (\frac{1}{2} + \frac{1}{2})(Z + z) \]  
\[ z = (\frac{1}{2} + \frac{1}{2})(X + x) \]

The second part of the epigram states that

\[ X + Y = \text{a rectangular number} \]  
\[ Z + W = \text{a triangular number} \]

This was solved by J. F. Wurm, and the solution is given by A. Amthor, Zeitschrift für Math. u. Physik. (Ilist.-litt. Abtheilung), xxv. (1880), pp. 153-171, and by Heath, The Works of Archimedes, pp. 319-326. For reasons of space, only the results can be noted here.

Equations (1) to (7) give the following as the values of the unknowns in terms of an unknown integer \( n \):

\[ X = 10366482n \]  
\[ Y = 7460514n \]  
\[ Z = 4149387n \]  
\[ W = 7358060n \]

\[ x = 7206360n \]  
\[ y = 4893246n \]  
\[ z = 5439213n \]  
\[ w = 3515820n \]

We have now to find a value of \( n \) such that equation (9) is also satisfied—equation (8) will then be simultaneously satisfied. Equation (9) means that

\[ Z + W = \frac{p(p + 1)}{2}, \]

where \( p \) is some positive integer, or

\[ (4149387 + 7358060)n = \frac{p(p + 1)}{2} \]

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ARCHIMEDES

(i) Mechanics: Centres of Gravity

(i.) Postulates


1. I postulate that equal weights at equal distances balance, and equal weights at unequal distances do not balance, but incline towards the weight which is at the greater distance.

\[ \begin{align*}
2471 \cdot 4657n &= \frac{p(p + 1)}{2}.
\end{align*} \]

This is found to be satisfied by

\[ n = 3^3 \cdot 4349, \]

and the final solution is

\[ \begin{align*}
X &= 1217263415886 \\
Y &= 876035935422 \\
Z &= 487233469701 \\
W &= 864005479380
\end{align*} \]

and the total is 5916837175686.

If equation (8) is taken to be that \( X + Y = \) a square number, the solution is much more arduous; Amthor found that in this case,

\[ W = 1598 \langle 206541 \rangle, \]

where \( \langle 206541 \rangle \) means that there are 206541 more digits to follow, and the whole number of cattle = 7766 \( \langle 206541 \rangle \).

Merely to write out the eight numbers, Amthor calculates, would require a volume of 660 pages, so we may reasonably doubt whether the problem was really framed in this more difficult form, or, if it were, whether Archimedes solved it.

* This is the earliest surviving treatise on mechanics; it presumably had predecessors, but we may doubt whether mechanics had previously been developed by rigorous geometrical principles from a small number of assumptions. References to the principle of the lever and the parallelogram of velocities in the Aristotelian Mechanics have already been given (vol. i. pp. 430-433).
GREEK MATHEMATICS

β'. eι κα βαρέων ἰσορροπεύοντον ἀπὸ τίων μακέων ποτὶ τὸ ἐτερον τῶν βαρέων ποτιτεθῇ, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος ἐκεῖνο, ὃ ποτετέθη.

γ'. 'Ομοίως δὲ καὶ, εἰ κα ἀπὸ τοῦ ἐτεροῦ τῶν βαρέων ἀφαίρεθη τι, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος, ἀφ’ οὐ οὐκ ἀφηρέθη.

δ'. Τῶν ἵσων καὶ ὁμοίων σχημάτων ἐπιπέδων ἐφαρμοζόμενων ἐπ’ ἀλλαλα καὶ τὰ κέντρα τῶν βαρέων ἐφαρμόζει ἐπ’ ἀλλαλα.

ε’. Τῶν δὲ ἀνίσων, ὁμοίων δὲ, τὰ κέντρα τῶν βαρέων ὁμοίως ἔσσεῖται κείμενα. ὁμοίως δὲ λέγομες σαμεία κέσσαι ποτὶ τὰ ὁμοία σχῆματα, ἀφ’ ὃν ἐπὶ τὰς ἴσας γωνίας ἀγόμεναι εὔθειαι ποιέοντι γωνίας ἴσας ποτὶ τὰς ὁμολόγους πλευράς.

ζ’. Εἰ κα μεγέθεα ἀπὸ τίων μακέων ἰσορροπεύοντι, καὶ τὰ ἴσα αὐτοῖς ἀπὸ τῶν αὐτῶν μακέων ἰσορροπήσει.

ζ’. Πάντως σχῆματος, οὐ κα ἀ περίμετρος ἐπὶ τὰ αὐτὰ κοίλα ᾗ, τὸ κέντρον τοῦ βάρεος ἐντὸς ἐλμὲν δεὶ τοῦ σχῆματος.

(ii.) Principle of the Lever

Ibid., Props. 6 et 7, Archim. ed. Heiberg ii. 132. 13–138. 8

ζ' Τὰ σύμμετρα μεγέθεα ἰσορροπεύοντι ἀπὸ μακέων ἀντιπεπονθότως τῶν αὐτῶν λόγον ἐχόντων τοῖς βάρεσιν.

'Εστω σύμμετρα μεγέθεα τὰ Α, Β, ὅν κέντρα τὰ Α, Β, καὶ μάκος ἔστω τι τὸ ΕΔ, καὶ ἔστω, ὡς τὸ Α ποτὶ τὸ Β, οὕτως τὸ ΔΓ μάκος ποτὶ τὸ ΓΕ 208
2. If weights at certain distances balance, and something is added to one of the weights, they will not remain in equilibrium, but will incline towards that weight to which the addition was made.

3. Similarly, if anything be taken away from one of the weights, they will not remain in equilibrium, but will incline towards the weight from which nothing was subtracted.

4. When equal and similar plane figures are applied one to the other, their centres of gravity also coincide.

5. In unequal but similar figures, the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures, I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.

6. If magnitudes at certain distances balance, magnitudes equal to them will also balance at the same distances.

7. In any figure whose perimeter is concave in the same direction, the centre of gravity must be within the figure.

(ii.) Principle of the Lever

_Ibid., Props. 6 and 7, Archim. ed. Heiberg_ 
_ii. 132. 13–138. 8_

Prop. 6

_Commensurable magnitudes balance at distances reciprocally proportional to their weights._

Let A, B be commensurable magnitudes with centres [of gravity] A, B, and let EΔ be any distance, and let $A : B = \Delta \Gamma : \Gamma E$;
μάκος· δεικτέον, ὅτι τοῦ ἐξ ἀμφοτέρων τῶν Α, Β
συγκεκριμένου μεγέθεος κέντρον ἔστι τοῦ βάρεος
tοῦ Γ.

Επει γάρ ἐστιν, ὡς τὸ Α ποτὶ τὸ Β, οὔτως τὸ
ΔΓ ποτὶ τὸ ΓΕ, τὸ δὲ Α τῷ Β σύμμετρον, καὶ
tὸ ΓΔ ἀρα τῷ ΓΕ σύμμετρον, τοντέστων εὐθεία
tὰ εὐθεία· ὡστε τῶν ΕΓ, ΓΔ ἐστὶ κοινὸν μέτρον.
ἔστω δὴ τὸ Ν, καὶ κείσθω τὰ μὲν ΕΓ ἵσα ἐκάτερα
tῶν ΔΗ, ΔΚ, τὰ δὲ ΔΓ ἵσα ἐκ ΕΛ. καὶ ἐπεὶ ἵσα

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it is required to prove that the centre of gravity of
the magnitude composed of both A, B is $\Gamma$.

Since $A : B = \Delta \Gamma : \Gamma E$,
and A is commensurate with B, therefore $\Gamma \Delta$ is com-
mensurate with $\Gamma E$, that is, a straight line with a
straight line [Eucl. x. 11]; so that $\Gamma \Gamma$, $\Gamma \Delta$ have a
common measure. Let it be N, and let $\Delta H$, $\Delta K$ be
each equal to $\Gamma \Gamma$, and let $\Gamma \Lambda$ be equal to $\Delta \Gamma$. Then
since $\Delta H = \Gamma E$, it follows that $\Delta \Gamma = \Gamma \Gamma$; so that
$\Lambda \varepsilon = H$. Therefore $\Lambda H = 2 \Delta \Gamma$ and $HK = 2 \Gamma E$; so
that N measures both $\Lambda H$ and $HK$, since it measures
their halves [Eucl. x. 12]. And since

$$A : B = \Delta \Gamma : \Gamma E,$$

while

$$\Delta \Gamma : \Gamma E = \Lambda H : HK,$$

for each is double of the other—

therefore

$$A : B = \Lambda H : HK.$$

Now let Z be the same part of A as N is of $\Lambda H$;
GREEK MATHEMATICS

taplaioin este to A to Z. estoin arra, os a LH poti N, outws to A poti Z. esto de kai, os a KH poti LH, outws to B poti A. di issou arra estoin, os a KH poti N, outws to B poti Z. isakis arra polllaaplaioin estoin a KH tais N kai to B to Z. edeichthe de to Z kai to A polla-
plasiou eon. woste to Z twn A, B koinon esto metron. diairetheias oiv tais men LH eis tais ta N ous, tou de A eis ta Z ous, ta en ta LH tmamata isomegheia ta N isa esseitai tw plithei
tois en tw A tmamateson isois eouson tw Z. woste, an ef ekastou twon tmamaton twon en ta
LH epitebh megebos ison tw Z to kentrou tou baresos exon epit mesou tou tmaamatos, ta te pantta
megethea isa enti tw A, kai tou ek panton sunkei-
mvenou kentrou esseitai tou baresos to E. artia
te gar esto ta pantta tw plithei, kai ta ef ekatera
tou E isa tw plithei dia to ousan emen twon LE
ta HE.

'Omoiws de deichthesetai, oti kain, eik ka ef'
kekastou twon en ta KH tmamaton epitebh megebos
ison tw Z kentrou tou baresos exon epit tou mesou
tou tmamatos, ta te pantta megethea isa esseitai
tw B, kai tou ek panton sunkeimenvon kentrou tou
baresos esseitai to D. esseitai oiv to men A episkei-
mevon kata to E, to de B kata to D. esseitai
di megeheia isa allalouss eiv eutheias keimeva, oiv
ta kentra tou baresos isa apt allalovn dieostaken,
[sunkeimeva]i artia tw plithei. deallon oiv, oti tou
ek panton sunkeimenvon megebos kentrou esto to
baresos a dihotomia tas eivtheias tas ehoosas ta
kentra ton meson megebedon. epei d' ousai enti
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then $\Delta H : N = A : Z$.  [Eucl. v., Def. 5]

And $KH : \Delta H = B : A$;  [Eucl. v. 7, coroll.]

therefore, *ex aequo,*

$KH : N = B : Z$;  [Eucl. v. 22]

therefore $Z$ is the same part of $B$ as $N$ is of $KH$. Now $A$ was proved to be a multiple of $Z$; therefore $Z$ is a common measure of $A, B$. Therefore, if $\Delta H$ is divided into segments equal to $N$ and $A$ into segments equal to $Z$, the segments in $\Delta H$ equal in magnitude to $N$ will be equal in number to the segments of $A$ equal to $Z$. It follows that, if there be placed on each of the segments in $\Delta H$ a magnitude equal to $Z$, having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to $A$, and the centre of gravity of the figure compounded of them all will be $E$; for they are even in number, and the numbers on either side of $E$ will be equal because $\Delta E = HE$.  [Prop. 5, coroll. 2.]

Similarly it may be proved that, if a magnitude equal to $Z$ be placed on each of the segments [equal to $N$] in $KH$, having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to $B$, and the centre of gravity of the figure compounded of them all will be $\Delta$ [Prop. 5, coroll. 2]. Therefore $A$ may be regarded as placed at $E$, and $B$ at $\Delta$. But they will be a set of magnitudes lying on a straight line, equal one to another, with their centres of gravity at equal intervals, and even in number; it is therefore clear that the centre of gravity of the magnitude compounded of them all is the point of bisection of the line containing the centres [of gravity] of the middle magnitudes [from Prop. 5, coroll. 2].

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1 συγκελμενα om. Heiberg.
ἀ μὲν ΛΕ τὰ ΓΔ, ἀ δὲ ΕΓ τὰ ΔΚ, καὶ ὅλα ἄρα ἀ ΛΓ ἵσα τὰ ΓΚ. ὡστε τοῦ ἐκ πάντων μεγέθεως κέντρον τοῦ βάρεος τὸ Γ σαμεῖον. τοῦ μὲν ἄρα Α κειμένου κατὰ τὸ Ε, τοῦ δὲ Β κατὰ τὸ Δ, ἴσορροπησοῦντι κατὰ τὸ Γ.

ζ′

Καὶ τοῖνυν, εἴ κα αὐσύμμετρα ἐσώτερα τὰ μεγέθεα, ὁμοίως ἴσορροπησοῦντι ἀπὸ μακέων ἀντιπεπονθότως τὸν αὐτὸν λόγον ἐχόντων τοῖς μεγέθεσιν.

"Ἕστω αὖσύμμετρα μεγέθεα τὰ ΑΒ, Γ, μάκεα δὲ τὰ ΔΕ, ΕΖ, ἔχετω δὲ τὸ ΑΒ ποτὲ τὸ Γ τῶν αὐτῶν λόγων, δν καὶ τὸ ΕΔ ποτὲ τὸ ΕΖ μάκοσ· λέγω, ὅτι τοῦ ἑς ἀμφοτέρων τῶν ΑΒ, Γ κέντρον τοῦ βάρεος ἐστὶ τὸ Ε.

Εἰ γὰρ μὴ ἴσορροπήσει τὸ ΑΒ τεθέν ἐπὶ τῷ Ζ τῶ Γ τεθέντι ἐπὶ τῷ Δ, ἦτοι μεῖζόν ἐστι τὸ ΑΒ

\[\text{Diagram with labeled points and shapes for Greek text explanation.}\]

tοῦ Γ ἥ ὡστε ἴσορροπεῖν [τῷ Γ]\(^{1}\) ἥ οὐ. ἕστω μεῖζον, καὶ ἀφηρήσθω ἀπὸ τοῦ ΑΒ ἔλασσον τὰς ύπεροχὰς, ᾧ μεῖζὸν ἐστὶ τὸ ΑΒ τοῦ Γ ἥ ὡστε ἴσορροπεῖν, ὡστε [τὸ]\(^{2}\) λοιπὸν τὸ Α σύμμετρον

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And since $\Delta E = \Gamma \Delta$ and $\Gamma \Gamma = \Delta \Gamma$, therefore $\Lambda \Gamma = \Gamma \Delta$; so that the centre of gravity of the magnitude compounded of them all is the point $\Gamma$. Therefore if $A$ is placed at $E$ and $B$ at $\Delta$, they will balance about $\Gamma$.

Prop. 7

And now, if the magnitudes be incommensurable, they will likewise balance at distances reciprocally proportional to the magnitudes.

Let $(A + B)$, $\Gamma$ be incommensurable magnitudes, and let $\Delta E$, $EZ$ be distances, and let

$$(A + B) : \Gamma = E \Delta : E Z ;$$

I say that the centre of gravity of the magnitude composed of both $(A + B)$, $\Gamma$ is $E$.

For if $(A + B)$ placed at $Z$ do not balance $\Gamma$ placed at $\Delta$, either $(A + B)$ is too much greater than $\Gamma$ to balance or less. Let it [first] be too much greater, and let there be subtracted from $(A + B)$ a magnitude less than the excess by which $(A + B)$ is too much greater than $\Gamma$ to balance, so that the remainder $A$ is

$^6$ As becomes clear later in the proof, the first magnitude is regarded as made up of two parts—$A$, which is commensurate with $\Gamma$ and $B$, which is not commensurate; if $(A + B)$ is too big for equilibrium with $\Gamma$, then $B$ is so chosen that, when it is taken away, the remainder $A$ is still too big for equilibrium with $\Gamma$. Similarly if $(A + B)$ is too small for equilibrium.

$^1$ τοῖς $\Gamma$ om. Eutocius.  
$^2$ τὸ om. Eutocius.

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(iii.) Centre of Gravity of a Parallelogram

\[\text{Ibid., Props. 9 et 10, Archim. ed. Heiberg ii. 140. 16–144.}\]

\[\theta'\]

\[\text{Pantos parallhlogrammou to kentron tou báreos estin epi tás evthéias tás epízeugnuóssas tás dichotomías tavn kat' enantion tòu parallhlogrammou pléurán.}\]

\[\text{*Estw parallhlogrammou to ABGD, epi de tavn dichotomían tavn AB, GD è EZ: faini ëh, òti toù ABGD parallhlogrammou to kentron tou báreos éssaitai epi tás EZ.}\]

\[\text{Mē gár, álλ', ei dyvatóv, ëstw to Θ, kai áxhòv parà tavn AB à ΘI. tás [de]' ëh EB dichotomou-ménav aiei éssaitai poika à kataléuropéna élássoın}\]

\[1 \text{ðe om. Heiberg.}\]

\[\text{* The proof is incomplete and obscure; it may be thus completed.}\]

\[\text{Since } \frac{A}{\Gamma} < \frac{\Delta E}{EZ},\]

\[\Delta \text{ will be depressed, which is impossible, since there has been taken away from } (A + B) \text{ a magnitude less than the deduc-}\]

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commensurate with \( \Gamma \). Then, since \( A, \Gamma \) are commensurable magnitudes, and

\[ A : \Gamma < \Delta E : EZ, \]

\( A, \Gamma \) will not balance at the distances \( \Delta E, EZ \), \( A \) being placed at \( Z \) and \( \Gamma \) at \( \Delta \). By the same reasoning, they will not do so if \( \Gamma \) is greater than the magnitude necessary to balance \( (A + B) \).

(iii.) **Centre of Gravity of a Parallelogram**


11. 140. 16–144. 4

Prop. 9

The centre of gravity of any parallelogram is on the straight line joining the points of bisection of opposite sides of the parallelogram.

Let \( AB\Gamma\Delta \) be a parallelogram, and let \( EZ \) be the straight line joining the mid-points of \( AB, \Gamma\Delta \); then I say that the centre of gravity of the parallelogram \( AB\Gamma\Delta \) will be on \( EZ \).

For if it be not, let it, if possible, be \( \Theta \), and let \( \Theta I \) be drawn parallel to \( AB \). Now if \( EB \) be bisected, and the half be bisected, and so on continually, there will be left some line less than \( I\Theta \); let \( EK \) be less than

the necessary to produce equilibrium, so that \( Z \) remains depressed. Therefore \( (A + B) \) is not greater than the magnitude necessary to produce equilibrium; in the same way it can be proved not to be less; therefore it is equal.

The centres of gravity of a triangle and a trapezium are also found by Archimedes in the first book; the second book is wholly devoted to finding the centres of gravity of a parabolic segment and of a portion of it cut off by a parallel to the base.

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τὰς ΙΘ. καὶ διηρήσθω ἐκατέρα τῶν ΑΕ, ΕΒ εἰς τὰς τὰ EΚ ἴσας, καὶ ἀπὸ τῶν κατὰ τὰς διαφέρουσιν ἱσαργράμμοιν ἐστὶν ἡμεῖς τῶν ΕΖ· διαπερθήσεται δὴ τὸ ὁλον παραλληλόγραμμον εἰς παραλληλόγραμμα τὰ ἴσα καὶ ὁμοία τῷ ΚΖ. τῶν οὖν παραλληλογράμμων τῶν ἴσων καὶ ὁμοίων τῷ ΚΖ ἐφαρμοζο-μένων ἐπʼ ἄλλαλα καὶ τὰ κέντρα τοῦ βάρεος αὐτῶν ἐπʼ ἄλλαλα πεσοῦνται. ἐσούνται δὴ μεγέθεα τινα, παραλληλόγραμμα ἴσα τῷ ΚΖ, ἄρτια τῷ πλήθει, καὶ τὰ κέντρα τοῦ βάρεος αὐτῶν ἐπʼ εὐθείας κείμενα, καὶ τὰ μέσα ἴσα, καὶ πάντα τὰ ἐφʼ ἐκάτερα τῶν μέσων αὐτά τε ἴσα ἐντὸ καὶ αἱ μεταξὺ τῶν κέντρων εὐθείαι ἴσαι· τοῦ ἐκ πάντων αὐτῶν ἀρα συγκεκήμενον μεγέθεος τὸ κέντρον ἐσοεῖται τοῦ βάρεος ἐπὶ τὰς εὐθείας τὰς ἐπιζευγνοῦσας τὰ κέντρα τοῦ βάρεος τῶν μέσων χωρίων. οὐκ ἔστι δὲ· τὸ γὰρ Θ ἐκτὸς ἐστὶ τῶν μέσων παραλληλο-γράμμων. φανερὸν οὖν, ὅτι ἐπὶ τὰς ΕΖ εὐθείας τὸ κέντρον ἐστὶ τοῦ βάρεος τοῦ ΑΒΓΔ παραλληλο-γράμμου.

Πάντως παραλληλογράμμοι τὸ κέντρον τοῦ βάρεος ἐστὶ τὸ σαμείον, καὶ δαὶ διαμέτροι συμπίπτοντι.

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I0,] and let each of AE, EB be divided into parts equal to EK, and from the points of division let straight lines be drawn parallel to EZ; then the whole parallelogram will be divided into parallelograms equal and similar to KZ. Therefore, if these parallelograms equal and similar to KZ be applied to each other, their centres of gravity will also coincide [Post. 4]. Thus there will be a set of magnitudes, being parallelograms equal to KZ, which are even in number and whose centres of gravity lie on a straight line, and the middle magnitudes will be equal, and the magnitudes on either side of the middle magnitudes will also be equal, and the straight lines between their centres [of gravity] will be equal; therefore the centre of gravity of the magnitude compounded of them all will be on the straight line joining the centres of gravity of the middle areas [Prop. 5, coroll. 2]. But it is not; for Θ lies without the middle parallelograms. It is therefore manifest that the centre of gravity of the parallelogram ABΓΔ will be on the straight line EZ.

Prop. 10

The centre of gravity of any parallelogram is the point in which the diagonals meet.
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"Εστιν παραλληλόγραμμον τὸ ΑΒΓΔ καὶ ἐν αὐτῷ ἀ ΕΖ δίχα τέμνουσα τὰς ΑΒ, ΓΔ, ἀ δὲ ΚΛ
tὰς ΑΓ, ΒΔ. ἔστιν δὴ τοῦ ΑΒΓΔ παραλληλο-
γράμμον τὸ κέντρον τοῦ βάρεος ἐπὶ τὰς ΕΖ-
δέδεικται γὰρ τούτο. διὰ ταύτα δὲ καὶ ἐπὶ τὰς
ΚΛ· τὸ Θ ἁρα σαμεῖον κέντρον τοῦ βάρεος. κατὰ
dὲ τὸ Θ αἱ διαμέτρου τοῦ παραλληλογράμμου
συμπίπτοντι· ὥστε δέδεικται τὸ προτεθέν.

(j) Mechanical Method in Geometry

ii. 426. 3–430. 22

'Ἀρχιμηδῆς Ἐρατοσθένει εὗ πράττεν . . .
'Ορῶν δὲ σε, καθάπερ λέγω, σπουδαῖον καὶ
φιλοσοφίας προεστῶτα ἀξιολόγως καὶ τὴν ἐν τοῖς

* According to Heath (H.G.M. ii. 21), Wallis has observed that Archimedes might seem, "as it were of set purpose to have covered up the traces of his investigation, as if he had grudged posterity the secret of his method of inquiry, while he wished to extort from them assent to his results." A comparison of the Method with other treatises now reveals to us how Archimedes found the areas and volumes of certain figures. His method was to balance elements of the figure against elements of another figure whose mensuration was 220
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For let $AB\Gamma\Delta$ be a parallelogram, and in it let $EZ$ bisect $AB$, $\Gamma\Delta$ and let $K\Lambda$ bisect $A\Gamma$, $B\Delta$; now the centre of gravity of the parallelogram $AB\Gamma\Delta$ is on $EZ$—for this has been proved. By the same reasoning it lies on $K\Lambda$; therefore the point $\Theta$ is the centre of gravity. And the diagonals of the parallelogram meet at $\Theta$; so that the proposition has been proved.


Archimedes to Eratosthenes\footnote{Archimedes to Eratosthenes v. infra, pp. 260-273 and vol. i. pp. 100-103, 256-261, and 290-299.} greeting . . .

Moreover, seeing in you, as I say, a zealous student and a man of considerable eminence in philosophy, known. This gave him the result, and then he proved it by rigorous geometrical methods based on the principle of \textit{reductio ad absurdum}.

The case of the parabola is particularly instructive. In the \textit{Method}, Prop. 1, Archimedes conceives a segment of a parabola as made up of straight lines, and by his mechanical method he proves that the segment is four-thirds of the triangle having the same base and equal height. In his \textit{Quadrature of a Parabola}, Prop. 14, he conceives the parabola as made up of a large number of trapezia, and by mechanical methods again reaches the same result. This is more satisfactory, but still not completely rigorous, so in Prop. 24 he proves the theorem without any help from mechanics by \textit{reductio ad absurdum}.

\footnote{The Method had to be classed among the lost works of Archimedes until 1906, when it was discovered at Constantinople by Heiberg in the ms. which he has termed C. Unfortunately the ms. is often difficult to decipher, and students of the text should consult Heiberg's edition. Moreover, the diagrams have to be supplied as they are undecipherable in the ms.}
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μαθήμασιν κατὰ τὸ ύποπτπτον θεωρίαν τετυπηκότα ἐδοκίμασα γράψαι σοι καὶ εἰς τὸ αὐτὸ βιβλίον ἐξορίσαι τρόπου τινὸς ἱδίότητα, καθ’ ὅν σοι παρεχόμενον ἔσται λαμβάνειν ἀφορμὰς εἰς τὸ δύνασθαι τινα τῶν ἐν τοῖς μαθήμασι θεωρεῖν διὰ τῶν μηχανικῶν. τούτῳ δὲ πέπεισμαι χρήσιμον εἶναι οὐδὲν ἥσσον καὶ εἰς τὴν ἀπόδειξιν αὐτῶν τῶν θεωρημάτων. καὶ γάρ τινα τῶν πρότερον μοι φανέρων μηχανικώς ὑστερον γεωμετρικῶς ἀπεδείχθη διὰ τὸ χωρίς ἀποδείξεως εἶναι τὴν διὰ τούτου τοῦ τρόπου θεωρίαν· έτοιμότερον γάρ ἐστι προλαβόντα διὰ τοῦ τρόπου γνώσιν τινα τῶν ξητημάτων πορίσασθαι τὴν ἀπόδειξιν μᾶλλον ἢ μηδενὸς ἐγνωσμένου ζητεῖν. . . . γράφομεν οὖν πρῶτον τὸ καὶ πρῶτον φανῇ διὰ τῶν μηχανικῶν, ὅτι πᾶν τμήμα ὀρθογώνιον κώνου τομῆς ἐπίτριτόν ἐστιν τριγώνου τοῦ βάσιν ἔχοντος τὴν αὐτὴν καὶ ὕψος ἰσον.

Ibid., Prop. 1, Archim. ed. Heiberg ii. 434. 14–438. 21

"Εστώ τμήμα τὸ ΑΒΓ περιεχόμενον ύπὸ εὐθείας τῆς ΑΓ καὶ ὀρθογώνιου κώνου τομῆς τῆς ΑΒΓ, καὶ τετμήσθω δίχα ἡ ΑΓ τῷ Δ, καὶ παρὰ τὴν διάμετρον ᾗχθω ἡ ΔΒΕ, καὶ ἐπεξεύχθωσαν αἱ ΑΒ, ΒΓ.

Λέγω, ὅτι ἐπίτριτόν ἐστιν τὸ ΑΒΓ τμήμα τοῦ ΑΒΓ τριγώνου.

"Ηχθωσαν ἀπὸ τῶν Α, Γ σημεῖων ἢ μὲν ΑΖ παρὰ τὴν ΔΒΕ, ἢ δὲ ΓΖ ἐπιφανέστατα τῆς τομῆς, καὶ ἐκβεβλήσθω ἡ ΓΒ ἐπὶ τὸ Κ, καὶ κείσθω τῇ ΓΚ ἵση ἡ ΚΘ. νοείσθω ζυγὸς ὁ ΓΘ καὶ μέσον 222
who gives due honour to mathematical inquiries when they arise, I have thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, with which furnished you will be able to make a beginning in the investigation by mechanics of some of the problems in mathematics. I am persuaded that this method is no less useful even for the proof of the theorems themselves. For some things first became clear to me by mechanics, though they had later to be proved geometrically owing to the fact that investigation by this method does not amount to actual proof; but it is, of course, easier to provide the proof when some knowledge of the things sought has been acquired by this method rather than to seek it with no prior knowledge. . . . At the outset therefore I will write out the very first theorem that became clear to me through mechanics, that any segment of a section of a right-angled cone is four-thirds of the triangle having the same base and equal height.

\textit{Ibid.}, Prop. 1, Archim. ed. Heiberg ii. 434. 14–438. 21

Let \( \triangle AB\Gamma \) be a segment bounded by the straight line \( \triangle \Gamma \) and the section \( \triangle AB\Gamma \) of a right-angled cone, and let \( \triangle \Gamma \) be bisected at \( \Delta \), and let \( \triangle \Gamma \) be drawn parallel to the axis, and let \( \triangle AB, \beta \Gamma \) be joined.

I say that the segment \( \triangle AB\Gamma \) is four-thirds of the triangle \( \triangle AB\Gamma \).

From the points \( A, \Gamma \) let \( AZ \) be drawn parallel to \( \triangle \Gamma \), and let \( \Gamma Z \) be drawn to touch the section, and let \( \Gamma B \) be produced to \( K \), and let \( K\Theta \) be placed equal to \( \Gamma K \). Let \( \Gamma \Theta \) be imagined to be a balance
αὐτοῦ τὸ Κ καὶ τῇ ΕΔ παράλληλος τυχοῦσα ἡ ΜΕ.

"Επει οὖν παραβολή ἐστιν ἡ ΓΒΑ, καὶ ἐφάπτεται ἡ ΓΖ, καὶ τεταγμένως ἡ ΓΔ, ἵση ἐστὶν ἡ EB τῇ ΒΔ. τούτῳ γὰρ ἐν τοῖς στοιχείοις δείκνυται. διὰ δὴ τούτῳ, καὶ διότι παράλληλοι εἰσιν αἱ ZA, ΜΕ τῇ ΕΔ, ἵση ἐστὶν καὶ ἡ μὲν MN τῇ ΝΕ, η δὲ ZK τῇ KA. καὶ ἐπεὶ ἐστιν, ως ἡ GA πρὸς ΑΞ, οὕτως ἡ ΜΕ πρὸς ΞΟ [τούτῳ γὰρ ἐν λήμματι δείκνυται], ως δὲ ἡ GA πρὸς ΑΞ, οὕτως ἡ ΓΚ πρὸς KN, καὶ ἵση ἐστὶν ἡ ΓΚ τῇ KΘ, ως ἄρα ἡ ΘΚ πρὸς KN, οὕτως ἡ ΜΕ πρὸς ΞΟ. καὶ ἐπεὶ τὸ Ν σημεῖον κέντρον τοῦ βάρους τῆς ΜΕ εὐθείας ἐστίν, ἐπείπερ ἵση ἐστὶν ἡ MN τῇ ΝΕ, ἐὰν ἄρα τῇ ΞΟ ἴσην θάμεν τὴν ΘΗ καὶ κέντρον τοῦ βάρους αὐτῆς τὸ Θ, ὀπωσὶ ἵση ἡ ΘΟ τῇ ΘΗ, ἱσορροπήσει ἡ ΘΗΚ τῇ ΜΕ αὐτοῦ μενοῦση διὰ τὸ ἀντιπεπονθότως τετμῆσθαι.
with mid-point $K$, and let $ME$ be drawn parallel to $ED$.

Then since $TBA$ is a parabola, and $TZ$ touches it, and $TA$ is a semi-ordinate, $EB = BA$—for this is proved in the elements; for this reason, and because $ZA$, $ME$ are parallel to $ED$, $MN = NE$ and $ZK = KA$ [Eucl. vi. 4, v. 9]. And since

$$\Gamma A : A\Xi = ME : \Xi O,$$

[Quad. parab. 5, Eucl. v. 18]

and

$$\Gamma A : A\Xi = \Gamma K : KN,$$

[Eucl. vi. 2, v. 18]

while

$$\Gamma K = K\Theta,$$

therefore

$$\Theta K : KN = ME : \Xi O.$$

And since the point $N$ is the centre of gravity of the straight line $ME$, inasmuch as $MN = NE$ [Lemma 4], if we place $TH = \Xi O$, with $\Theta$ for its centre of gravity, so that $T\Theta = \Theta H$ [Lemma 4], then $T\Theta H$ will balance $ME$ in its present position, because $\Theta N$ is cut

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1. Archimedes would have said "section of a right-angled cone"—ὁρθογωνίου κώνου τομά.

2. The reference will be to the Elements of Conics by Euclid and Aristaeus for which v. vol. i. pp. 486-491 and infra, p. 280 n. a; cf. similar expressions in On Conoids and Spheroids, Prop. 3 and Quadrature of a Parabola, Prop. 3; the theorem is Quadrature of a Parabola, Prop. 2.

1. τοῦτο ... δείκνυται om. Heiberg. It is probably an interpolator's reference to a marginal lemma.
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tήν ΘΝ τοὺς ΘΗ, ΜΕ βάρεσιν, καὶ ὡς τὴν ΘΚ πρὸς ΚΝ, οὐτως τὴν ΜΕ πρὸς τὴν ΗΤ· ὥστε τοῦ εξ ἀμφοτέρων βάρους κέντρον ἐστὶν τοῦ βάρους τὸ Κ. ὁμοίως δὲ καὶ, ὅσαν ἄν ἀχθώσων ἐν τῷ ΖΑΓ τριγώνῳ παράλληλοι τῇ ΕΔ· ἱσορρόπησον αὐτοῦ μένουσι ταῖς ἀπολαμβανομέναις ἀπ' αὐτῶν ὑπὸ τῆς τομῆς μετενεχθείσας ἐπὶ τὸ Θ, ὥστε εἶναι τοῦ εξ ἀμφοτέρων κέντρον τοῦ βάρους τὸ Κ. καὶ ἔπει ἐκ μὲν τῶν ἐν τῷ ΓΖΑ τριγώνῳ τὸ ΓΖΑ τρίγωνον συνέστηκεν, ἐκ δὲ τῶν ἐν τῇ τομῇ ὁμοίως τῇ ΕΘ λαμβανομένων συνέστηκε τὸ ΑΒΓ τμῆμα, ἱσορρόπησει ἄρα τὸ ΖΑΓ τρίγωνον αὐτοῦ μένον τῷ τμῆματι τῆς τομῆς τεθέντι περὶ κέντρου τοῦ βάρους τὸ Θ κατὰ τὸ Κ σημείον, ὥστε τοῦ εξ ἀμφοτέρων κέντρον εἶναι τοῦ βάρους τὸ Κ. τεθείσον δὴ ἡ ΖΚ τῷ Χ, ὥστε τριπλασιάν εἶναι τὴν ΓΚ τῆς ΚΧ· ἔσται ἄρα τὸ Χ σημείον κέντρου βάρους τοῦ ΑΖΓ τριγώνου· δεδεικται γάρ ἐν τοῖς Ἰσορροπικοῖς. ἔπει οὖν ἱσορροποῦν τὸ ΖΑΓ τρίγωνον αὐτοῦ µένον τῷ ΒΑΓ τμῆματι κατὰ τὸ Κ τεθέντι περὶ τὸ Θ κέντρον τοῦ βάρους, καὶ ἔστων τοῦ ΖΑΓ τριγώνου κέντρου βάρους τὸ Χ, ἔστων ἄρα, ὡς τὸ ΑΖΓ τρίγωνον πρὸς τὸ ΑΒΓ τμῆμα κεῖµενον περὶ τὸ Θ κέντρον, οὕτως ἡ ΘΚ πρὸς ΧΚ. τριπλασιά δὲ ἔστων ἡ ΘΚ τῆς ΚΧ· τριπλάσιον ἄρα καὶ τὸ ΑΖΓ τρίγωνον τοῦ ΑΒΓ τμῆματος. ἔστη δὲ καὶ τὸ ΖΑΓ τρίγωνον τετραπλάσιον τοῦ ΑΒΓ τριγώνου διὰ τὸ ἰσχὺν εἶναι τὴν µέν ΖΚ τῇ KA, τὴν δὲ ΑΔ τῇ ΔΓ· ἐπίτρωτον ἄρα ἔστων τὸ ΑΒΓ τμῆμα τοῦ ΑΒΓ τριγώνου. [τότῳ οὖν φανερὸν ἔστων].

1 τούτῳ . . . ἔστω om. Heiberg.
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in the inverse proportion of the weights TH, \( \text{ME} \), and 
\[ \frac{\Theta K}{KN} = \frac{\text{ME}}{HT} ; \]
therefore the centre of gravity of both [TH, \( \text{ME} \)] taken together is K. In the same way, as often as parallels to \( E\Delta \) are drawn in the triangle \( Z\Delta\Gamma \), these parallels, remaining in the same position, will balance the parts cut off from them by the section and transferred to \( \Theta \), so that the centre of gravity of both together is K. And since the triangle \( \Gamma ZA \) is composed of the [straight lines drawn] in \( \Gamma ZA \), and the segment \( AB\Gamma \) is composed of the lines in the section formed in the same way as \( \Xi \Xi \), therefore the triangle \( Z\Delta\Gamma \) in its present position will be balanced about K by the segment of the section placed with \( \Theta \) for its centre of gravity, so that the centre of gravity of both combined is K. Now let \( \Gamma K \) be cut at X so that \( \Gamma K = 3 \Xi X \); then the point X will be the centre of gravity of the triangle \( AZ\Gamma \); for this has been proved in the books *On Equilibriums*. Then since the triangle \( Z\Delta\Gamma \) in its present position is balanced about K by the segment \( B\Delta\Gamma \) placed so as to have \( \Theta \) for its centre of gravity, and since the centre of gravity of the triangle \( Z\Delta\Gamma \) is X, therefore the ratio of the triangle \( AZ\Gamma \) to the segment \( AB\Gamma \) placed about \( \Theta \) as its centre [of gravity] is equal to \( \frac{\Theta K}{XK} \). But \( \Theta K = 3 \Xi X \); therefore

\[ \text{triangle } AZ\Gamma = 3 \cdot \text{segment } AB\Gamma. \]
And \[ \text{triangle } Z\Delta\Gamma = 4 \cdot \text{triangle } AB\Gamma, \]
because \( ZK = KA \) and \( A\Delta = \Delta \Gamma \);
therefore \[ \text{segment } AB\Gamma = \frac{4}{3} \text{triangle } AB\Gamma. \]

* Cf. De Plan. Equil. i. 15. \[ 227 \]
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'Τούτο δὴ διὰ μὲν τῶν νῦν εἰρημένων οὐκ ἀποδεικταῖ, ἐμφασιν δὲ τινα πεποιηκε τὸ συμπέρασμα ἀληθὲς εἶναι. διὸπερ ἣμείς ὅρωντες μὲν οὐκ ἀποδειγμένον, ὑπονοοῦντες δὲ τὸ συμπέρασμα ἀληθὲς εἶναι, τάξομεν τὴν γεωμετρομενὴν ἀπόδειξιν ἐξευρόντει αὐτοὶ τὴν ἐκδοθεῖσαν πρῶτερον.¹

Archim. Quadr. Parab., Praef., Archim. ed. Heiberg ii. 262. 2–266. 4

'Αρχιμήδης Λοσιθέω εὖ πράττειν. 'Ακοὺσας Κόνωνα μὲν τετελευτηκέναι, δε ἢν οὔδεν ἐπιλείπον ἀμῖν ἐν φιλία, τὶν δὲ Κόνωνος γνώριμον γεγενηθαι καὶ γεωμετρίας οἰκεῖον εἶμεν τοῦ μὲν τετελευτηκότος εἶνεκεν ἐλυπήθημες ὡς καὶ φίλου τοῦ ἀνδρὸς γεναμένου καὶ ἐν τοῖς μαθηματοῦ θαυμαστοῦ τυνος, ἐπροχειριζάμεθα δὲ ἀποστείλαι τοι γράφαντες, ὡς Κόνωνι γράφεων ἐγνωκότες ἤμες, γεωμετρικῶν θεωρημάτων, δὲ πρῶτερον μὲν οὐκ ἢν τεθεωρημένον, νῦν δὲ υφ' ἀμῖν τεθεώρηται, πρῶτερον μὲν διὰ μηχανικῶν εὑρεθέν, ἐπειτα δὲ καὶ διὰ τῶν γεωμετρικῶν ἐπι- δεικθέν. τῶν μὲν οὖν πρῶτον περὶ γεωμετριῶν πραγματευθέντων ἐπεχείρησαν τινες γράφεων ὡς δυνατὸν ἐδώ κύκλω τῷ δοθέντι καὶ κύκλου τριματι τῷ δοθέντι χωρίον εὑρεῖν εὐθύγραμμον ἵσον, καὶ μετὰ ταύτα τὸ περιεχόμενον χωρίον ὑπὸ τῇ τὰς

¹ τούτο ... πρῶτερον. In the ms. the whole paragraph from τούτο to πρῶτερον comes at the beginning of Prop. 2; it is more appropriate at the end of Prop. 1.
This, indeed, has not been actually demonstrated by the arguments now used, but they have given some indication that the conclusion is true; seeing, therefore, that the theorem is not demonstrated, but suspecting that the conclusion is true, we shall have recourse a to the geometrical proof which I myself discovered and have already published. b

Archimedes, Quadrature of a Parabola, Preface, Archim. ed. Heiberg ii. 262. 2-266. 4

Archimedes to Dositheus greeting.

On hearing that Conon, who fulfilled in the highest degree the obligations of friendship, was dead, but that you were an acquaintance of Conon and also versed in geometry, while I grieved for the death of a friend and an excellent mathematician, I set myself the task of communicating to you, as I had determined to communicate to Conon, a certain geometrical theorem, which had not been investigated before, but has now been investigated by me, and which I first discovered by means of mechanics and later proved by means of geometry. Now some of those who in former times engaged in mathematics tried to find a rectilineal area equal to a given circle c and to a given segment of a circle, and afterwards they tried to square the area bounded by the section

a I have followed Heath's rendering of τάξιομεν, which seems more probable than Heiberg's "suo loco proponemus," though it is a difficult meaning to extract from τάξιομεν.

b Presumably Quadr. Parab. 24, the second of the proofs now to be given. The theorem has not been demonstrated, of course, because the triangle and the segment may not be supposed to be composed of straight lines.

c This seems to indicate that Archimedes had not at this time written his own book On the Measurement of a Circle. For attempts to square the circle, v. vol. i. pp. 308-347.
όλου τοῦ κώνου τομᾶς καὶ εὐθείας τετραγωνίζειν ἐπειρώντο λαμβάνοντες οὐκ εὑπαραχώρητα λήμματα, διότε οὗτοι ὑπὸ τῶν πλεῖστων οὐκ εὑρισκόμενα ταῦτα κατεγνώσθεν. τὸ δὲ ὑπ’ εὐθείας τε καὶ ὀρθογωνίου κώνου τομᾶς τμῆμα περιεχόμενον οὐδένα τῶν προτέρων ἐγχειρήσαντα τετραγωνίζειν ἐπιστάμεθα, ὃ δὴ νῦν ύφ’ ἁμῶν εὐρήται δείκνυται γάρ, ὅτι πάν τιμᾶμα περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς ἐπίτρυτον ἐστὶ τοῦ τριγώνου τοῦ βάσιν ἔχοντος τὰν αὐτάν καὶ ύψος ἵσον τῷ τμάματι λαμβανομένου τοῦ τοῦ λήμματος ἐς τάν ἀπὸδειξαν αὐτοῦ τῶν ἁνίσων χωρίων τῶν ὑπεροχῶν, ὃ ὑπερέχει τὸ μεῖζον τοῦ ἐλάσσουν, δυνατὸν εἴμεν αὐτὰν ῥαυταὶ συντιθεμέναι παντὸς ὑπερέχειν τοῦ προτεθέντος πεπερασμένον χωρίου.· κέχρηνται δὲ καὶ οἱ προτερον γεωμέτραι τῶδε τῷ λήμματι τοῦ τε γὰρ κύκλους διπλασίονα λόγον ἔχειν ποιεῖσθαι ἀλλὰ τῶν διαμέτρων ἀπὸδειχάσαι αὐτῷ τούτῳ τῷ λήμματι χρωμένοι, καὶ τὰς σφαιρὰς ὅτι τριπλασίονα λόγον ἔχοντι ποὶ ἀλλὰ τῶν διαμέτρων, ἐτεὶ δὲ καὶ ὅτι πάσα πυράμις τριῶν μέρος ἐστὶ τοῦ πρίσματος τοῦ ταῦτα βάσιν ἔχοντος ταῦτα πυράμιδι καὶ ύψος ἵσον· καὶ διὸτε πᾶσα κώνος τρίτον μέρος ἐστὶ τοῦ κυλίνδρου τοῦ ταῦτα βάσιν ἔχοντος τῷ κώνῳ καὶ ύψος ἵσον, ὁμοίων τῷ προειρημένῳ λήμματι τοι τοῖς λαμβάνοντες ἐγγραφον. συμβαίνει δὲ τῶν προειρημένων ψευδημάτων ἐκαστὸν μηδενὸς ἡσυχόν τῶν ἄνευ τοῦτον τὸν λήμματος ἀπὸδεδειγμένων πεπιστευκέναι· ἀρκεῖ δὲ ἐς τὰν ὁμοίων πίστιν τοῦτοις ἀναγμένων τῶν ύφ’ ἁμῶν ἐκδιδομένων. ἀναγράφαντες οὐν αὐτοῦ τὰς ἀποδείξεις ἀποστέλλομες 230
of the whole cone and a straight line, assuming lemmas far from obvious, so that it was recognized by most people that the problem had not been solved. But I do not know that any of my predecessors has attempted to square the area bounded by a straight line and a section of a right-angled cone, the solution of which problem I have now discovered; for it is shown that any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle which has the same base and height equal to the segment, and for the proof this lemma is assumed: given [two] unequal areas, the excess by which the greater exceeds the less can, by being added to itself, be made to exceed any given finite area. Earlier geometers have also used this lemma: for, by using this same lemma, they proved that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and also that any pyramid is a third part of the prism having the same base as the pyramid and equal height; and, further, by assuming a lemma similar to that aforesaid, they proved that any cone is a third part of the cylinder having the same base as the cone and equal height. In the event, each of the aforesaid theorems has been accepted, no less than those proved without this lemma; and it will satisfy me if the theorems now published by me obtain the same degree of acceptance. I have therefore written out the proofs, and now send them, first

a A "section of the whole cone" is probably a section cutting right through it, i.e., an ellipse, but the expression is odd.

b For this lemma, v. supra, p. 46 n. a.
πρώτον μὲν, ὡς διὰ τῶν μηχανικῶν ἑθεωρήθη, μετὰ ταῦτα δὲ καὶ, ὡς διὰ τῶν γεωμετρουμένων ἀποδείκνυται. προγράφεται δὲ καὶ στοιχεία κω-

νικά χρείαν ἔχουντα ἐσ τὰς ἀπόδειξιν. ἔρρωσο.


"Εστι τὸ τμάμα τὸ ΒΘΓ περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογώνιου κώνου τομᾶς. ἔστω δὴ πρῶτον

ά ΒΓ ποτ’ ὀρθὰς τὰ διαμέτρῳ, καὶ ἄχθων ἀπὸ μὲν τοῦ Β σαμείου ἀ ΒΔ παρὰ τὰν διάμετρον, ἀπὸ δὲ τοῦ Γά ΓΔ ἐπισαλύουσα τὰς τοῦ κώνου τομᾶς κατὰ τὸ Γ. ἐσσεῖται δὴ τὸ ΒΓΔ τρίγωνον ὀρθο-

γώνιον. διηρήσω δὴ ἃ ΒΓ ἐσ ἵσα τμάματα ὀποσαοῦν τὰ ΒΕ, ΕΖ, ΖΗ, ΗΓ, καὶ ἀπὸ τῶν τομῶν ἄχθωσαν παρὰ τὰν διάμετρον αἱ ΕΣ, ΖΤ, ἘΥ, ΕΣ, ἀπὸ δὲ τῶν σαμείων, καθ’ ἃ τέμνοντι

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as they were investigated by means of mechanics, and also as they may be proved by means of geometry. By way of preface are included the elements of conics which are needed in the demonstration. Farewell.


Let BΩΓ be a segment bounded by a straight line and a section of a right-angled cone. First let BΓ be at right angles to the axis, and from B let BΔ be drawn parallel to the axis, and from Γ let ΓΔ be drawn touching the section of the cone at Γ; then the triangle BΓΔ will be right-angled [Eucl. i. 29]. Let BΓ be divided into any number of equal segments BE, EZ, ZH, HI, IT, and from the points of section let EZ, ZT, HY, IE be drawn parallel to the axis, and from the points in which these cut the
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αὐταὶ τὰν τοῦ κόνου τομὰν, ἐπεζεύχθωσαν ἐπὶ τὸ Γ καὶ ἐκβεβλήσθωσαν. φαμὶ δὴ τὸ τρίγωνον τὸ ΒΔΓ τῶν μὲν τραπεζίων τῶν ΚΕ, ΔΖ, ΜΗ, ΝΙ καὶ τοῦ ΞΓ τριγώνου ἐλασσον εἰμεν ἡ τριπλάσιον, τῶν δὲ τραπεζίων τῶν ΖΦ, ΗΘ, ΙΠ καὶ τοῦ ΙΟΓ τριγώνου μείζον [ἐστιν] ἡ τριπλάσιον.

Διάχω γὰρ εὐθείᾳ ἀ ΑΒΓ, καὶ ἀπολελάφθω ἀ ΑΒ ἵσα τὰ ἩΓ, καὶ νοείσθω ζύγιον τὸ ΑΓ. μέσον δὲ αὐτοῦ ἐσσεῖται τὸ Β. καὶ κρημάσθω ἐκ τοῦ Β, κρημάσθω δὲ καὶ τὸ ΒΔΓ ἐκ τοῦ ζύγιον κατὰ τὰ Β, Γ, ἐκ δὲ τοῦ βατέρου μέρεος τοῦ ζύγιον κρημάσθω τὰ Ρ, Χ, Ψ, Ω, Δ χωρία κατὰ τὸ Α, καὶ ἵσορροπεῖτω τὸ μὲν Ρ χωρίον τῷ ΔΕ τραπεζίῳ οὔτως ἔχοντι, τὸ δὲ Χ τῷ ΖΣ τραπεζίῳ, τὸ δὲ Ψ τῷ ΘΗ, τὸ δὲ Ω τῷ ΥΙ, τὸ δὲ Δ τῷ ΞΓ τριγώνῳ. ἵσορροπήσει δὴ καὶ τὸ ὁλον τῷ ὅλω. ὁστε τριπλάσιον ἂν εἴῃ τὸ ΒΔΓ τρίγωνον τοῦ ΡΧΥΩΔ χωρίου. καὶ ἐπεῖ ἐστιν τμᾶμα τὸ ΒΓΘ, ὁ περιέχεται ὑπὸ τε εὐθείας και ὀρθογωνίου κόνου τομᾶς, καὶ ἀπὸ μὲν τοῦ Β παρὰ τὰν διάμετρον ἄκται αὐτὸ τὸ ΒΔ, ἀπὸ δὲ τοῦ Γ αὐτὸν ἐπιμακρύσσεσαι τὰς τοῦ κόνου τομᾶς κατὰ τὸ Γ, ἄκται δὲ τὶς καὶ ἄλλα παρὰ τὰν διάμετρον αὐτὸν ἔχει λόγον ἀ ΒΓ ποτὶ τὰν ΒΕ, ὅν αὐτὸν ἔχει λόγον, τὸ δὲ ΔΕ τραπεζίον ποτὶ τὸ ΚΕ. ῥαμοῖος δὲ δειχθήσεται αὐτὸν ἔχουσα τὸν ποτὶ τὰν ΒΖ τὸν αὐτὸν ἔχουσα λόγον, ὅν τὸ ΖΣ τραπεζίον ποτὶ τὸ ΔΖ, ποτὶ δὲ τὰν ΒΗ, ὅν τὸ ΘΗ ποτὶ τὸ ΜΗ, ποτὶ δὲ τὰν ΒΙ, ὅν τὸ ΥΙ ποτὶ τὸ ΝΙ. ἐπεὶ οὖν ἐστὶ τραπεζίον τὸ ΔΕ τὰς

1 ἐστιν om. Heiberg.
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section of the cone let straight lines be drawn to $\Gamma$ and produced. Then I say that the triangle $B\Delta\Gamma$ is less than three times the trapezia $KE$, $AZ$, $MH$, $NI$ and the triangle $\Xi\Pi\Gamma$, but greater than three times the trapezia $Z\Phi$, $H\Theta$, $I\Pi$ and the triangle $I\Omega\Gamma$.

For let the straight line $AB\Gamma$ be drawn, and let $AB$ be cut off equal to $B\Gamma$, and let $A\Gamma$ be imagined to be a balance; its middle point will be $B$; let it be suspended from $B$, and let the triangle $B\Delta\Gamma$ be suspended from the balance at $B$, $\Gamma$, and from the other part of the balance let the areas $P$, $X$, $\Psi$, $\Omega$, $\Delta$ be suspended at $A$, and let the area $P$ balance the trapezium $\Delta E$ in this position, let $X$ balance the trapezium $Z\Sigma$, let $\Psi$ balance $TH$, let $\Omega$ balance $YI$, and let $\Delta$ balance the triangle $\Xi\Pi\Gamma$; then the whole will balance the whole; so that the triangle $B\Delta\Gamma$ will be three times the area $P + X + \Psi + \Omega + \Delta$ [Prop. 6]. And since $B\Gamma\Theta$ is a segment bounded by a straight line and a section of a right-angled cone, and $B\Delta$ has been drawn from $B$ parallel to the axis, and $\Gamma\Delta$ has been drawn from $\Gamma$ touching the section of a cone at $\Gamma$, and another straight line $\Sigma E$ has been drawn parallel to the axis,

$$B\Gamma : BE = \Sigma E : E\Phi;$$

[Prop. 5]

therefore $BA : BE = \text{trapezium } \Delta E : \text{trapezium } KE$.a

Similarly it may be proved that

$$AB : BZ = \Sigma Z : AZ,$$

$$AB : BH = TH : MH,$$

$$AB : BI = YI : NI.$$

Therefore, since $\Delta E$ is a trapezium with right angles

- For $BA = B\Gamma$ and $\Delta E : KE = \Sigma E : E\Phi$.  

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μὲν ποτὶ τοῖς Β, Ε σάμελοις γωνίαις ὀρθᾶς ἔχον, τὰς δὲ πλευρὰς ἐπὶ τὸ Γ νευώσας, ἱσορροπεῖ δὲ τὶ χωρίον αὐτῷ τὸ Ρ κρεμάμενον ἐκ τοῦ ξυγοῦ κατὰ τὸ Α οὐτως ἔχοντος τοῦ τραπεζίου, ὡς νῦν κεῖται, καὶ ἔστιν, ὡς ἄ ΒΑ ποτὶ τὰν ΒΕ, οὕτως τὸ ΔΕ τραπέζιον ποτὶ τὸ ΚΕ, μείζον ἁρα ἔστιν τὸ ΚΕ χωρίον τοῦ Ρ χωρίου· δέδεικται γὰρ τοῦτο. πάλιν δὲ καὶ τὸ ΖΣ τραπέζιον τὰς μὲν ποτὶ τοῖς Ζ, Ε γωνίαις ὀρθᾶς ἔχον, τὰν δὲ ΣΤ νεύσας ἐπὶ τὸ Γ, ἱσορροπεῖ δὲ αὐτῷ χωρίον τὸ Χ ἐκ τοῦ ξυγοῦ κρεμάμενον κατὰ τὸ Α οὕτως ἔχοντος τοῦ τραπεζίου, ὡς νῦν κεῖται, καὶ ἔστιν, ὡς μὲν ἄ ΑΒ ποτὶ τὰν ΒΕ, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΖΦ, ὡς δὲ ἄ ΑΒ ποτὶ τὰν ΒΖ, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΛΖ· εἴη οὖν καὶ τὸ Χ χωρίον τοῦ μὲν ΛΖ τραπεζίου ἠλάσσον, τοῦ δὲ ΖΦ μείζον· δεδεικται γὰρ καὶ τοῦτο. διὰ τὰ αὐτὰ δὴ καὶ τὸ Ψ χωρίον τοῦ μὲν ΜΗ τραπεζίου ἠλάσσον, τοῦ δὲ ΘΗ μείζον, καὶ τὸ Ω χωρίον τοῦ μὲν ΝΟΙΗ τραπεζίου ἠλάσσον, τοῦ δὲ ΠΙ μείζον, ὁμοίως δὲ καὶ τὸ Δ χωρίον τοῦ μὲν ΞΙΓ τριγώνου ἠλάσσον, τοῦ δὲ ΓΙΟ μείζον. ἐπεὶ οὖν τὸ μὲν ΚΕ τραπεζίου μείζον ἔστι τοῦ Ρ χωρίου, τὸ δὲ ΛΖ τοῦ Χ, τὸ δὲ ΜΗ τοῦ Ψ, τὸ δὲ ΝΙ τοῦ Ω, τὸ δὲ ΞΙΓ τριγώνου τοῦ Δ, φανερόν, ὅτι καὶ πάντα τὰ εἰρημένα χωρία μείζονά ἔστι τοῦ ΡΧΨΩΔ χωρίου. ἐστιν δὲ τὸ ΡΧΨΩΔ τρίτον μέρος τοῦ ΒΓΔ τριγώνου· δὴλον ἁρα, ὅτι τὸ ΒΓΔ τριγώνων ἠλάσσον ἐστιν η τριπλάσιον τῶν ΚΕ, ΛΖ, ΜΗ, ΝΙ τραπεζίων καὶ τοῦ ΞΙΓ τριγώνου. πάλιν, ἐπεὶ τὸ μὲν ΖΦ τραπεζίον ἠλάσσον ἐστὶ τοῦ Χ χωρίου, τὸ δὲ ΘΗ τοῦ Ψ, τὸ δὲ ΠΙ τοῦ Ω, τὸ δὲ ΙΟΓ τριγώνων τοῦ Δ, φανερόν, ὅτι καὶ πάντα 236
at the points $B$, $E$ and with sides converging on $\Gamma$, and it balances the area $P$ suspended from the balance at $A$, if the trapezium be in its present position, while

$$BA : BE = \Delta E : KE,$$

therefore

$$KE > P;$$

for this has been proved [Prop. 10]. Again, since $Z\Sigma$ is a trapezium with right angles at the points $Z$, $E$ and with $\Sigma T$ converging on $\Gamma$, and it balances the area $X$ suspended from the balance at $A$, if the trapezium be in its present position, while

$$AB : BE = Z\Sigma : Z\Phi,$$

$$AB : BZ = Z\Sigma : \Lambda Z,$$

therefore

$$\Lambda Z > X > Z\Phi;$$

for this also has been proved [Prop. 12]. By the same reasoning

$$MH > \Psi > \Theta H,$$

and

$$NOI H > \Omega > \Pi I,$$

and similarly

$$\Xi \Gamma > \Delta > \Gamma \Omega.$$

Then, since $KE > P$, $\Lambda Z > X$, $MH > \Psi$, $NI > \Omega$, $\Xi \Gamma > \Delta$, it is clear that the sum of the aforesaid areas is greater than the area $P + X + \Psi + \Omega + \Delta$. But

$$P + X + \Psi + \Omega + \Delta = \frac{1}{3} \quad B\Gamma \Delta;$$

[Prop. 6]

it is therefore plain that

$$B\Gamma \Delta < 3(KE + \Lambda Z + MH + NI + \Xi \Gamma).$$

Again, since $Z\Phi < X$, $\Theta H < \Psi$, $\Pi I < \Omega$, $\Omega \Gamma < \Delta$, it is
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τὰ εἰρημένα ἐλάσσονά ἐστὶ τοῦ ΔΩΨΧ χωρίου·
φανερὸν οὖν, ὅτι καὶ τὸ ΒΔΓ τρίγωνον μειζὸν
ἐστὶν ἡ τριπλάσιον τῶν ΦΖ, ΘΗ, ΠΠ τραπεζίων
καὶ τοῦ ΙΓΟ τριγώνου, ἐλάσσον δὲ ἡ τριπλάσιον
τῶν προγεγραμμένων.

Ibid., Prop. 24, Archim. ed. Heiberg ii. 312. 2–314. 27

Πάν τμῆμα τὸ περιεχόμενον ὑπὸ εὐθείας καὶ
ὀρθογωνίου κόνου τομᾶς ἐπίτριτον ἐστὶ τριγώνου
τοῦ τῶν αὐτῶν βάσιν ἔχοντος αὐτῶ καὶ ὑψὸς ἴσον.
"Εστώ γὰρ τὸ ΑΔΒΕΓ τμῆμα περιεχόμενον ὑπὸ
eὐθείας καὶ ὀρθογωνίου κόνου τομᾶς, τὸ δὲ ΑΒΓ
τρίγωνον ἐστὶν τῶν αὐτῶν βάσιν ἔχον τῷ τμῆματι

καὶ ὑψὸς ἴσον, τοῦ δὲ ΑΒΓ τριγώνου ἐστὶ ἐπὶ-
τριτον τὸ Κ χωρίου. δεικτέον, ὅτι ἴσον ἐστὶ τῷ
ΑΔΒΕΓ τμῆματι.

Εἰ γὰρ μὴ ἐστὶν ἴσον, ἢτοι μειζὸν ἐστὶν ἡ ἐλασσον.
ἐστὶν πρότερον, εἰ δυνατόν, μειζὸν τὸ ΑΔΒΕΓ
τμῆμα τοῦ Κ χωρίου. ἐνέγραψα δὴ τὰ ΑΔΒ,
ΒΕΓ τρίγωνα, ὡς εἰρηται, ἐνέγραψα δὲ καὶ εἰς τὰ
περιλειπόμενα τμήματα ἄλλα τρίγωνα τῶν αὐτῶν

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clear that the sum of the aforesaid areas is greater than the area \( \Delta + \Omega + \Psi + X \); it is therefore manifest that

\[
B\Delta \Gamma > 3(\Phi Z + \Theta H + \Pi I + \Gamma O),
\]

but is less than thrice the aforementioned areas.


Any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle having the same base and equal height.

For let \( \Delta \Delta \beta \gamma \) be a segment bounded by a straight line and a section of a right-angled cone, and let \( \Delta \beta \gamma \) be a triangle having the same base as the segment and equal height, and let the area \( K \) be four-thirds of the triangle \( \Delta \beta \gamma \). It is required to prove that it is equal to the segment \( \Delta \Delta \beta \gamma \).

For if it is not equal, it is either greater or less. Let the segment \( \Delta \Delta \beta \gamma \) first be, if possible, greater than the area \( K \). Now I have inscribed the triangles \( \Delta \Delta \beta, \beta \gamma \), as aforesaid, and I have inscribed in the remaining segments other triangles having the same

\[\text{For } B\Delta \Gamma = 3(P + X + \Psi + \Omega + \Delta) > 3(\Delta + \Omega + \Psi + X).\]

\[\text{In Prop. 15 Archimedes shows that the same theorem holds good even if } \beta \gamma \text{ is not at right angles to the axis. It is then proved in Prop. 16, by the method of exhaustion, that the segment is equal to one-third of the triangle } \beta \gamma \Delta. \text{ This is done by showing, on the basis of the "Axiom of Archimedes," that by taking enough parts the difference between the circumscribed and the inscribed figures can be made as small as we please. It is equivalent to integration. From this it is easily proved that the segment is equal to four-thirds of a triangle with the same base and equal height (Prop. 17).}\]

\[\text{In earlier propositions Archimedes has used the same procedure as he now describes. } \Delta, E \text{ are the points in which the diameter through the mid-points of } \Delta B, \beta \gamma \text{ meet the curve.}\]
This was proved geometrically in Prop. 23, and is proved generally in Eucl. ix. 35. It is equivalent to the summation

\[ 1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^{n-1} = \frac{\frac{1}{4} - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}}. \]
base as the segments and equal height, and so on continually I inscribe in the resulting segments two

\[\begin{array}{c}
  K \\
\end{array}\]

\[\begin{array}{c}
  Z \\
\end{array}\]

\[\begin{array}{c}
  H \\
\end{array}\]

\[\begin{array}{c}
  \oplus \\
\end{array}\]

\[\begin{array}{c}
  I \\
\end{array}\]

triangles having the same base as the segments and equal height; then there will be left [at some time] segments less than the excess by which the segment $A\Delta B\Gamma$ exceeds the area $K$ [Prop. 20, coroll.]. Therefore the inscribed polygon will be greater than $K$; which is impossible. For since the areas successively formed are each four times as great as the next, the triangle $A\Gamma$ being four times the triangles $A\Delta B$, $B\Gamma$ [Prop. 21], then these last triangles four times the triangles inscribed in the succeeding segments, and so on continually, it is clear that the sum of all the areas is less than four-thirds of the greatest [Prop. 23], and $K$ is equal to four-thirds of the greatest area. Therefore the segment $A\Delta B\Gamma$ is not greater than the area $K$.

Now let it be, if possible, less. Then let

\[Z = A\Gamma, \ H = \frac{1}{4} Z, \ \Theta = \frac{1}{4} H,\]

and so on continually, until the last [area] is less than
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υπεροχάς, ἃ υπερέχει τὸ Κ χωρίον τοῦ τμάματος, καὶ ἔστω ἐλασσόν τὸ Ι. ἔστω δὴ τὰ Ζ, Η, Θ, I χωρία καὶ τὸ τρίτον τοῦ I ἐπίτρυτα τοῦ Ζ. ἔστων δὲ καὶ τὸ K τοῦ Z ἐπίτρυτον. ἵσον ἢ ρὰ τὸ K τοῖς Ζ, Η, Θ, I καὶ τῷ τρίτῳ μέρει τοῦ I. ἐπεὶ οὖν τὸ K χωρίον τῶν μὲν Ζ, Η, Θ, I χωρίων ὑπερέχει ἐλασσόν τοῦ Ι, τοῦ δὲ τμάματος μεῖζον τοῦ Ι, δήλον, ὡς μείζονά ἐντι τὰ Ζ, Η, Θ, I χωρία τοῦ τμάματος. ὁπερ ἄδυνατον ἐδείχθη γάρ, ὦτι, ἐὰν ἤ ὁποσαούν χωρία ἐξῆς κείμενα ἐν τετραπλασίονι λόγῳ, τὸ δὲ μέγιστον ἵσον ἢ τῷ εἰς τὸ τμάμα ἐγγραφομένῳ τριγώνῳ, τὰ σύμπαντα χωρία ἐλάσσονα ἐσσεῖται τοῦ τμάματος. οὐκ ἢρα τὸ ΑΔΒΕΓ τμάμα ἐλασσόν ἢστι τοῦ Κ χωρίου. ἐδείχθη δὲ, ὅτι οὐδὲ μεῖζον ἵσον ἢρα ἢστιν τῷ Κ. τὸ δὲ Κ χωρίον ἐπίτρυτον ἢστι τοῦ τριγώνου τοῦ ΑΒΓ. καὶ τὸ ΑΔΒΕΓ ἢρα τμάμα ἐπίτρυτον ἢστι τοῦ ΑΒΓ τριγώνου.

(k) Hydrostatics

(i.) Postulates

ii. 318. 2-8

*The Greek text of the book On Floating Bodies, the earliest extant treatise on hydrostatics, first became available in 1906 when Heiberg discovered at Constantinople the ms. which he terms C. Unfortunately many of the readings are doubtful, and those who are interested in the text should consult the Teubner edition. Still more unfortunately, it is incomplete; but, as the whole treatise was translated into Latin in 1269 by William of Moerbeke from a Greek ms. 242
the excess by which the area $K$ exceeds the segment [Eucl. x. 1], and let $I$ be [the area] less [than this excess]. Now
\[ Z + H + \Theta + I + \frac{1}{3}I = \frac{4}{3}Z. \]  
[Prop. 23]
But
\[ K = \frac{4}{3}Z; \]
therefore
\[ K = Z + H + \Theta + I + \frac{1}{3}I. \]
Therefore since the area $K$ exceeds the areas $Z$, $H$, $\Theta$, $I$ by an excess less than $I$, and exceeds the segment by an excess greater than $I$, it is clear that the areas $Z$, $H$, $\Theta$, $I$ are greater than the segment; which is impossible; for it was proved that, if there be any number of areas in succession such that each is four times the next, and the greatest be equal to the triangle inscribed in the segment, then the sum of the areas will be less than the segment [Prop. 22]. Therefore the segment $\Delta\text{BET}$ is not less than the area $K$. And it was proved not to be greater; therefore it is equal to $K$. But the area $K$ is four-thirds of the triangle $\text{ABI}$; and therefore the segment $\Delta\text{BET}$ is four-thirds of the triangle $\text{ABI}$.

(k) Hydrostatics

(i.) Postulates

Archimedes, On Floating Bodies a 1., Archim. ed. Heiberg ii. 318. 2-8

Let the nature of a fluid be assumed to be such that, of its parts which lie evenly and are continuous, since lost, it is possible to supply the missing parts in Latin, as is done for part of Prop. 2. From a comparison with the Greek, where it survives, William's translation is seen to be so literal as to be virtually equivalent to the original. In each case Heiberg's figures are taken from William's translation, as they are almost unrecognizable in C; for convenience in reading the Greek, the figures are given the appropriate Greek letters in this edition.
GREEK MATHEMATICS


eχέων εώντων ἐξωθεῖσθαι τὸ ἱσσον θλιβόμενον ὑπὸ τοῦ μάλλον θλιβομένου, καὶ ἐκαστὸν δὲ τῶν μερῶν αὐτοῦ θλιβεσθαι τῷ ὑπεράνω αὐτοῦ ύγρῷ κατὰ κάθετον ἐόντι, εἴ καὶ μὴ τὸ ύγρὸν ἢ καθεργ-μένον ἐν τινι καὶ ὑπὸ ἄλλου τινὸς θλιβóμενον.

Ibid. i., Archim. ed. Heiberg ii. 336. 14-16

Ὑποκεῖσθω, τῶν ἐν τῷ ύγρῷ ἄνω φερομένων ἐκαστὸν ἀναφέρεσθαι κατὰ τὰν κάθετον τὰν διὰ τοῦ κέντρου τοῦ βάρεος αὐτοῦ ἀγμέναν.

(ii.) Surface of Fluid at Rest

Ibid. i., Prop. 2, Archim. ed. Heiberg ii. 319. 7–320. 30

Omnis humidī consistentis ita, ut maneat inmotum, superficies habebit figuram sperae habentis centrum idem cum terra.

Intelligatur enim humidum consistens ita, ut maneat non motum, et secetur ipsius superficies plano per centrum terrae, sit autem terrae centrum K, superficieī autem sectio linea ABGD. Dico itaque,

lineam ABGD circuli esse periferiam, centrum autem ipsius K.

Si enim non est, rectae a K ad lineam ABGD

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that which is under the lesser pressure is driven along by that under the greater pressure, and each of its parts is under pressure from the fluid which is perpendicularly above it, except when the fluid is enclosed in something and is under pressure from something else.

*Ibid. i., Archim. ed. Heiberg ii. 336. 14-16*

Let it be assumed that, of bodies which are borne upwards in a fluid, each is borne upwards along the perpendicular drawn through its centre of gravity.a

(ii.) *Surface of Fluid at Rest*

*Ibid. i., Prop. 2, Archim. ed. Heiberg ii. 319. 7-320. 30*

The surface of any fluid at rest is the surface of a sphere having the same centre as the earth.

For let there be conceived a fluid at rest, and let its surface be cut by a plane through the centre of the earth, and let the centre of the earth be K, and let the section of the surface be the curve ABΓΔ. Then I say that the curve ABΓΔ is an arc of a circle whose centre is K.

For if it is not, straight lines drawn from K to the

- These are the only assumptions, other than the assumptions of Euclidean geometry, made in this book by Archimedes; if the object of mathematics be to base the conclusions on the fewest and most "self-evident" axioms, Archimedes' treatise *On Floating Bodies* must indeed be ranked highly.

- The earlier part of this proposition has to be given from William of Moerbeke's translation. The diagram is here given with the appropriate Greek letters.
occurrentes non erunt aequales. Sumatur itaque aliqua recta, quae est quarundam quidem a K occurrentium ad lineam ABGD maior, quarundam autem minor, et centro quidem K, distantia autem sumptae lineae circulus describatur; cadet igitur periferia circuli habenshoc quidem extra lineam ABGD, hoc autem intra, quoniam quae ex centro quarundam quidem a K occurrentium ad lineam ABGD est maior, quarundam autem minor. Sit igitur descripti circuli periferia quae ZBH, et a B ad K recta ducatur, et copulentur quae ZK, KEL aequales facientes angulos, describatur autem et centro K periferia quaedam quae XOP in plano et in humido; partes itaque humidi quae secundum XOP periferiam ex aequo sunt positae et continuae inuicem. Et premuntur quae quidem secundum XO periferiam humido quod secundum ZB locum, quae autem secundum periferiam OP humido quod secundum BE locum; inaequaliter igitur premuntur partes humidi quae secundum periferiam XO ei quae [ὁ]¹ κατὰ τὰν ΟΠ· ὡστε ἐξωθήσονται τὰ ἱσον ὥστε ὥστε μένει αἱ κῦκλοι περιφέρειαι εἰμεν ὥστε μένειν ἀκίνητον· ἀναγκαῖον ἀρὰ τὰν ἈΒΓΔ γραμμάριον κῦκλον περιφέρειαν εἰμεν καὶ κέντρον αὐτὰς τὸ K. ὁμοίως δὴ δειχθῆσαι καὶ, ὅπως καὶ ἄλλως ἡ ἐπιφάνεια τοῦ ὑγροῦ ἐπιπέδῳ τμαθῇ διὰ τοῦ κέντρου τὰς γᾶς, ὅτι ἀ τομα ἐσσείται κῦκλον περιφέρειαν, καὶ κέντρον αὐτὰς ἐσσείται, δ καὶ τὰς γᾶς ἢστὶ κέντρον. δῆλον οὖν, ὅτι ἡ ἐπιφάνεια τοῦ ὑγροῦ καθεστακότος ἀκίνητον σφαῖρας ἔχει τὸ σχῆμα τὸ αὐτὸ κέντρον ἐχούσας τὰ γὰ, ἐπειδὴ

1 ὁ om. Heiberg.
curve $AB\Gamma\Delta$ will not be equal. Let there be taken, therefore, any straight line which is greater than some of the straight lines drawn from $K$ to the curve $AB\Gamma\Delta$, but less than others, and with centre $K$ and radius equal to the straight line so taken let a circle be described; the circumference of the circle will fall partly outside the curve $AB\Gamma\Delta$, partly inside, inasmuch as its radii are greater than some of the straight lines drawn from $K$ to the curve $AB\Gamma\Delta$, but less than others. Let the arc of the circle so described be $ZBH$, and from $B$ let a straight line be drawn to $K$, and let $ZK$, $KE\Lambda$ be drawn making equal angles [with $KB$], and with centre $K$ let there be described, in the plane and in the fluid, an arc $\Xi\Omega\Pi$; then the parts of the fluid along $\Xi\Omega\Pi$ lie evenly and are continuous [v. supra, p. 243]. And the parts along the arc $\Xi O$ are under pressure from the portion of the fluid between it and $ZB$, while the parts along the arc $\Omega\Pi$ are under pressure from the portion of the fluid between it and $BE$; therefore the parts of the fluid along $\Xi O$ and the parts of the fluid along $\Omega\Pi$ are under unequal pressures; so that the parts under the lesser pressure are thrust along by the parts under the greater pressure [v. supra, p. 245]; therefore the fluid will not remain at rest. But it was postulated that the fluid would remain unmoved; therefore the curve $AB\Gamma\Delta$ must be an arc of a circle with centre $K$. Similarly it may be shown that, in whatever other manner the surface be cut by a plane through the centre of the earth, the section is an arc of a circle and its centre will also be the centre of the earth. It is therefore clear that the surface of the fluid remaining at rest has the form of a sphere with the same centre as the earth, since it is such
(iii.) Solid immersed in a Fluid


Τὰ βαρύτερα τοῦ ὕγροῦ ἀφεθέντα εἰς τὸ ὕγρον οὐσεῖται κάτω, ἐστὶν ἀν καταβάντι, καὶ ἐσοῦνται κουφότερα ἐν τῷ ὕγρῳ τοσοῦτον, ὅσον ἔχει τὸ βάρος τοῦ ὕγροῦ τοῦ ταλικοῦτον ὄγκον ἔχοντος, ἀλλὰς ἐστὶν ὁ τοῦ στερεοῦ μεγέθεος ὄγκος.

"Ὅτι μὲν οὖν οὐσεῖται ἐς τὸ κάτω, ἐστὶν ἀν καταβάντι, δῆλον· τὰ γὰρ ὑποκάτω αὐτοῦ μέρεα τοῦ ὕγροῦ θλιβησοῦνται μᾶλλον τῶν ἐς ἵσον αὐτοῖς κειμένων μερέων, ἐπειδὴ βαρύτερον ὑπόκειται τὸ στερεὸν μέγεθος τοῦ ὕγροῦ· ὅτι δὲ κουφότερα ἐσοῦνται, ὡς εἴρηται, δειχθῆσται.

"Εστὼ τι μέγεθος τὸ Α, ὃ ἐστὶ βαρύτερον τοῦ ὕγροῦ, βάρος δὲ ἐστὶν τοῦ μὲν ἐν φ Α μεγέθεος τὸ ΒΓ, τοῦ δὲ ὕγροῦ τοῦ ἵσον ὄγκον ἔχοντος τῶν Α τὸ Β. δεικτέον, ὅτι τὸ Α μέγεθος ἐν τῷ ὕγρῳ ἐὸν βάρος ἐξεί ἵσον τῶν Γ.

Δελάφθω γὰρ τι μέγεθος τὸ ἐν φ τὸ Δ κουφότερον τοῦ ὕγροῦ τοῦ ἵσον ὄγκον ἔχοντος αὐτῶ, ἐστὼ δὲ τοῦ μὲν ἐν φ τὸ Δ μεγέθεος βάρος ἵσον τῶν Β βάρει, τοῦ δὲ ὕγροῦ τοῦ ἵσον ὄγκον ἔχοντος τῶν Δ μεγέθει τὸ βάρος ἐστὶν ἵσον τῶν ΒΓ βάρει.

* Or, as we should say, “lighter by the weight of fluid displaced.”
that, when it is cut [by a plane] always passing through the same point, the section is an arc of a circle having for centre the point through which it is cut by the plane [Prop. 1].

(iii.) *Solid immersed in a Fluid*


*Solids heavier than a fluid will, if placed in the fluid, sink to the bottom, and they will be lighter [if weighed] in the fluid by the weight of a volume of the fluid equal to the volume of the solid.*

That they will sink to the bottom is manifest; for the parts of the fluid under them are under greater pressure than the parts lying evenly with them, since it is postulated that the solid is heavier than water; that they will be lighter, as aforesaid will be [thus] proved.

Let A be any magnitude heavier than the fluid, let the weight of the magnitude A be $B + \Gamma$, and let the weight of fluid having the same volume as A be $B$. It is required to prove that in the fluid the magnitude A will have a weight equal to $\Gamma$.

For let there be taken any magnitude $\Delta$ lighter than the same volume of the fluid such that the weight of the magnitude $\Delta$ is equal to the weight $B$, while the weight of the fluid having the same volume as the magnitude $\Delta$ is equal to the weight $B + \Gamma$. 249
This proposition suggests a method, alternative to that given by Vitruvius (v. supra, pp. 36-39, especially p. 38 n. a), whereby Archimedes may have discovered the proportions of gold and silver in King Hiero’s crown.

Let \( w \) be the weight of the crown, and let \( w_1 \) and \( w_2 \) be the weights of gold and silver in it respectively, so that \( w = w_1 + w_2 \).

Take a weight \( w \) of gold and weigh it in a fluid, and let the loss of weight be \( P_1 \). Then the loss of weight when a weight \( w_1 \) of gold is weighed in the fluid, and consequently the weight of fluid displaced, will be \( \frac{w_1}{w} \cdot P_1 \).
Then if we combine the magnitudes $\Lambda$, $\Delta$, the combined magnitude will be equal to the weight of the same volume of the fluid; for the weight of the combined magnitudes is equal to the weight $(B + \Gamma) + B$, while the weight of the fluid having the same volume as both the magnitudes is equal to the same weight. Therefore, if the [combined] magnitudes are placed in the fluid, they will balance the fluid, and will move neither upwards nor downwards [Prop. 3]; for this reason the magnitude $\Lambda$ will move downwards, and will be subject to the same force as that by which the magnitude $\Delta$ is thrust upwards, and since $\Delta$ is lighter than the fluid it will be thrust upwards by a force equal to the weight $\Gamma$; for it has been proved that when solid magnitudes lighter than the fluid are forcibly immersed in the fluid, they will be thrust upwards by a force equal to the difference in weight between the magnitude and an equal volume of the fluid [Prop. 6]. But the fluid having the same volume as $\Delta$ is heavier than the magnitude $\Delta$ by the weight $\Gamma$; it is therefore plain that the magnitude $\Lambda$ will be borne upwards by a force equal to $\Gamma$.

Now take a weight $\omega$ of silver and weigh it in the fluid, and let the loss of weight be $P_3$. Then the loss of weight when a weight $\omega_2$ of silver is weighed in the fluid, and consequently the weight of fluid displaced, will be $\frac{\omega_2}{\omega} \cdot P_3$.

Finally, weigh the crown itself in the fluid, and let the loss of weight, and consequently the weight of fluid displaced, be $P$.

It follows that $\frac{\omega_1}{\omega} \cdot P_1 + \frac{\omega_2}{\omega} \cdot P_3 = P$,

whence $\frac{\omega_1}{\omega} = \frac{P_3 - P}{P - P_1}$. 251
(iv.) Stability of a Paraboloid of Revolution


To ὁ ὀρθὸν τμῆμα τοῦ ὀρθογωνίου κωνοειδέος, ὅταν τὸν ἄξονα ἔχῃ μὴ μείζονα ἡ ἡμόλιον τᾶς μέχρι τοῦ ἄξονος, πάντα λόγον ἔχον ποτὲ τὸ ὑγρὸν τῷ βάρει, ἀφεθέν εἰς τὸ ὑγρὸν οὕτως, ὥστε τὰν βάσιν αὐτοῦ ἀπτεσθαι τοῦ ὑγροῦ, τεθὲν κεκλιμένον οὐ μενεὶ κεκλιμένον, ἀλλὰ ἀποκαταστασεῖται ὁρθὸν. ὁρθὸν δὲ λέγω καθεστάκειν τὸ τουῷ τμῆμα, ὁπόταν τὸ ἀποτετμακὸς αὐτὸ ἐπίπεδον παρὰ τὰν ἐπιφάνειαν ἢ τοῦ ὑγροῦ.

"Εστώ τμῆμα ὀρθογωνίου κωνοειδέος, οὐν ἕρηται, καὶ κεῖσθω κεκλιμένον. δεικτέον, ὅτι οὐ μενεὶ, ἀλλ’ ἀποκαταστασεῖται ὁρθὸν.

Ὑμαθέντος δὴ αὐτοῦ ἐπιπέδῳ διὰ τοῦ ἄξονος ὁρθῶν ποτὲ τὸ ἐπίπεδον τὸ ἐπὶ τὰς ἐπιφάνειας τοῦ ὑγροῦ τμάματος ἑστω τομὰ ἀ ΑΠΟΛ ὀρθογωνίου κώνου τομά, ἄξων δὲ τοῦ τμάματος καὶ διάμετρος τὰς τομάς ᾧ ΝΟ, τὰς δὲ τοῦ ὑγροῦ ἐπιφάνειας τομά ᾧ ΙΣ. ἐπεὶ οὖν τὸ τμῆμα οὐκ ἐστὶν ὁρθὸν, οὐκ ἂν εἴη παράλληλος ᾧ ΑΛ ταῖς ΙΣ· ὥστε οὗ ποὺσεὶ ὁρθὰν γωνίαν ᾧ ΝΟ ποτὲ τὰν ΙΣ. ἄχθω

* Writing of the treatise On Floating Bodies, Heath (H.G.M. ii. 94-95) justly says: "Book ii., which investigates fully the conditions of stability of a right segment of a paraboloid of revolution floating in a fluid for different values of the specific gravity and different ratios between the axis or height of the segment and the principal parameter of the generating parabola, is a veritable tour de force which must be read in full to be appreciated."

b In this technical term the "axis" is the axis of the
(iv.) Stability of a Paraboloid of Revolution

If there be a right segment of a right-angled conoid, whose axis is not greater than one-and-a-half times the line drawn as far as the axis, and whose weight relative to the fluid may have any ratio, and if it be placed in the fluid in an inclined position in such a manner that its base do not touch the fluid, it will not remain inclined but will return to the upright position. I mean by returning to the upright position the figure formed when the plane cutting off the segment is parallel to the surface of the fluid.

Let there be a segment of a right-angled conoid, such as has been stated, and let it be placed in an inclined position. It is required to prove that it will not remain there but will return to the upright position.

Let the segment be cut by a plane through the axis perpendicular to the plane which forms the surface of the fluid, and let APOL be the section of the segment, being a section of a right-angled cone [De Con. et Sphaer. 11], and let NO be the axis of the segment and the axis of the section, and let IΣ be the section of the surface of the liquid. Then since the segment is not upright, ΛΛ will not be parallel to IΣ; and therefore NO will not make a right angle

right-angled cone from which the generating parabola is derived. The latus rectum is “the line which is double of the line drawn as far as the axis” (ἀ διπλασία τὰς μέχρι τοῦ ἄξονος); and so the condition laid down by Archimedes is that the axis of the segment of the paraboloid of revolution shall not be greater than three-quarters of the latus rectum or principal parameter of the generating parabola.
οὖν παράλληλος ἀ ἐφαπτομένα ἀ ΚΩ τᾶς τοῦ κόνου τομᾶς κατὰ τὸ Π, καὶ ἀπὸ τοῦ Π παρὰ τὰν

ΝΟ ἄχθω ἀ ΠΦ· τέμνει δὴ ἀ ΠΦ δίχα τὰν ΙΣ·

δεδεικται γὰρ ἐν τοῖς κωνικοῖς. τετμᾶσθω ἀ
ΠΦ, ὥστε εἴμεν διπλασιὰν τὰν ΠΒ τὰς ΒΦ, καὶ
ἀ ΝΟ κατὰ τὸ Ρ τετμᾶσθω, ὡστε καὶ τὰν ΟΡ τὰς
ΡΝ διπλασιὰν εἴμεν· ἐσσεῖται δὴ τοῦ μείζονος
ἀποτμάματος τοῦ στερεοῦ κέντρον τοῦ βάρεος τὸ
Ρ, τοῦ δὲ κατὰ τὰν ΠΠΟΣ τὸ Β· δεδεικται γὰρ
ἐν ταῖς Ἰσορροπίαις, ὦτι παντὸς ὀρθογώνιον
κωνοειδέος τμάματος τὸ κέντρον τοῦ βάρεος ἐστὶν
ἐπὶ τοῦ ἄξονος διηρημένου οὐτῶς, ὥστε τὸ ποτὶ
tὰ κορυφὰ τοῦ ἄξονος τμάμα διπλάσιον εἴμεν τοῦ
λοιποῦ. ἀφαιρεθέντος δὴ τοῦ κατὰ τὰν ΠΠΟΣ
τμάματος στερεοῦ ἀπὸ τοῦ ὅλου τοῦ λοιποῦ κέντρον
ἐσσεῖται τοῦ βάρεος ἐπὶ τὰς ΒΓ εὐθείας· δεδεικται
γὰρ τοῦτο ἐν τοῖς Στοιχείοις τῶν μηχανικῶν, ὦτι,
εἰ καὶ μέγεθος ἀφαιρεθῇ μὴ τὸ αὐτὸ κέντρον ἔχω
τοῦ βάρεος τῷ ὅλῳ μεγέθει, τοῦ λοιποῦ τὸ κέντρον
ἐσσεῖται τοῦ βάρεος ἐπὶ τὰς εὐθείας τὰς ἐπιζευ-

γνυσας τὰ κέντρα τοῦ τε ὅλου μεγέθεος καὶ τοῦ

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with Σ. Therefore let ΚΟ be drawn parallel [to Σ] and touching the section of the cone at Π, and from Π let ΠΦ be drawn parallel to NO; then ΠΦ bisects Σ—for this is proved in the [Elements of] Conics.\(^a\)

Let ΠΦ be cut so that ΠΒ = 2ΒΦ, and let NO be cut at P so that OP = 2PN; then P will be the centre of gravity of the greater segment of the solid, and B that of ΠΟΣ; for it is proved in the books On Equilibriums that the centre of gravity of any segment of a right-angled conoid is at the point dividing the axis in such a manner that the segment towards the vertex of the axis is double of the remainder.\(^b\)

Now if the solid segment ΠΟΣ be taken away from the whole, the centre of gravity of the remainder will lie upon the straight line BV; for it has been proved in the Elements of Mechanics that if any magnitude be taken away not having the same centre of gravity as the whole magnitude, the centre of gravity of the remainder will be on the straight line joining the centres [of gravity] of the whole magnitude and of the part

\(^a\) Presumably in the works of Aristaeus or Euclid, but it is also Quad. Parab. 1.

\(^b\) The proof is not in any extant work by Archimedes.
If the normal at $\Pi$ meets the axis in $M$, then $OM$ is greater than "the line drawn as far as the axis" except in the case where $\Pi$ coincides with the vertex, which case is excluded by the conditions of this proposition. Hence $OM$ is always greater than $OP$; and because the angle $\Omega\Pi\Pi$ is right, the angle $\Omega\Pi\Pi$ must be acute.
taken away, produced from the extremity which is the centre of gravity of the whole magnitude [De Plan. Aequil. i. 8]. Let BP then be produced to Γ, and let Γ be the centre of gravity of the remaining magnitude. Then, since $NO = \frac{3}{2} \cdot OP$, and $NO > \frac{3}{2} \cdot (\text{the line drawn as far as the axis})$, it is clear that $PO > (\text{the line drawn as far as the axis})$; therefore $ΠP$ makes unequal angles with $KΩ$, and the angle $ΠΠΩ$ is acute\(^a\); therefore the perpendicular drawn from $P$ to $ΠΩ$ will fall between $Π, Ω$. Let it fall as $PΘ$; then $PΘ$ is perpendicular to the cutting plane containing $ΣΙ$, which is on the surface of the fluid. Now let lines be drawn from $B, Γ$ parallel to $PΘ$; then the portion of the magnitude outside the fluid will be subject to a downward force along the line drawn through $Γ$—for it is postulated that each weight is subject to a downward force along the perpendicular drawn through its centre of gravity\(^b\); and since the magnitude in the fluid is lighter than the fluid,\(^c\) it will be subject to an upward force along the perpendicular drawn through $B$.\(^d\) But, since they are not subject to contrary forces along the same perpendicular, the figure will not remain at rest but the portion on the side of $A$ will move upwards and the portion on the side of $Δ$ will move downwards, and this will go on continually until it is restored to the upright position.

\(^a\) Cf. supra, p. 245; possibly a similar assumption to this effect has fallen out of the text.

\(^b\) A tacit assumption, which limits the generality of the opening statement of the proposition that the segment may have any weight relative to the fluid.

\(^c\) v. supra, p. 251.
XVIII. ERATOSTHENES
Several of Eratosthenes' achievements have already been described—his solution of the Delian problem (vol. i. pp. 290-297), and his sieve for finding successive odd numbers (vol. i. pp. 100-103). Archimedes, as we have seen, dedicated the Method to him, and the Cattle Problem, as we have also seen, is said to have been sent through him to the Alexandrian mathematicians. It is generally supposed that Ptolemy credits him with having calculated the distance between the tropics (or twice the obliquity of the ecliptic) at 11/83rds. of a complete circle or 47° 29′ 39″, but Ptolemy's meaning is not clear. Eratosthenes also calculated the distances of the sun and moon from the earth and the size of the sun. Fragments of an astronomical poem which he wrote under the title
XVIII. ERATOSTHENES

(a) General

Suidas, s.v. Eratosthenes

Eratosthenes, son of Aglaus, others say of Ambrosius; a Cyrenean, a pupil of the philosopher Ariston of Chios, of the grammarian Lysanias of Cyrene and of the poet Callimachus; he was sent for from Athens by the third Ptolemy and stayed till the fifth. Owing to taking second place in all branches of learning, though approaching the highest excellence, he was called Beta. Others called him a Second or New Plato, and yet others Pentathlon. He was born in the 126th Olympiad and died at the age Hermes have survived. He was the first person to attempt a scientific chronology from the siege of Troy in two separate works, and he wrote a geographical work in three books. His writings are critically discussed in Bernhardy's Eratosthenica (Berlin, 1822).

Callimachus, the famous poet and grammarian, was also a Cyrenean. He opened a school in the suburbs of Alexandria and was appointed by Ptolemy Philadelphus chief librarian of the Alexandrian library, a post which he held till his death c. 240 B.C. Eratosthenes later held the same post.

Euergetes I (reigned 246-221 B.C.), who sent for him to be tutor to his son and successor Philopator (v. vol. 1. pp. 256, 296).

Epiphanes (reigned 204-181 B.C.).
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πιάδι καὶ ἐτελεύτησεν ἔτων γεγονός, ἀποσχόμενος τροφῆς διὰ τὸ ἄμβλυώττειν, μαθητὴν ἐπίσημον καταληπτῶν Ἀριστοφάνην τὸν Βυζάντιον· οὗ πάλιν Ἀρίσταρχος μαθητής. μαθηταὶ δὲ αὐτοῦ Μνασέας καὶ Μένανδρος καὶ Ἀριστις. ἔγραψε δὲ φιλόσοφα καὶ ποιήματα καὶ ἱστορίας, Ἀστρονομίαν ἡ Καταστερισμοῦ;¹ Περὶ τῶν κατὰ φιλοσοφίαν αἱρέσεων, Περὶ ἀλυπίας, διαλόγους πολλοὺς καὶ γραμματικὰ συχνά.

(b) ON MEANS

Papp. Coll. vii. 3, ed. Hultsch 636. 18-25

Τῶν δὲ προειρημένων τοῦ Ἀναλυομένου βιβλίων ἡ τάξις ἐστὶν τοιαύτη . . . Ἐρατοσθένους περὶ μεσοτήτων δύο.

Papp. Coll. vii. 21, ed. Hultsch 660. 18-662. 18

Τῶν τόπων καθόλου οἱ μὲν εἰσὶν ἐφεκτικοί, ὡς καὶ Ἀπολλώνιος πρὸ τῶν ἰδίων στοιχείων λέγει σημείον μὲν τόπον σημείον, γραμμῆς δὲ τόπον γραμμῆν, ἐπιφανείας δὲ ἐπιφάνειαν, στερεοῦ δὲ στερεόν, οἱ δὲ διεξοδικοί, ὡς σημείον μὲν γραμμῆν, γραμμῆς δ' ἐπιφάνειαν, ἐπιφανείας δὲ στερεόν,

¹ Καταστερισμοῦ cons. Portus, Καταστηρίγμους codd.

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¹ Not, of course, Aristarchus of Samos, the mathematician, but the celebrated Samothracian grammarian.

² Mnaseas was the author of a work entitled Περίπλοις, whose three sections dealt with Europe, Asia and Africa, and a collection of oracles given at Delphi.

³ This work is extant, but is not thought to be genuine in
of eighty of voluntary starvation, having lost his sight; he left a distinguished pupil, Aristophanes of Byzantium; of whom in turn Aristarchus was a pupil. Among his pupils were Mnaseas, Menander and Aristis. He wrote philosophical works, poems and histories, *Astronomy or Placings Among the Stars,* *On Philosophical Divisions,* *On Freedom from Pain,* many dialogues and numerous grammatical works.

(b) On Means

Pappus, *Collection* vii. 3, ed. Hultsch 636. 18-25

The order of the aforesaid books in the *Treasury of Analysis* is as follows...the two books of Eratosthenes *On Means."

Pappus, *Collection* vii. 21, ed. Hultsch 660. 18-662. 18

Loci in general are termed fixed, as when Apollonius at the beginning of his own *Elements* says the locus of a point is a point, the locus of a line is a line, the locus of a surface is a surface and the locus of a solid is a solid; or progressive, as when it is said that the locus of a point is a line, the locus of a line is a surface and the locus of a surface is a solid; or circumambient as its extant form; it contains a mythology and description of the constellations under forty-four heads. The general title Αστρονομία may be a mistake for Αστρονομεία; elsewhere it is alluded to under the title Κατάλογοι.

The inclusion of this work in the *Treasury of Analysis*, along with such works as those of Euclid, Aristaeus and Apollonius, shows that it was a standard treatise. It is not otherwise mentioned, but the loci with reference to means referred to in the passage from Pappus next cited were presumably discussed in it.
oί δὲ ἀναστροφικοί, ὡς σημείου μὲν ἐπιφάνειαν, γραμμὴς δὲ στερεόν. [. . . oί δὲ ύπὸ 'Ἐρατοσθέ
νους ἐπιγραφέντες τόποι πρὸς μεσόττας ἐκ τῶν
προειρημένων εἰσών τῷ γένει, ἀπὸ δὲ τῆς ἰδιότητος
tῶν ὑποθέσεων . . . ἐκείνως.]

(c) The “Platonicus”

Theon Smyr., ed. Hiller 81. 17–82. 5

"Ἐρατοσθένης δὲ ἐν τῷ Πλατωνικῷ φησι, μὴ
tαύτὸν εἶναι διάστημα καὶ λόγον. ἐπειδὴ λόγος
μὲν ἔστι δύο μεγεθῶν ἤ πρὸς ἄλληλα ποιά σχέσις,
γίνεται δ’ αὐτὴ καὶ ἐν διαφόροις (καὶ ἐν ἀδιαφό-
ροις). 1 οἶνον ἐν ὧ λόγῳ ἐστὶ τὸ αἰσθητὸν πρὸς
tὸ νοητὸν, ἐν τούτῳ δόξα πρὸς ἐπιστήμην, καὶ
dιαφέρει καὶ τὸ νοητὸν τοῦ ἐπιστήμου ὡς καὶ ἡ
dόξα τοῦ αἰσθητοῦ. διάστημα δὲ ἐν διαφέρουσι
μόνον, ἡ κατὰ τὸ μέγεθος ἡ κατὰ ποιότητα ἡ κατὰ
θέσιν ἡ ἄλλως ὀπώσοιν. δὴ λοι ἐν καὶ ἐνεύθεν,

1 The passage of which this forms the concluding sentence
is attributed by Hultsch to an interpolator. To fill the
lacuna before ἐκείνως he suggests ἀνόμου ἐκείνως, following
Halley’s rendering, “diversa sunt ab illis.”

2 καὶ ἐν ἀδιαφόροις add. Hiller.

* Tannery conjectured that these were the loci of points
such that their distances from three fixed lines provided a
“médiaité,” i.e., loci (straight lines and conics) which can be
represented in trilinear co-ordinates by such equations as

\[
2y = x + z, \quad y^2 = xz, \quad y(x + z) = 2xz, \quad x(x - y) = z(y - z),
\]

\[
x(x - y) = y(y - z);
\]

these represent respectively the arithmetic, geometric and
harmonic means, and the means subcontrary to the harmonic
and geometric means (v. vol. i. pp. 122-125). Zeuthen has
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when it is said that the locus of a point is a surface and
the locus of a line is a solid. [. . . the loci described
by Eratosthenes as having reference to means belong
to one of the aforesaid classes, but from a peculiarity
in the assumptions are unlike them.]

(c) The "Platonicus"

Theon of Smyrna, ed. Hiller 81. 17–82. 5

Eratosthenes in the Platonicus \(^b\) says that interval
and ratio are not the same. Inasmuch as a ratio is a
sort of relationship of two magnitudes one towards
the other,\(^c\) there exists a ratio both between terms
that are different and also between terms that are not
different. For example, the ratio of the perceptible
to the intelligible is the same as the ratio of opinion
to knowledge, and the difference between the intel-
ligible and the known is the same as the difference of
opinion from the perceptible.\(^d\) But there can be an
interval only between terms that are different,
according to magnitude or quality or position or in
some other way. It is thence clear that ratio is

an alternative conjecture on similar lines (Die Lehre von den
Kegelschnitten im Altertum, pp. 320-321).

\(^b\) Theon cites this work in one other passage (ed. Hiller 2.
3-12) telling how Plato was consulted about the doubling of
the cube; it has already been cited (vol. i. p. 256). Eratost-
thenes' own solution of the problem has already been given
in vol. i. pp. 290-297, and a letter purporting to be from
Eratosthenes to Ptolemy Euergetes is given in vol. i. pp. 256-
261. Whether the Platonicus was a commentary on Plato
or a dialogue in which Plato was an interlocutor cannot be
decided.

\(^c\) Cf. Eucl. v. Def. 3, cited in vol. i. p. 444.

\(^d\) A reference to Plato, Rep. vi. 509 d—511 e, vii. 517 \(\Delta\)—
518 b.
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ὅτι λόγος διαστήματος ἐτερον τὸ γὰρ ἦμιον πρὸς τὸ διπλάσιον (καὶ τὸ διπλάσιον πρὸς τὸ ἦμιον)¹ λόγον μὲν οὐ τὸν αὐτὸν ἔχει, διάστημα δὲ τὸ αὐτὸ.

(d) MEASUREMENT OF THE EARTH

Cleom. De motu cirr. i. 10. 52, ed. Ziegler 94. 23-100. 23

Καὶ ἡ μὲν τοῦ Ποσειδωνίου ἑφόδος περὶ τοῦ κατὰ τὴν γῆν μεγέθους τοιαύτην, ἡ δὲ τοῦ Ἐρατοσθένους γεωμετρικῆς ἑφόδου ἐχομένη, καὶ δοκοῦσα τι ἀσαφέστερον ἔχειν. Ποιήσει δὲ σαφῆ τὰ λεγόμενα ὑπ’ αὐτοῦ τάδε προϋποτεθέμενων ἦμων. ὑποκείσθω ἦμῖν πρῶτον μὲν κάνταῦθα, ὑπὸ τῷ αὐτῷ μεσημβρινῷ κείσθαι Συήνην καὶ Ἀλεξάνδρειαν, καὶ δεύτερον, τὸ διάστημα τὸ μεταξὺ τῶν πόλεων πεντακισχιλίων σταδίων εἶναι, καὶ τρίτον, τὰς καταπεμπομένας ἀκτίνας ἀπὸ διαφόρων μερῶν τοῦ ἠλίου ἐπὶ διάφορα τῆς γῆς μέρη παραλλήλους εἶναι. οὕτως γὰρ ἔχειν αὐτὸς οἱ γεωμέτραι ὑπότιθενται. Τέταρτον ἐκείνῳ ὑποκείσθω, δεικνύμενον παρὰ τοῖς γεωμέτραις, τὰς εἰς παραλλήλους ἐμπιπτούσας εὐθείας τὰς ἐναλλαξ γωνίας ἵσας πουεῖν, πέμπτον, τὰς ἐπὶ ἕσων γωνίων βεβηκυίας περιφερεῖας ὀμοίας εἶναι, τούτεστιν τὴν αὐτὴν ἀναλογίαν καὶ τὸν αὐτὸν λόγον ἔχειν πρὸς τοὺς ὁικείους κύκλους, δεικνυμένου καὶ τούτου παρὰ τοῖς γεωμέτραις. ὅποταν γὰρ περιφερείαν ἐπὶ ἕσων γωνίων ὧσι βεβηκυία, ἂν μία ἡτισοῦν

¹ καὶ ... ἦμιον add. Hiller.

* The difference between ratio and interval is explained a little more neatly by Theon himself (ed. Hiller 81. 6-9): 266
ERATOSTHENES

different from interval; for the relationship of the half to the double and of the double to the half does not furnish the same ratio, but it does furnish the same interval.\textsuperscript{a}

\noindent (d) Measurement of the Earth

Cleomedes,\textsuperscript{b} \textit{On the Circular Motion of the Heavenly Bodies} i. 10. 52, ed. Ziegler 94. 23–100. 23

Such then is Posidonius’s method of investigating the size of the earth, but Eratosthenes’ method depends on a geometrical argument, and gives the impression of being more obscure. What he says will, however, become clear if the following assumptions are made. Let us suppose, in this case also, first that Syene and Alexandria lie under the same meridian circle; secondly, that the distance between the two cities is 5000 stades; and thirdly, that the rays sent down from different parts of the sun upon different parts of the earth are parallel; for the geometers proceed on this assumption. Fourthly, let us assume that, as is proved by the geometers, straight lines falling on parallel straight lines make the alternate angles equal, and fifthly, that the arcs subtended by equal angles are similar, that is, have the same proportion and the same ratio to their proper circles—this also being proved by the geometers. For whenever arcs of circles are subtended by equal angles, if any one of these is (say) one-tenth

\begin{align*}
&\textit{διαφέρει \ δὲ \ διάστημα καὶ \ λόγος, \ ἐπειδὴ \ διάστημα μὲν \ ἐστὶ \ τὸ \ μεταξὺ τῶν \ ὁμογενῶν \ τε \ καὶ \ ἀνέσων \ ὄρων, \ λόγος \ δὲ \ ἀπλῶς \ ἡ \ τῶν \ ὁμογενῶν \ ὄρων \ πρὸς \ ἄλληλους \ σχέσις.}
\end{align*}

\textsuperscript{b} Cleomedes probably wrote about the middle of the first century B.C. His handbook \textit{De motu circulari corporum caelestium} is largely based on Posidonius.

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αυτῶν δέκατον ὢ μέρος τοῦ ὁικείου κύκλου, καὶ αἱ λοιπαὶ πᾶσαι δέκατα μέρη γενήσονται τῶν ὁικείων κύκλων.

Τούτων ὁ κατακρατήσας οὐκ ἂν χαλεπῶς τὴν ἔφοδον τοῦ Ἐρατοσθένους καταμάθω ἔχουσαν οὕτως. ὅπο τῷ αὐτῷ κείσθαι μεσημβρίνῳ φησί Συήνην καὶ Ἀλεξάνδρειαν. ἐπεὶ οὖν μέγιστοι τῶν ἐν τῷ κόσμῳ οἱ μεσημβρινοί, δεῖ καὶ τοὺς ὑποκειμένους τούτους τῆς γῆς κύκλους μεγίστους εἶναι ἁναγκαῖος. ὡστε ἥλικον ἂν τὸν διὰ Συήνης καὶ Ἀλεξάνδρείας ἦκοντα κύκλον τῆς γῆς ἢ ἔφοδος ἀποδείξει αὐτή, τηλικοῦτος καὶ ὁ μέγιστος ἔσται τῆς γῆς κύκλος. φησί τοῖς, καὶ ἔχει οὕτως, τὴν Συήνην ὑπὸ τῷ θερινῷ τροπικῷ κείσθαι κύκλω. ὁπότεν οὖν ἐν καρκίνῳ γενόμενος ὁ ἦλιος καὶ θερινᾶς ποιῶν τροπᾶς ἀκριβῶς μεσουρανήσῃ, ἀσκὶοι γίνονται οἱ τῶν ὄρολογίων γνώμονες ἁναγκαῖος, κατὰ κάθετον ἀκριβῆ τοῦ ἦλιου ὑπερκειμένου· καὶ τούτῳ γίνεσθαι λόγος ἐπὶ στάδιοις τριακοσίους τὴν διάμετρον. ἐν Ἀλεξάνδρείᾳ δὲ τῇ αὐτῇ ὡρα ἀποβάλλουσιν οἱ τῶν ὄρολογίων γνώμονες σκιάν, ἀτε πρὸς ἀρκτῷ μᾶλλον τῆς Συήνης ταύτης τῆς πόλεως κειμένης. ὑπὸ τῷ αὐτῷ μεσημβρίνῳ τοίνυν καὶ μεγίστῳ κύκλῳ τῶν πόλεως κειμένων, ἂν περιαγάγωμεν περιφέρειαν ἀπὸ τοῦ ἀκροῦ τῆς τοῦ γνώμονος σκιῶν ἐπὶ τὴν βάσιν αὐτῆς τοῦ γνώμονος τοῦ ἐν Ἀλεξάνδρείᾳ ὄρολογίου, αὐτῇ ἢ περιφέρεια τιμήμα γενήσεται τοῦ μεγίστου τῶν ἐν τῇ σκάφη κύκλων, ἐπεὶ μεγίστῳ κύκλῳ ὑπόκειται ἡ τοῦ ὄρολογίου σκάφη. εἰ οὖν ἐξῆς νοῆσαμεν εὐθείας διὰ τῆς γῆς ἐκβαλλομένας ἀφ’ ἐκατέρου τῶν γνωμόνων, πρὸς τῷ κέντρῳ τῆς γῆς 268
of its proper circle, all the remaining arcs will be tenth parts of their proper circles.

Anyone who has mastered these facts will have no difficulty in understanding the method of Eratosthenes, which is as follows. Syene and Alexandria, he asserts, are under the same meridian. Since meridian circles are great circles in the universe, the circles on the earth which lie under them are necessarily great circles also. Therefore, of whatever size this method shows the circle on the earth through Syene and Alexandria to be, this will be the size of the great circle on the earth. He then asserts, as is indeed the case, that Syene lies under the summer tropic. Therefore, whenever the sun, being in the Crab at the summer solstice, is exactly in the middle of the heavens, the pointers of the sundials necessarily throw no shadows, the sun being in the exact vertical line above them; and this is said to be true over a space 300 stades in diameter. But in Alexandria at the same hour the pointers of the sundials throw shadows, because this city lies farther to the north than Syene. As the two cities lie under the same meridian great circle, if we draw an arc from the extremity of the shadow of the pointer to the base of the pointer of the sundial in Alexandria, the arc will be a segment of a great circle in the bowl of the sundial, since the bowl lies under the great circle. If then we conceive straight lines produced in order from each of the pointers through the earth, they
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συμπεσοῦνται. ἔπει ὁ ὑ Ῥνή ὠρολόγιον κατὰ κάθετον ὑπόκειται τῷ ἡλίῳ, ἃν ἐπινοήσωμεν οὐθείαν ἀπὸ τοῦ ἡλίου ἠκούσαν ἐπὶ ἀκρον τὸν ὠρολογίον γνώμονα, μία γενήσεται οὐθεία ἡ ἀπὸ τοῦ ἡλίου μέχρι τοῦ κέντρου τῆς γῆς ἦκουσα. ἦν οὖν ἔτεραν οὐθείαν νοήσωμεν ἀπὸ τοῦ ἀκρον τῆς σκιᾶς τοῦ γνώμονος δι’ ἀκρον τοῦ γνώμονος ἐπὶ τοῦ ἡλίου ἀναγομένην ἀπὸ τῆς ἐν Ἀλεξάνδρεια σκάφης, αὕτη καὶ ἡ προειρημένη οὐθεία παράλληλοι γενήσονται ἀπὸ διαφόρων γε τοῦ ἡλίου μερῶν ἐπὶ διάφορα μέρη τῆς γῆς διήκουσαν. εἰς ταύτας τούς παράλληλους οὕσας ἐμπίπτει οὐθεία ἡ ἀπὸ τοῦ κέντρου τῆς γῆς ἐπὶ τοῦ ἐν Ἀλεξάνδρεια γνώμονα ἦκουσα, ὥστε τὰς ἐναλλαζέ γωνίας ἵσας ποιεῖν ὅπερ ἢ μὲν ἐστι πρὸς τῷ κέντρῳ τῆς γῆς κατὰ σύμπτωσιν τῶν οὐθείων, ἀν ἀπὸ τῶν ὠρολογίων ἠχθησαν ἐπὶ τὸ κέντρον τῆς γῆς, γνωμένη, ἡ δὲ κατὰ σύμπτωσιν ἀκρον τοῦ ἐν Ἀλεξάνδρεια γνώμονος καὶ τῆς ἀπ’ ἀκρον τῆς σκιᾶς αὐτοῦ ἐπὶ τοῦ ἡλίου διὰ τῆς πρὸς αὐτὸν ψαύσεως ἀναχείρετης γεγενημένη. καὶ ἐπὶ μὲν ταύτης βέβηκε περιφέρεια ἡ ἀπ’ ἀκρον τῆς σκιᾶς τοῦ γνώμονος ἐπὶ τῆς βάσιν αὐτοῦ περιαχθεῖσα, ἐπὶ δὲ τῆς πρὸς τῷ κέντρῳ τῆς γῆς ἡ ἀπὸ Ἔνθης διήκουσα εἰς Ἀλεξάνδρειαν. δομοῖο τοῖνυν αἱ περιφέρειαι εἰσὶν ἀλλήλαις ἐπὶ ίσων γε γωνιῶν βεβηκύται. ὃν ἄρα λόγον ἔχει ἢ ἐν τῇ σκάφη πρὸς τὸν οἰκεῖον κύκλου, τούτον ἔχει τὸν λόγον καὶ ἡ ἀπὸ Ἐνθῆς εἰς Ἀλεξάνδρειαν ἦκουσα. ἡ δὲ γε ἐν τῇ σκάφῃ πεντηκοστὸν μέρος εὑρίσκεται τοῦ οἰκείου κύκλου. δεῖ οὖν ἀναγκαῖος καὶ τὸ ἀπὸ Ἐνθῆς εἰς Ἀλεξάνδρειαν διάστημα πεντηκοστὸν εἶναι μέρος τοῦ 270
will meet at the centre of the earth. Now since the sundial at Syene is vertically under the sun, if we conceive a straight line drawn from the sun to the top of the pointer of the sundial, the line stretching from the sun to the centre of the earth will be one straight line. If now we conceive another straight line drawn upwards from the extremity of the shadow of the pointer of the sundial in Alexandria, through the top of the pointer to the sun, this straight line and the aforesaid straight line will be parallel, being straight lines drawn through from different parts of the sun to different parts of the earth. Now on these parallel straight lines there falls the straight line drawn from the centre of the earth to the pointer at Alexandria, so that it makes the alternate angles equal; one of these is formed at the centre of the earth by the intersection of the straight lines drawn from the sundials to the centre of the earth; the other is at the intersection of the top of the pointer in Alexandria and the straight line drawn from the extremity of its shadow to the sun through the point where it meets the pointer. Now this latter angle subtends the arc carried round from the extremity of the shadow of the pointer to its base, while the angle at the centre of the earth subtends the arc stretching from Syene to Alexandria. But the arcs are similar since they are subtended by equal angles. Whatever ratio, therefore, the arc in the bowl of the sundial has to its proper circle, the arc reaching from Syene to Alexandria has the same ratio. But the arc in the bowl is found to be the fiftieth part of its proper circle. Therefore the distance from Syene to Alexandria must necessarily be a fiftieth part of the great
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μεγίστου τῆς γῆς κύκλου καὶ ἐστὶ τοῦτο σταδίων πεντακισχιλίων. ὦ ἀρὰ σύμπας κύκλος γίνεται μυριάδων εἴκοσι πέντε. καὶ ἡ μὲν Ἐρατοσθένους ἕφοδος τοιαύτη.

Heron, Dioptra 36, ed. H. Schöne 302. 10-17

Δέον δὲ ἔστω, εἰ τύχων, τὴν μεταξὺ Ἀλεξανδρείας καὶ Ῥώμης ὁδὸν ἐκμετρῆσαι τὴν ἐπὶ εὐθείας, τὴν γε ἐπὶ κύκλου περιφερείας μεγίστου τοῦ ἐν τῇ γῆ, προσομολογούμενον τοῦ ὅτι περίμετρος τῆς γῆς σταδίων ἐστὶ μὲ καὶ ἔτι β, ὡς ὁ μάλιστα τῶν ἄλλων ἀκριβέστερον πεπραγματευμένος Ἐρατοσθένης δείκνυσιν ἐν (τῶ) ἐπιγραφομένῳ Περὶ τῆς ἀναμετρήσεως τῆς γῆς.

1 τῶ add. H. Schöne.

"The attached figure will help to elucidate Cleomedes. S is Syene and A Alexandria; the centre of the earth is O. The sun's rays at the two places are represented by the broken straight lines. If a be the angle made by the sun's rays with the pointer of the sundial at Alexandria (OA produced), the angle SOA is also equal to a, or one-fiftieth of four right angles. The arc SA is known to be 5000 stades and it follows that the whole circumference of the earth must be 250000 stades."
ERATOSTHENES

circle of the earth. And this distance is 5000 stades. Therefore the whole great circle is 250000 stades. Such is the method of Eratosthenes.

Heron, *Dioptra* 36, ed. H. Schöne 302. 10-17

Let it be required, perchance, to measure the distance between Alexandria and Rome along the arc of a great circle, on the assumption that the perimeter of the earth is 252000 stades, as Eratosthenes, who investigated this question more accurately than others, shows in the book which he wrote *On the Measurement of the Earth.*

Lit. "along the circumference of the greatest circle on the earth."

Strabo (ii. 5. 7) and Theon of Smyrna (ed. Hiller 124. 10-12) also give Eratosthenes' measurement as 252000 stades against the 250000 of Cleomedes. "The reason of the discrepancy is not known; it is possible that Eratosthenes corrected 250000 to 252000 for some reason, perhaps in order to get a figure divisible by 60 and, incidentally, a round number (700) of stades for one degree. If Pliny (*N.H.* xii. 13. 53) is right in saying that Eratosthenes made 40 stades equal to the Egyptian σχοῖνος, then, taking the σχοῖνος at 12000 Royal cubits of 0.525 metres, we get 300 such cubits, or 157.5 metres, i.e., 516.73 feet, as the length of the stade. On this basis 252000 stades works out to 24662 miles, and the diameter of the earth to about 7850 miles, only 50 miles shorter than the true polar diameter, a surprisingly close approximation, however much it owes to happy accidents in the calculation" (Heath, *H.G.M.* ii. 107).
XIX. APOLLONIUS OF PERGA
XIX. APOLLONIUS OF PERGA

(a) The Conic Sections

(i.) Relation to Previous Works


‘Απολλώνιος ὁ γεωμέτρης, ὃς ἔλεγε ἐταίρη Ἀν-θέως, γέγονε μὲν ἐκ Πέργης τῆς ἐν Παμφυλίᾳ ἐν χρόνοις τοῦ Εὐεργέτου Πτολεμαίου, ὦς ἱστορεῖ Ἡράκλειος ὃ τὸν βίον Ἄρχιμήδους γράφων, ὦς καὶ φησι τὰ κωνικὰ θεωρήματα ἐπινοήσας μὲν πρῶτον τὸν Ἀρχιμήδη, τὸν δὲ Ἀπολλώνιον αὐτὰ εὐρόντα ὑπὸ Ἄρχιμήδους μὴ ἐκδοθέντα ἰδιοποιήσασθαι, οὐκ ἄληθεύων κατὰ γε τὴν ἐμὴν. ὃ τε γὰρ Ἀρχιμήδης ἐν πολλοῖς φαίνεται ὡς παλαιοτέρας τῆς στοιχείωσεως τῶν κωνικῶν μεμνημένος, καὶ ὃ Ἀπολλώνιος οὐχ ὡς ἰδίας ἐπινοίας γράφει· οὐ γὰρ ἄν ἐφη ἐπὶ πλέον καὶ καθόλου μᾶλλον

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a Scarcely anything more is known of the life of one of the greatest geometers of all time than is stated in this brief reference. From Pappus, Coll. vii., ed. Hultsch 67 (quoted in vol. i. p. 488), it is known that he spent much time at Alexandria with Euclid's successors. Ptolemy Euergetes reigned 246–221 B.C., and as Ptolemaeus Chennus (apud Photii Bibl., cod. exc., ed. Bekker 151 b 18) mentions an astro-

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XIX. APOLLONIUS OF PERGA

(a) The Conic Sections

(i.) Relation to Previous Works

Eutocius, Commentary on Apollonius's Conics, Apoll. Perg. ed. Heiberg ii. 168. 5-170. 26

Apollonius the geometer, my dear Anthemius, flourished at Perga in Pamphylia during the time of Ptolemy Euergetes, as is related in the life of Archimedes written by Heraclius, who also says that Archimedes first conceived the theorems in conics and that Apollonius, finding they had been discovered by Archimedes but not published, appropriated them for himself, but in my opinion he errs. For in many places Archimedes appears to refer to the elements of conics as an older work, and moreover Apollonius does not claim to be giving his own discoveries; otherwise he would not have described his purpose as "to investigate these properties more fully and more

nomer named Apollonius who flourished in the time of Ptolemy Philopator (221-204 B.C.), the great geometer is probably meant. This fits in with Apollonius's dedication of Books iv.-viii. of his Conics to King Attalus I (247-197 B.C.). From the preface to Book i., quoted infra (p. 281), we gather that Apollonius visited Eudemus at Pergamum, and to Eudemus he dedicated the first two books of the second edition of his work.

More probably Heraclides, v. supra, p. 18 n. a.
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εξειργάσθαι ταῦτα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα." ἀλλὰ ὑπὲρ φησθον ὁ Γέμωνος ἀληθὲς ἔστιν, ὅτι οἱ παλαιοὶ κάωνον ὁμοίωμενοι τὴν τοῦ ὄρθογωνίου τριγώνου περιφορὰν μενοῦσης μᾶς τῶν περὶ τὴν ὀρθὴν εἰκότως καὶ τοὺς κώνους πάντας ὀρθοὺς ὑπελάμβανον γίνεσθαι καὶ μίαν τομὴν ἐν ἐκάστῳ, ἐν μὲν τῷ ὀρθογωνίῳ τὴν νῦν καλομένην παραβολῆν, ἐν δὲ τῷ ἀμβλυγωνίῳ τὴν ὑπερβολῆν, ἐν δὲ τῷ ὄξυγωνίῳ τὴν ἐλλειψιν καὶ ἐστὶν πάρ’ αὐτοῖς εὐρέων οὗτως ὀνομαζομένας τὰς τομὰς. ὥστερον οὖν τῶν ἀρχαίων ἐπὶ ἐνὸς ἐκάστου εἴδους τριγώνου θεωρησάντων τὰς δύο ὀρθὰς πρότερον ἐν τῷ ἱσοπλεύρῳ καὶ πάλιν ἐν τῷ ἱσο-σκελεί καὶ ύστερον ἐν τῷ σκαληνῷ ὁι μεταγενεστέροι καθολικόν θεώρημα ἀπεδείξαν τοιούτῳ παντὸς τριγώνου αἱ ἐντὸς τρεῖς γνώσις δυσών ὀρθαῖς ἵσαι εἰσὶν οὔτως καὶ ἐπὶ τῶν τοῦ κώνων τομῶν τῆς μὲν γὰρ λειομένην ὀρθογωνίου κώνων τομὴν ἐν ὀρθογωνίῳ μόνον κώνων ἐδεώρουν τεμνομένω ἐπιπέδῳ ὀρθῷ πρὸς μίαν πλευρὰν τοῦ κώνου, τὴν δὲ τοῦ ἀμβλυγωνίου κώνου τομὴν ἐν ἀμβλυγωνίῳ γνωμένην κώνῳ ἀπεδείκνυσαν, τὴν δὲ τοῦ ὄξυ-γωνίου ἐν ὄξυγωνίῳ, ὁμοίως ἐπὶ πάντων τῶν κώνων ἀγοντες τὰ ἐπίπεδα ὀρθὰ πρὸς μίαν πλευρὰν τοῦ κώνου δηλοὶ δὲ καὶ αὐτὰ τὰ ἀρχαία ὀνοματα τῶν γραμμῶν. ὑστερον δὲ Ἀπελλώνιος ὁ Περγαῖος καθόλου τι ἐδεώρησεν, ὅτι ἐν παντὶ κώνῳ καὶ ὀρθῷ καὶ σκαληνῷ πάσαι αἱ τομαι εἰσὶ κατὰ διάφορον τοῦ ἐπιπέδου πρὸς τὸν κώνον προσβολῆν. διν καὶ θαυμάσαντες οἱ κατ’ αὐτὸν γενόμενοι διὰ τὸ θαυμάσιον τῶν ὑπ’ αὐτοῦ δεδειγμένων κωνικῶν θεωρημάτων μέγαν γεωμέτρησιν ἐκάλουν. ταῦτα 278
generally than is done in the works of others." But what Geminus says is correct: defining a cone as the figure formed by the revolution of a right-angled triangle about one of the sides containing the right angle, the ancients naturally took all cones to be right with one section in each—in the right-angled cone the section now called the parabola, in the obtuse-angled the hyperbola, and in the acute-angled the ellipse; and in this may be found the reason for the names they gave to the sections. Just as the ancients, investigating each species of triangle separately, proved that there were two right angles first in the equilateral triangle, then in the isosceles, and finally in the scalene, whereas the more recent geometers have proved the general theorem, that in any triangle the three internal angles are equal to two right angles, so it has been with the sections of the cone; for the ancients investigated the so-called section of a right-angled cone in a right-angled cone only, cutting it by a plane perpendicular to one side of the cone, and they demonstrated the section of an obtuse-angled cone in an obtuse-angled cone and the section of an acute-angled cone in the acute-angled cone, in the cases of all the cones drawing the planes in the same way perpendicularly to one side of the cone; hence, it is clear, the ancient names of the curves. But later Apollonius of Perga proved generally that all the sections can be obtained in any cone, whether right or scalene, according to different relations of the plane to the cone. In admiration for this, and on account of the remarkable nature of the theorems in conics proved by him, his contemporaries called him the "Great

* This comes from the preface to Book i., v. infra, p. 283.
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\[ \mu\varepsilon\nu\ \sigma\nu\nu\ \hat{o}\ \Gamma\acute{e}m\nu\nu\ \epsilon\nu\ \tau\omega\ \varepsilon\kappa\tau\omega\ \varphi\eta\sigma\iota\ \tau\acute{h}s\ \mathrm{T} \acute{h}n\ \mu\alpha\theta\eta\mu\acute{a}\tau\omega\nu\ \theta\varepsilon\varphi\acute{r}i\acute{a}. \]

(ii.) Scope of the Work

i. 2. 2–4. 28

'Απολλώνιος Εὐδήμων χαίρειν.

Εἰ τῷ τε σώματι εὖ ἐπανάγεις καὶ τὰ ἄλλα κατὰ γνώμην ἐστὶ σοι, καλῶς ἂν ἔχοι, μετρίως δὲ ἔχομεν καὶ αὐτοί. καθ' οὖν δὲ καὶρὸν ἕμην μετὰ σοῦ ἐν Περγάμῳ, ἑθεώρουν σε σπεύδοντα μετασχεῖν τῶν πεπραγμένων ἕμιν κωνικῶν· πέπομοφα σοὶ τὸ πρῶτον βιβλίον διορθωσάμενος, τα δὲ λοιπά, ὅταν εὐαρεστήσωμεν, ἕξαποστελοῦμεν· οὐκ ἁμημονεῖν γάρ οἴμαι σε παρ' ἐμοῦ ἀκηκοῦτα, διότι τὴν περὶ ταῦτα ἐφοδον ἐποιησάμην ἅξιοθεῖς ὑπὸ Ναυκρά-

touς τοῦ γεωμέτρου, καθ' οὖν καὶρὸν ἐσχόλαζε

* Menaechmus, as shown in vol. i. pp. 278–283, and more particularly p. 283 n. a, solved the problem of the doubling of the cube by means of the intersection of a parabola with a hyperbola, and also by means of the intersection of two parabolas. This is the earliest mention of the conic sections in Greek literature, and therefore Menaechmus (fl. 360–350 B.C.) is generally credited with their discovery; and as Eratosthenes' epigram (vol. i. p. 296) speaks of "cutting the cone in the triads of Menaechmus," he is given credit for discovering the ellipse as well. He may have obtained them all by the method suggested by Geminus, but Heath (H.G.M. ii. 111–116) gives cogent reasons for thinking that he may have obtained his rectangular hyperbola by a section of a right-angled cone parallel to the axis.

A passage already quoted (vol. i. pp. 486–489) from Pappus (ed. Hultsch 672. 18–678. 24) informs us that treatises on the conic sections were written by Aristaeus and Euclid. Aristaeus' work, in five books, was entitled Solid Loci; Euclid's
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Geometer.” Geminus relates these details in the sixth book of his Theory of Mathematics.¹

(ii.) Scope of the Work

Apollonius, Conics i., Preface, Apoll. Perg. ed. Heiberg i. 2. 2–4. 28

Apollonius to Eudemus b greeting.

If you are in good health and matters are in other respects as you wish, it is well; I am pretty well too. During the time I spent with you at Pergamum, I noticed how eager you were to make acquaintance with my work in conics; I have therefore sent to you the first book, which I have revised, and I will send the remaining books when I am satisfied with them. I suppose you have not forgotten hearing me say that I took up this study at the request of Naucrates the geometer, at the time he came

Conics was in four books. The work of Aristaeus was obviously more original and more specialized; that of Euclid was admittedly a compilation largely based on Aristaeus. Euclid flourished about 300 B.C. As noted in vol. i. p. 495 n. a, the focus-directrix property must have been known to Euclid, and probably to Aristaeus; curiously, it does not appear in Apollonius’s treatise.

Many properties of conics are assumed in the works of Archimedes without proof and several have been encountered in this work; they were no doubt taken from the works of Aristaeus or Euclid. As the reader will notice, Archimedes’ terminology differs in several respects from that of Apollonius, apart from the fundamental difference on which Geminus laid stress.

The history of the conic sections in antiquity is admirably treated by Zeuthen, Die Lehre von den Kegelschnitten im Altertum (1886) and Heath, Apollonius of Perga, xvii-clvi.

b Not, of course, the pupil of Aristotle who wrote the famous History of Geometry, unhappily lost.
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παρ’ ἦμῖν παραγενηθεὶς εἰς Ἀλεξάνδρειαν, καὶ διότι πραγματεύσαντες αὐτὰ ἐν ὅκτῳ βιβλίοις ἔξ ἀυτῆς μεταδεδώκαμεν αὐτὰ εἰς τὸ σπουδαῖοτέρον διὰ τὸ πρὸς ἐκπλω ἀυτὸν εἶναι ὦ διακαθάραντες, ἀλλὰ πάντα τὰ ὑποπτότα ἦμῖν θέντες ὡς ἔσχατον ἐπελευσόμενοι. Ὁθεν καὶ μόνον ὕν λαβόντες ἀεὶ τὸ τυγχάνον διορθώσεως ἐκδίδομεν. καὶ ἐπεὶ συμβέβηκε καὶ ἄλλους τινὰς τῶν συμμειρχῶν ἡμῖν μετεληφθέναι τὸ πρῶτον καὶ τὸ δεύτερον βιβλίον πρὶν ἦ διορθωθῆναι, μὴ θαυμάσῃς, ἐὰν περιπτάτησ αὐτοῖς ἑτέρως ἔχουσιν.

Ἀπὸ δὲ τῶν ὅκτω βιβλίων τὰ πρῶτα τέσσαρα πέπτωκεν εἰς ἁγωγὴν στοιχειώδη, περιέχει δὲ τὸ μὲν πρῶτον τὰς γενέσεις τῶν τριῶν τομῶν καὶ τῶν ἀντικειμένων καὶ τὰ ἐν αὐταῖς ἀρχικὰ συμπτώματα ἐπὶ πλέον καὶ καθόλου μᾶλλον ἐξειργασμένα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα, τὸ δὲ δεύτερον τὰ περὶ τὰς διαμέτρους καὶ τῶς ἄξονας τῶν τομῶν συμβαίνοντα καὶ τὰς ἀποξιώσεως καὶ ἄλλα γενικὴν καὶ ἀναγκαίαν χρείαν παρεχόμενα πρὸς τοὺς διορισμοὺς· τίνας δὲ διαμέτρους καὶ τίνας ἄξονας καλῶ, εἰδήσεις ἐκ τοῦτο τοῦ βιβλίου. τὸ δὲ τρίτον πολλὰ καὶ παράδοξα θεωρήματα χρήσιμα πρὸς τε τὰς συνθέσεις τῶν στερεῶν τόπων καὶ τοὺς διορισμοὺς, ὥν τὰ πλείστα καὶ κάλλιστα ἰένα, δὲ καὶ κατανόησαντες συνείδομεν μὴ συντιθέμενον ὑπὸ Εὐκλείδου τὸν ἐπὶ τρεῖς καὶ τέσσαρας γραμμὰς τόπον, ἀλλὰ μόριον τὸ τυχόν αὐτοῦ καὶ τοῦτο οὐκ εὐτυχῶς· οὐ γὰρ ἦν δυνατὸν ἀνευ τῶν προσευρημένων ἦμῖν τελειωθῆναι τὴν

* A necessary observation, because Archimedes had used the terms in a different sense. 282
to Alexandria and stayed with me, and that, when I had completed the investigation in eight books, I gave them to him at once, a little too hastily, because he was on the point of sailing, and so I was not able to correct them, but put down everything as it occurred to me, intending to make a revision at the end. Accordingly, as opportunity permits, I now publish on each occasion as much of the work as I have been able to correct. As certain other persons whom I have met have happened to get hold of the first and second books before they were corrected, do not be surprised if you come across them in a different form.

Of the eight books the first four form an elementary introduction. The first includes the methods of producing the three sections and the opposite branches [of the hyperbola] and their fundamental properties, which are investigated more fully and more generally than in the works of others. The second book includes the properties of the diameters and the axes of the sections as well as the asymptotes, with other things generally and necessarily used in determining limits of possibility; and what I call diameters and axes you will learn from this book.\textsuperscript{a} The third book includes many remarkable theorems useful for the syntheses of solid loci and for determining limits of possibility; most of these theorems, and the most elegant, are new, and it was their discovery which made me realize that Euclid had not worked out the synthesis of the locus with respect to three and four lines, but only a chance portion of it, and that not successfully; for the synthesis could not be completed without the theorems discovered by me.\textsuperscript{b}

\textsuperscript{a} For this locus, and Pappus's comments on Apollonius's claims, \textit{v. vol. i. pp. 486-489.}

\textsuperscript{b}
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σύνθεσιν. τὸ δὲ τέταρτον, ποσακῶς αἱ τῶν κύκλων τομαὶ ἀλλήλαις τε καὶ τῇ τοῦ κύκλου περιφερείᾳ συμβάλλουσι, καὶ ἄλλα ἕκ περισσόν, ὧν οὐδέτερον ὑπὸ τῶν πρὸ ἡμῶν γέγραπται, κύκλου περιφερεία κατὰ πόσα σημεῖα συμβάλλουσι.

Τὰ δὲ λοιπά ἐστὶ περιουσιαστικῶτερα: ἐστὶ γὰρ τὸ μὲν περὶ ἐλαχίστων καὶ μεγίστων ἐπὶ πλέον, τὸ δὲ περὶ ἴσων καὶ ὀμοίων κύκλων τομῶν, τὸ δὲ περὶ διοριστικῶν θεωρημάτων, τὸ δὲ προβλημάτων κωνικῶν διωρισμένων. οὐ μὴν ἄλλα καὶ πάντων ἐκδοθέντων ἐξετάστω τοὺς περιτυχάνουσι κρίνεις αὐτά, ὡς ἂν αὐτῶν ἔκαστος αἰρήσεται. εὐτύχει.

(iii.) Definitions

Ibid., Deff., Apoll. Perg. ed. Heiberg l. 6. 2–8. 20

Ἐὰν ἀπὸ τινος σημείου πρὸς κύκλου περιφερείαν, ὡς ὧν ἐστὶν ἐν τῷ αὐτῷ ἐπιπέδῳ τῷ σημείῳ, εὐθεῖα ἐπίξενχθείσα ἐφ’ ἐκάτερα προσεκβληθή, καὶ μένοντος τοῦ σημείου ἡ εὐθεία περιενεχθεῖσα περὶ τῆς τοῦ κύκλου περιφερείας εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἢρέει ἄρα, τῇ γραφείσαν ὑπὸ τῆς εὐθείας ἐπιφάνειαν, ἡ σύγκειται ἐκ δύο ἐπιφανειῶν κατὰ κορύφην ἀλλήλαις κειμένων, ὡς ἐκάτερα εἰς ἀπειρὸν αὐξηταὶ τῆς

* Only the first four books survive in Greek. Books v.–vii. have survived in Arabic, but Book viii. is wholly lost. Halley (Oxford, 1710) edited the first seven books, and his edition is still the only source for Books vi. and vii. The first four books have since been edited by Heiberg (Leipzig, 1891–1893) and Book v. (up to Prop. 7) by L. Nix (Leipzig, 1889). The
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The fourth book investigates how many times the sections of cones can meet one another and the circumference of a circle; in addition it contains other things, none of which have been discussed by previous writers, namely, in how many points a section of a cone or a circumference of a circle can meet [the opposite branches of hyperbolas].

The remaining books are thrown in by way of addition: one of them discusses fully minima and maxima, another deals with equal and similar sections of cones, another with theorems about the determinations of limits, and the last with determinate conic problems. When they are all published it will be possible for anyone who reads them to form his own judgement. Farewell.

(iii.) Definitions

Ibid., Definitions, Apoll. Perg. ed. Heiberg i. 6. 2–8. 20

If a straight line be drawn from a point to the circumference of a circle, which is not in the same plane with the point, and be produced in either direction, and if, while the point remains stationary, the straight line be made to move round the circumference of the circle until it returns to the point whence it set out, I call the surface described by the straight line a conical surface; it is composed of two surfaces lying vertically opposite to each other, of which each surviving books have been put into mathematical notation by T. L. Heath, Apollonius of Perga (Cambridge, 1896) and translated into French by Paul Ver Eecke, Les Coniques d'Apollonius de Perga (Bruges, 1923).

In ancient times Eutocius edited the first four books with a commentary which still survives and is published in Heiberg's edition. Serenus and Hypatia also wrote commentaries, and Pappus a number of lemmas.
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γραφούσης εὐθείας εἰς ἀπειρον προσεκβαλλομένης, καλῶ κωνικῆς ἐπιφάνειαν, κορυφήν δὲ αὐτῆς τὸ μεμενηκός σημεῖον, ἄξονα δὲ τὴν διὰ τοῦ σημείου καὶ τοῦ κέντρου τοῦ κύκλου ἀγομένην εὐθείαν.

Κώνον δὲ τὸ περιεχόμενον σχῆμα ὑπὸ τὸ τοῦ κύκλου καὶ τῆς μεταξὺ τῆς τε κορυφῆς καὶ τῆς τοῦ κύκλου περιφερείας κωνικῆς ἐπιφανείας, κορυφήν δὲ τοῦ κώνου τὸ σημεῖον, δὲ καὶ τῆς ἐπιφανείας ἑστὶ κορυφή, ἄξονα δὲ τὴν ἀπὸ τῆς κορυφῆς ἑπὶ τὸ κέντρον τοῦ κύκλου ἀγομένην εὐθείαν, βάσων δὲ τὸν κύκλον.

Τῶν δὲ κώνων ὀρθοὺς μὲν καλῶ τοὺς πρὸς ὀρθὰς ἔχοντας ταῖς βάσεις τοὺς ἄξονας, σκαληνοὺς δὲ τοὺς μὴ πρὸς ὀρθὰς ἔχοντας ταῖς βάσεις τοὺς ἄξονας.

Πάσης καμπύλης γραμμῆς, ἦτης ἐστὶν ἐν ἐνὶ ἐπιπέδῳ, διάμετρον μὲν καλῶ εὐθείαν, ἦτης ἡγμένη ἀπὸ τῆς καμπύλης γραμμῆς πάσας τὰς ἀγομένας ἐν τῇ γραμμῇ εὐθείᾳς εὐθείας τυί παραλλήλους δίχα διαφέρει, κορυφήν δὲ τῆς γραμμῆς τὸ πέρας τῆς εὐθείας το πρὸς τῇ γραμμῇ, τεταγμένως δὲ ἑπὶ τὴν διάμετρον κατήχθαι ἕκαστη τῶν παραλλήλων.

Ὅμοιος δὲ καὶ δύο καμπύλων γραμμῶν ἐν ἐνὶ ἐπιπέδῳ κειμένων διάμετρον καλῶ πλαγίαν μὲν, ἦτης εὐθεία τέμνουσα τὰς δύο γραμμὰς πάσας τὰς ἀγομένας ἐν ἐκατέρα τῶν γραμμῶν παρά τινα εὐθείαν δίχα τέμνει, κορυφὰς δὲ τῶν γραμμῶν τὸ πρὸς ταῖς γραμμαῖς πέρατα τῆς διαμέτρου, ὀρθίαν δὲ, ἦτης κειμένη μεταξὺ τῶν δύο γραμμῶν πάσας τὰς ἀγομένας παραλλήλους εὐθείας εὐθεία ἑπὶ καὶ ἀπολαμβανομένας μεταξὺ τῶν γραμμῶν δίχα

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extends to infinity when the straight line which describes them is produced to infinity; I call the fixed point the vertex, and the straight line drawn through this point and the centre of the circle I call the axis.

The figure bounded by the circle and the conical surface between the vertex and the circumference of the circle I term a cone, and by the vertex of the cone I mean the point which is the vertex of the surface, and by the axis I mean the straight line drawn from the vertex to the centre of the circle, and by the base I mean the circle.

Of cones, I term those right which have their axes at right angles to their bases, and scalene those which have their axes not at right angles to their bases.

In any plane curve I mean by a diameter a straight line drawn from the curve which bisects all straight lines drawn in the curve parallel to a given straight line, and by the vertex of the curve I mean the extremity of the straight line on the curve, and I describe each of the parallels as being drawn ordinate-wise to the diameter.

Similarly, in a pair of plane curves I mean by a transverse diameter a straight line which cuts the two curves and bisects all the straight lines drawn in either curve parallel to a given straight line, and by the vertices of the curves I mean the extremities of the diameter on the curves; and by an erect diameter I mean a straight line which lies between the two curves and bisects the portions cut off between the curves of all straight lines drawn parallel to a given
Construction of the Sections

Ibid., Props. 7-9, Apoll. Perg. ed. Heiberg i. 22. 26-36. 5

This proposition defines a conic section in the most general way with reference to any diameter. It is only much

1 δύο om. Heiberg.
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straight line; and I describe each of the parallels as drawn ordinate-wise to the diameter.

By conjugate diameters in a curve or pair of curves I mean straight lines of which each, being a diameter, bisects parallels to the other.

By an axis of a curve or pair of curves I mean a straight line which, being a diameter of the curve or pair of curves, bisects the parallels at right angles.

By conjugate axes in a curve or pair of curves I mean straight lines which, being conjugate diameters, bisect at right angles the parallels to each other.

(iv.) Construction of the Sections

Ibid., Props. 7-9, Apoll. Perg. ed. Heiberg i. 22. 26-36. 5

Prop. 7

If a cone be cut by a plane through the axis, and if it be also cut by another plane cutting the plane containing the base of the cone in a straight line perpendicular to the base of the axial triangle, or to the base produced, a section will be made on the surface of the cone by the cutting plane, and straight lines drawn in it parallel to the straight line perpendicular to the base of the axial triangle will meet the common section of the cutting plane and the axial

later in the work (i. 52-58) that the principal axes are introduced as diameters at right angles to their ordinates. The proposition is an excellent example of the generality of Apollonius’s methods.

Apollonius followed rigorously the Euclidean form of proof. In consequence his general enunciations are extremely long and often can be made tolerable in an English rendering only by splitting them up; but, though Apollonius seems to have taken a malicious pleasure in their length, they are formed on a perfect logical pattern without a superfluous word.

* Lit. “the triangle through the axis.”
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νοντος ἐπιπέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου καὶ προσεκβαλλόμεναι ἐστὶ τοῦ ἐτέρου μέρους τῆς τομῆς δίχα τμηθῆσονται ὑπ' αὐτῆς, καὶ ἕαν μὲν ὀρθὸς ἢ ὁ κώνος, ἢ ἐν τῇ βάσει εὐθείᾳ πρὸς ὀρθὰς ἐσται τῇ κοινῇ τομῇ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου, ἕαν δὲ σκαληνός, οὐκ αἰεὶ πρὸς ὀρθὰς ἐσται, ἀλλ' ὅταν τὸ διὰ τοῦ ἄξονος ἐπιπέδου πρὸς ὀρθὰς ἢ τῇ βάσει τοῦ κώνου.

"Εστω κώνος, οὗ κορυφή μὲν τὸ Α σημείον, βάσις δὲ τὸ ΒΓ κύκλος, καὶ τετμῆσθω ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τριγώνον. τετμῆσθω δὲ καὶ ἐτέρῳ ἐπιπέδῳ τέμνοντι τὸ ἐπιπέδου, ἐν ὃ ἐστὶν ὁ ΒΓ κύκλος, κατ' εὐθείαν τὴν ΔΕ ἢτοι πρὸς ὀρθὰς ὅπου ἐν τῇ ΒΓ ἢ τῇ ἐπ' εὐθείας αὐτῆς, καὶ ποιεῖτω τομήν ἐν τῇ ἐπιφάνειᾳ τοῦ κώνου τὴν ΔΖΕ· κοινῇ δὴ τομῇ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ ΑΒΓ τριγώνου ἢ ΖΗ. καὶ 290
triangle and, if produced to the other part of the section, will be bisected by it; if the cone be right, the straight line in the base will be perpendicular to the common section of the cutting plane and the axial triangle; but if it be scalene, it will not in general be perpendicular, but only when the plane through the axis is perpendicular to the base of the cone.

Let there be a cone whose vertex is the point $A$ and whose base is the circle $BG$, and let it be cut by a plane through the axis, and let the section so made be the triangle $ABG$. Now let it be cut by another plane cutting the plane containing the circle $BG$ in a straight line $DE$ which is either perpendicular to $BG$ or to $BG$ produced, and let the section made on the surface of the cone be $DE$; then the common section of the cutting plane and of the triangle $ABG$.

* This applies only to the first two of the figures given in the MSS.
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eίλήφθω τι σημείον ἐπί τῆς ΔΖΕ τομῆς τοῦ Θ, καὶ ἤχθω διὰ τοῦ Θ τῆς ΔΕ παράλληλος ή ΘΚ. λέγω, ὅτι ή ΘΚ συμβαλεῖ τῇ ΖΗ καὶ ἐκβαλλομένη ἦσσ ποὺ ἐτέρου μέρους τῆς ΔΖΕ τομῆς δίχα τμηθῆσται ὑπὸ τῆς ΖΗ εὐθείας.

"Επεῖ γὰρ κῶνος, οὐ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ τὸ ΒΓ κύκλος, τέτμηται ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιεῖ τομὴν τὸ ΑΒΓ τρίγωνον, εἰλήφτω δὲ τι σημείον ἐπὶ τῆς ἐπιφανείας, ὁ μὴ ἔστιν ἐπὶ πλευράς τοῦ ΑΒΓ τριγώνου, τὸ Θ, καὶ ἔστι κάθετος ἡ ΔΗ ἐπί τὴν ΒΓ, ἡ ἄρα διὰ τοῦ Θ τῆς ΔΗ παράλληλος ἀγομένη, τούτεστι ἡ ΘΚ, συμβαλεῖ τῷ ΑΒΓ τριγώνω καὶ προσεκβαλλομένη ἦσσ ποὺ ἐτέρου μέρους τῆς ἐπιφανείας δίχα τμηθῆσται ὑπὸ τοῦ τριγώνου. ἐπεὶ οὖν ἡ διὰ τοῦ Θ τῆς ΔΕ παράλληλος ἀγομένη συμβάλλει τῷ ΑΒΓ τριγώνω καὶ ἔστιν ἐν τῷ διὰ τῆς ΔΖΕ τομῆς ἐπιπέδῳ, ἐπὶ τὴν κοινὴν ἄρα τομὴν πεσεῖται τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ ΑΒΓ τριγώνου. κοινῆ δὲ τομὴ ἔστι τῶν ἐπιπέδων ἡ ΖΗ· ἡ ἄρα διὰ τοῦ Θ τῆς ΔΕ παράλληλος ἀγομένη πεσεῖται ἐπὶ τὴν ΖΗ· καὶ προσεκβαλλομένη ἦσσ ποὺ ἐτέρου μέρους τῆς ΔΖΕ τομῆς δίχα τμηθῆσται ὑπὸ τῆς ΖΗ εὐθείας.

"Ητοι δὴ ὁ κῶνος ὀρθὸς ἔστιν, ἡ τὸ διὰ τοῦ ἄξονος τρίγωνον τὸ ΑΒΓ ὀρθὸν ἔστι πρὸς τὸν ΒΓ κύκλον, ἡ οὐδέτερον.

"Εστιν πρῶτερον ὁ κῶνος ὀρθός· εἰὴ ἂν οὖν καὶ τὸ ΑΒΓ τρίγωνον ὀρθὸν πρὸς τὸν ΒΓ κύκλον. ἐπεὶ οὖν ἐπιπέδου τὸ ΑΒΓ πρὸς ἐπιπέδου τὸ ΒΓ ὀρθὸν ἔστι, καὶ τῇ κοινῇ αὐτῶν τομῇ τῇ ΒΓ ἐν ἐν τῶν ἐπιπέδων τῷ ΒΓ πρὸς ὀρθὰς ἡκται ἡ ΔΕ, 292
is ZH. Let any point Θ be taken on ΔZE, and through Θ let ΘK be drawn parallel to ΔE. I say that ΘK intersects ZH and, if produced to the other part of the section ΔZE, it will be bisected by the straight line ZH.

For since the cone, whose vertex is the point A and base the circle BG, is cut by a plane through the axis and the section so made is the triangle ABG, and there has been taken any point Θ on the surface, not being on a side of the triangle ABG, and ΔH is perpendicular to BG, therefore the straight line drawn through Θ parallel to ΔH, that is ΘK, will meet the triangle ABG and, if produced to the other part of the surface, will be bisected by the triangle [Prop. 6]. Therefore, since the straight line drawn through Θ parallel to ΔE meets the triangle ABG and is in the plane containing the section ΔZE, it will fall upon the common section of the cutting plane and the triangle ABG. But the common section of those planes is ZH; therefore the straight line drawn through Θ parallel to ΔE will meet ZH; and if it be produced to the other part of the section ΔZE it will be bisected by the straight line ZH.

Now the cone is right, or the axial triangle ABG is perpendicular to the circle BG, or neither.

First, let the cone be right; then the triangle ABG will be perpendicular to the circle BG [Def. 3; Eucl. xi. 18]. Then since the plane ABG is perpendicular to the plane BG, and ΔE is drawn in one of the planes perpendicular to their common section BG, therefore
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η ΔΕ ἀρα τῷ ΑΒΓ τριγώνῳ ἐστὶ πρὸς ὀρθᾶς. καὶ πρὸς πᾶσας ἀρα τᾶς ἀπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ ΑΒΓ τριγώνῳ ὀρθῇ ἐστίν. ὡστε καὶ πρὸς τὴν ΖΗ ἐστὶ πρὸς ὀρθᾶς.

Μὴ ἐστω δὴ ὁ κώνος ὀρθὸς. εἰ μὲν οὖν τὸ διὰ τοῦ ἀξονος τριγώνον ὀρθὸν ἐστὶ πρὸς τὸν ΒΓ κύκλον, ὡμοίως δείξομεν, ὅτι καὶ η ΔΕ τῇ ΖΗ ἐστὶ πρὸς ὀρθᾶς. μὴ ἐστω δὴ τὸ διὰ τοῦ ἀξονος τριγώνον τῷ ΑΒΓ ὀρθὸν πρὸς τὸν ΒΓ κύκλον.

λέγω, ὅτι οὐδὲ ἡ ΔΕ τῇ ΖΗ ἐστὶ πρὸς ὀρθᾶς. εἰ γὰρ δυνατόν, ἐστων ἐστὶ δὲ καὶ τῇ ΒΓ πρὸς ὀρθᾶς. η ἀρα ΔΕ ἐκατέρα τῶν ΒΓ, ΖΗ ἐστὶ πρὸς ὀρθᾶς. καὶ τῷ διὰ τῶν ΒΓ, ΖΗ ἐπιπέδῳ ἀρα πρὸς ὀρθᾶς ἐστί. τὸ δὲ διὰ τῶν ΒΓ, ΗΖ ἐπιπέδον ἐστὶ τῷ ΑΒΓ. καὶ η ΔΕ ἀρα τῷ ΑΒΓ τριγώνῳ ἐστὶ πρὸς ὀρθᾶς. καὶ πάντα ἀρα τὰ δι' αὐτῆς ἐπιπέδα τῷ ΑΒΓ τριγώνῳ ἐστὶ πρὸς ὀρθᾶς. ἐν δὲ τὶ τῶν διὰ τῆς ΔΕ ἐπιπέδων ἐστὶν ὁ ΒΓ κύκλος. ο ΒΓ ἀρα κύκλος πρὸς ὀρθᾶς ἐστὶ τῷ ΑΒΓ τριγώνῳ. ὡστε καὶ τὸ ΑΒΓ τριγώνον ὀρθὸν ἐσται πρὸς τὸν ΒΓ κύκλον. ὅπερ οὐχ ὑπόκειται. οὐκ ἀρα η ΔΕ τῇ ΖΗ ἐστὶ πρὸς ὀρθᾶς.

Πόρισμα

Ἐκ δὴ τούτου φανερόν, ὅτι τῆς ΔΖΕ τομῆς διάμετρος ἐστὶν η ΖΗ, ἐπείπερ τὰς ἀγομένας παραλληλίους εὐθείας τινὶ τῇ ΔΕ δίχα τέμνει, καὶ οτι δυνατὸν ἐστιν ὕπο τῆς διαμέτρου τῆς ΖΗ παραλληλίους τινὰς δίχα τέμνεσθαι καὶ μὴ πρὸς ὀρθᾶς.

η'

Ἐάν κώνος ἐπιπέδω τμηθῇ διὰ τοῦ ἀξονος.
Apollonius of Perga

$\Delta E$ is perpendicular to the triangle $AB\Gamma$ [Eucl. xi. Def. 4]; and therefore it is perpendicular to all the straight lines in the triangle $AB\Gamma$ which meet it [Eucl. xi. Def. 3]. Therefore it is perpendicular to $ZH$.

Now let the cone be not right. Then, if the axial triangle is perpendicular to the circle $\beta\Gamma$, we may similarly show that $\Delta E$ is perpendicular to $ZH$. Now let the axial triangle $AB\Gamma$ be not perpendicular to the circle $\beta\Gamma$. I say that neither is $\Delta E$ perpendicular to $ZH$. For if it is possible, let it be; now it is also perpendicular to $\beta\Gamma$; therefore $\Delta E$ is perpendicular to both $\beta\Gamma, ZH$. And therefore it is perpendicular to the plane through $\beta\Gamma, ZH$ [Eucl. xi. 4]. But the plane through $\beta\Gamma, HZ$ is $AB\Gamma$; and therefore $\Delta E$ is perpendicular to the triangle $AB\Gamma$. Therefore all the planes through it are perpendicular to the triangle $AB\Gamma$ [Eucl. xi. 18]. But one of the planes through $\Delta E$ is the circle $\beta\Gamma$; therefore the circle $\beta\Gamma$ is perpendicular to the triangle $AB\Gamma$. Therefore the triangle $AB\Gamma$ is perpendicular to the circle $\beta\Gamma$; which is contrary to hypothesis. Therefore $\Delta E$ is not perpendicular to $ZH$.

Corollary

From this it is clear that $ZH$ is a diameter of the section $\Delta ZE$ [Def. 4], inasmuch as it bisects the straight lines drawn parallel to the given straight line $\Delta E$, and also that parallels can be bisected by the diameter $ZH$ without being perpendicular to it.

Prop. 8

*If a cone be cut by a plane through the axis, and it be*
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τοιοῦτοι δὲ καὶ ἔτερω ἐπιπέδω τέμνοντι τὴν βάσιν τοῦ κώνου κατ’ εὐθείαν πρὸς ὅρθας οὖσαν τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου, ἢ δὲ διάμετρος τῆς γινομένης ἐν τῇ ἐπιφάνειᾳ τομῆς ἦτοι παρὰ μίαν ἢ τῶν τοῦ τριγώνου πλευρῶν ἡ συμπλήρωσις αὐτῆς ἕκτος τῆς κορυφῆς τοῦ κώνου, προσεκβάλλεται δὲ ἢ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπίπεδον εἰς ἄπειρον, καὶ ἢ τομὴ εἰς ἄπειρον αὐξηθῇ-σεται, καὶ ἀπὸ τῆς διαμέτρου τῆς τομῆς πρὸς τῇ κορυφῇ πάσῃ τῇ δοθείσῃ εὐθείᾳ ἵσθην ἀπολήψεται τῆς εὐθείας ἀγωμένη ἀπὸ τῆς τοῦ κώνου τομῆς παρὰ τὴν ἐν τῇ βάσει τοῦ κώνου εὐθείαν.

"Εστι τὸ κώνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθω ἐπιπέδῳ
diὰ τοῦ ἄξονος, καὶ ποιεῖτω τομῆν τὸ ΑΒΓ τρί-
also cut by another plane cutting the base of the cone in a line perpendicular to the base of the axial triangle, and if the diameter of the section made on the surface be either parallel to one of the sides of the triangle or meet it beyond the vertex of the cone, and if the surface of the cone and the cutting plane be produced to infinity, the section will also increase to infinity, and a straight line can be drawn from the section of the cone parallel to the straight line in the base of the cone so as to cut off from the diameter of the section towards the vertex an intercept equal to any given straight line.

Let there be a cone whose vertex is the point A and base the circle BΓ, and let it be cut by a plane through the axis, and let the section so made be the triangle
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gωνον· τετμήσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ τέμνοντι τὸν ΒΓ κύκλου καὶ ἐνδείχθαι τὴν ΔΕ πρὸς ὀρθάς οὕσαν τῇ ΒΓ, καὶ ποιεῖτο τοµήν ἐν τῇ ἐπιφανείᾳ τὴν ΔΖΕ γραμμῆν· ἢ δὲ διάμετρος τῆς ΔΖΕ τοµῆς ἢ ΖΗ ἦτοι παράλληλος ἐστὼ τῇ ΑΓ ἢ ἐκβαλλο-
μένη συμπεπτέτω αὐτῇ ἐκτὸς τοῦ Α σημείου. λέγω, ὅτι καὶ, ἐὰν ἢ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπίπεδον ἐκβάλληται εἰς ἀπειρον, καὶ ἢ ΔΖΕ τοµὴ εἰς ἀπειρον αὐξηθήσεται.

Ἐκβεβλήσθω γὰρ ἢ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπίπεδον· φανερὸν δὴ, ὅτι καὶ αἱ ΑΒ, ΑΓ, ΖΗ συνεκβληθήσονται. ἐπεὶ ἢ ΖΗ τῇ ΑΓ ἦτοι παράλληλος ἐστὼ ἢ ἐκβαλλομένη συμ-
πιπτεί αὐτῇ ἐκτὸς τοῦ Α σημείου, αἱ ΖΗ, ΑΓ ἁρὰ ἐκβαλλόμεναι ὡς ἐπὶ τὰ Γ, Η μέρη οὐδέποτε συμπεσοῦνται· ἐκβεβλήσθωσαν οὖν, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς ΖΗ τυχόν τὸ Θ, καὶ διὰ τοῦ Θ σημείου τῇ μὲν ΒΓ παράλληλος ἢχθων ἢ ΚΘΔ, τῇ δὲ ΔΕ παράλληλος ἢ ΜΘΝ· τὸ ἀρὰ διὰ τῶν ΚΛ, ΜΝ ἐπίπεδον παράλληλόν ἐστὶ τῷ διὰ τῶν ΒΓ, ΔΕ. κύκλος ἁρὰ ἐστὶ τὸ ΚΑΜΝ ἐπίπεδον. καὶ ἐπεὶ τὰ Δ, Ε, Μ, Ν σημεία ἐν τῷ τέμνοντι ἐστὶν ἐπιπέδῳ, ἐστὶ δὲ καὶ ἐν τῇ ἐπιφανείᾳ τοῦ κώνου, ἐπὶ τῆς κοψῆς ἁρὰ τοµῆς ἐστὶν· θυξηται ἁρὰ ἢ ΔΖΕ μέχρι τῶν Μ, Ν σημείων. αὐξηθεῖσα ἁρὰ τῆς ἐπιφανείας τοῦ κώνου καὶ τοῦ τέμνοντος ἐπιπέδου μέχρι τοῦ ΚΑΜΝ κύκλου θυξηται καὶ ἢ ΔΖΕ τοµὴ μέχρι τῶν Μ, Ν σημείων. ὅμως δὴ δείξομεν, ὅτι καὶ, ἐὰν εἰς ἀπειρον ἐκβάλληται ἢ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπίπεδον, καὶ ἢ ΜΔΖΕΝ τοµὴ εἰς ἀπειρον αὐξηθήσεται.

Καὶ φανερὸν, ὅτι πάση τῇ δοθείσῃ εὐθείᾳ ἐσθη

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$AB\Gamma$; now let it be cut by another plane cutting the circle $B\Gamma$ in the straight line $\Delta E$ perpendicular to $B\Gamma$, and let the section made on the surface be the curve $\Delta ZE$; let $ZH$, the diameter of the section $\Delta ZE$, be either parallel to $A\Gamma$ or let it, when produced, meet $A\Gamma$ beyond the point $A$. I say that if the surface of the cone and the cutting plane be produced to infinity, the section $\Delta ZE$ will also increase to infinity.

For let the surface of the cone and the cutting plane be produced; it is clear that the straight lines, $AB$, $A\Gamma$, $ZH$ are simultaneously produced. Since $ZH$ is either parallel to $A\Gamma$ or meets it, when produced, beyond the point $A$, therefore $ZH$, $A\Gamma$ when produced in the directions $H$, $\Gamma$, will never meet. Let them be produced accordingly, and let there be taken any point $\Theta$ at random upon $ZH$, and through the point $\Theta$ let $K\Theta\Lambda$ be drawn parallel to $B\Gamma$, and let $M\Theta N$ be drawn parallel to $\Delta E$; the plane through $K\Lambda$, $MN$ is therefore parallel to the plane through $B\Gamma$, $\Delta E$ [Eucl. xi. 15]. Therefore the plane $K\Lambda MN$ is a circle [Prop. 4]. And since the points $\Delta$, $E$, $M$, $N$ are in the cutting plane, and are also on the surface of the cone, they are therefore upon the common section; therefore $\Delta ZE$ has increased to $M$, $N$. Therefore, when the surface of the cone and the cutting plane increase up to the circle $K\Lambda MN$, the section $\Delta ZE$ increases up to the points $M$, $N$. Similarly we may prove that, if the surface of the cone and the cutting plane be produced to infinity, the section $M\Delta ZEN$ will increase to infinity.

And it is clear that there can be cut off from the
ἀπολήμεται τις ἀπὸ τῆς ΖΘ εὐθείας πρὸς τῷ Ζ σημείῳ. ἐὰν γὰρ τῇ δοθείσῃ ἴσῃ δώμεν τὴν ΖΞ καὶ διὰ τοῦ Ξ τῇ ΔΕ παράλληλον ἀγάγωμεν, συμπεσεῖται τῇ τομῇ, ὡσπερ καὶ ἡ διὰ τοῦ Θ ἀπεδέιξθη συμπίπτουσα τῇ τομῇ κατὰ τὰ Μ, Ν σημεία· ὡστε ἀγεταί τις εὐθεία συμπίπτουσα τῇ τομῇ παράλληλος οὐσα τῇ ΔΕ ἀπολαμβάνουσα ἀπὸ τῆς ΖΗ εὐθείαν ἴσην τῇ δοθείσῃ πρὸς τῷ Ζ σημείῳ.

θ'

'Εὰν κῶνος ἐπιπέδω τμηθῇ συμπίπτοντι μὲν ἑκατέρα πλευρά τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παρὰ τὴν βάσιν ἡγεμένῳ μήτε ὑπεναντίως, ἡ τομὴ οὐκ ἔσται κύκλος.

'Εστω κῶνος, οὐ κορυφὴ μὲν τὸ Α σημεῖον,
straight line $Z\Theta$ in the direction of the point $Z$ an intercept equal to any given straight line. For if we place $Z\Xi$ equal to the given straight line and through $\Xi$ draw a parallel to $\Delta E$, it will meet the section, just as the parallel through $\Theta$ was shown to meet the section at the points $M$, $N$; therefore a straight line parallel to $\Delta E$ has been drawn to meet the section so as to cut off from $ZH$ in the direction of the point $Z$ an intercept equal to the given straight line.

Prop. 9

* If a cone be cut by a plane meeting either side of the axial triangle, but neither parallel to the base nor subcontrary, the section will not be a circle.

Let there be a cone whose vertex is the point $A$ and base the circle $B\Gamma$, and let it be cut by a plane neither parallel to the base nor subcontrary, and let

- In the figure of this theorem, the section of the cone by the plane $\Delta E$ would be a subcontrary section ($\upsilon\epsilon\nu\alpha\nu\alpha\tau\iota \tau\omicron\upsilon$) if the triangle $A\Delta E$ were similar to the triangle $A\Gamma B$, but in a contrary sense, i.e., if angle $A\Delta E =$ angle $A\Gamma B$. Apollonius proves in i. 5 that subcontrary sections of the cone are circles; it was proved in i. 4 that all sections parallel to the base are circles.
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evantoiv, kai poieito tov tov en tη epifaneia tηn DKE grammηn. leγw, oti h DKE grammηn ouk estai kyklou.

Ei gar dynatohon, estw, kai synaptetw to temon epiponed tη basεi, kai estw twv epipenedon kouh tov h ZH, to de kentron tou BG kyclou estw to Θ, kai ap autou kαθetos ηξθω επi tηn ZH h ΘΗ, kai ekbeblisΘw dia tηs HΘ kai tou δεκονοσ epipedon kai poieito tovmaes en tη kwnikη epifaneia tas BA, AG euvtheias. eπei oiv tα Δ, E, H sηmeia en te tw dia tηs DKE epipedo estwn, esti de kai en tw δia twv A, B, Γ, tα ara Δ, E, H sηmeia epι tηs kouhηs tovmaes twn epipedωn estwn: euvtheia ara estin η HED. eilophw δη τι epι tηs DKE grammηs sηmeion to K, kai dia tou K tη ZH paralληlos ηξθω h ΚΛ. estai δη isη h KM tη ΜΛ. ἡ άra ΔΕ διαμετρου esti tou DΚΛΕ κυκλου. hξθω δη δia tou M tη BG paralληlos η NΜΕ. esti de kai h ΚΛ tη ZH paralληlos: άste tο δia twv NΕ, KM epipedon paralληlou esti δia twv BG, ZH, touteste tη basεi, kai estai h tovma kυκλου. estw o NΚΕ. kai eπei h ZH tη BH prοs orθas esti, kai h KM tη NΕ prοs orθas estin: άste tο υπο twv NΜΕ άsou esti tο υπο απo tηs KM. esti de tο υπο twv DΜΕ άsou tο υπο της KM: kyclou gar upokeita h DΚΕΛ grammη, kai διαμετρou autou h ΔΕ. tο ara υπο twv NΜΕ άsou esti tο υπο DΜΕ. estin ara ωs h MN prοs MD, oútωs h EM prοs ME. omoion ara esti to DΜΝ trignonon tο ΕΜΕ trigwvω, kai h υπο DΝΜ γwvia isη esti tη υπο ΜΕΣ. alla

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the section so made on the surface be the curve $\Delta KE$. I say that the curve $\Delta KE$ will not be a circle.

For, if possible, let it be, and let the cutting plane meet the base, and let the common section of the planes be $ZH$, and let the centre of the circle $B\Gamma$ be $\Theta$, and from it let $\Theta H$ be drawn perpendicular to $ZH$, and let the plane through $H\Theta$ and the axis be produced, and let the sections made on the conical surface be the straight lines $BA, AT$. Then since the points $\Delta, E, H$ are in the plane through $\Delta KE$, and are also in the plane through $A, B, \Gamma$, therefore the points $\Delta, E, H$ are on the common section of the planes; therefore $HE\Delta$ is a straight line [Eucl. xi. 3]. Now let there be taken any point $K$ on the curve $\Delta KE$, and through $K$ let $KA$ be drawn parallel to $ZH$; then $KM$ will be equal to $MA$ [Prop. 7]. Therefore $\Delta E$ is a diameter of the circle $\Delta KEA$ [Prop. 7, coroll.]. Now let $NM\Xi$ be drawn through $M$ parallel to $B\Gamma$; but $KA$ is parallel to $ZH$; therefore the plane through $N\Xi$, $KM$ is parallel to the plane through $B\Gamma, ZH$ [Eucl. xi. 15], that is to the base, and the section will be a circle [Prop. 4]. Let it be $NK\Xi$. And since $ZH$ is perpendicular to $BH$, $KM$ is also perpendicular to $N\Xi$ [Eucl. xi. 10]; therefore $NM \cdot M\Xi = KM^2$. But $\Delta M \cdot ME = KM^2$; for the curve $\Delta KEA$ is by hypothesis a circle, and $\Delta E$ is a diameter in it. Therefore $NM \cdot M\Xi = \Delta M \cdot ME$. Therefore $MN : M\Delta = EM : M\Xi$. Therefore the triangle $\Delta MN$ is similar to the triangle $\Xi ME$, and the angle $\Delta NM$ is equal to the angle $ME\Xi$. 

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Ἐὰν κώνος ἐπιπέδῳ τμηθῇ διὰ τοῦ ἀξονος, τμηθῇ δὲ καὶ έτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ’ εὐθείαν πρὸς ὀρθάς οὖσαν τῇ βάσει τοῦ διὰ τοῦ ἀξονος τριγώνου, ἐτὶ δὲ ἡ διάμετρος τῆς τομῆς παράλληλος ἢ μηδε πλευρα τοῦ διὰ τοῦ ἀξονος τριγώνου, ἢτης ἢν ἀπὸ τῆς τομῆς τοῦ κώνου παράλληλος ἀρνθῇ τῇ κοινῇ τομῇ τοῦ τεμνοντος ἐπιπέδου καὶ τῆς βάσεως τοῦ κώνου μέχρι τῆς διαμέτρου τῆς τομῆς, δυνησεται τὸ περιεχόμενον ὑπὸ τε τῆς ἀπολαμβανομένης ὑπ’ αὐτής ἀπὸ τῆς διαμετρου πρὸς τῇ κορυφῇ τῆς τομῆς καὶ ἀλλῆς τινος εὐθείας, ἢ λόγον ἔχει πρὸς τὴν μεταξὺ τῆς τοῦ κώνου γωνίας καὶ τῆς κορυφῆς τῆς τομῆς, ὅν τὸ τετράγωνον τὸ ἀπὸ τῆς βάσεως τοῦ διὰ τοῦ ἀξονος τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν καλείσθω δὲ ἡ τοιαύτη τομῇ παραβολή.

Εστίως κώνος, οὗ τὸ Α σημείον κορυφῆ, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσθων ἐπιπέδῳ διὰ τοῦ ἀξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τρίγωνον, τετμήσθω δὲ καὶ έτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ’ εὐθείαν τὴν ΔΕ πρὸς ὀρθάς.
But the angle \( \Delta NM \) is equal to the angle \( \Delta B \Gamma \); for \( N \Xi \) is parallel to \( BI' \); and therefore the angle \( \Delta B \Gamma \) is equal to the angle \( ME \Xi \). Therefore the section is subcontrary [Prop. 5]; which is contrary to hypothesis. Therefore the curve \( \Delta K \Xi \) is not a circle.

(v.) Fundamental Properties

Ibid., Props. 11-14, Apoll. Perg. ed. Heiberg i. 36. 26-58. 7

Prop. 11

Let a cone be cut by a plane through the axis, and let it be also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and further let the diameter of the section be parallel to one side of the axial triangle; then if any straight line be drawn from the section of the cone parallel to the common section of the cutting plane and the base of the cone as far as the diameter of the section, its square will be equal to the rectangle bounded by the intercept made by it on the diameter in the direction of the vertex of the section and a certain other straight line; this straight line will bear the same ratio to the intercept between the angle of the cone and the vertex of the segment as the square on the base of the axial triangle bears to the rectangle bounded by the remaining two sides of the triangle; and let such a section be called a parabola.

For let there be a cone whose vertex is the point \( A \) and whose base is the circle \( B \Gamma \), and let it be cut by a plane through the axis, and let the section so made be the triangle \( \Delta B \Gamma \), and let it be cut by another plane cutting the base of the cone in the straight line
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όυσαν τῇ ΒΓ, καὶ ποιεῖτω τομῆν ἐν τῇ ἑπιφανείᾳ τοῦ κώνου τῆν ΔΖΕ, ἡ δὲ διάμετρος τῆς τομῆς ἡ ΖΗ παράλληλος ἐστω μία πλευρὰ τοῦ διὰ τοῦ ἀξονος τριγώνου τῇ ΑΓ, καὶ ἀπὸ τοῦ Ζ σημείου τῇ ΖΗ εὐθείᾳ πρὸς ὀρθὰς ἡ χθω ἡ ΖΘ, καὶ πεποιήθω, ὡς τὸ ἀπὸ ΒΓ πρὸς τὸ ὑπὸ ΒΑΓ, οὕτως ἡ ΖΘ πρὸς ΖΑ, καὶ εἰλήφθω τι σημείου ἐπὶ τῆς τομῆς τυχόν τὸ Κ, καὶ διὰ τοῦ Κ τῇ ΔΕ παράλληλος ἡ ΚΛ. λέγω, ὅτι τὸ ἀπὸ τῆς ΚΛ ἵσον ἐστὶ τῷ ὑπὸ τῶν ΘΩΛ.

"Ἡχθω γὰρ διὰ τοῦ Α τῇ ΒΓ παράλληλος ἡ ΜΝ. ἐστὶ δὲ καὶ ἡ ΚΛ τῇ ΔΕ παράλληλος· τὸ ἀρα διὰ τῶν ΚΛ, ΜΝ ἐπίπεδον παράλληλον ἐστὶ τῷ διὰ τῶν ΒΓ, ΔΕ ἐπιπέδῳ, τούτεστι τῇ βάσει τοῦ κώνου. τὸ ἀρα διὰ τῶν ΚΛ, ΜΝ ἐπίπεδον κύκλος ἐστίν, οὐ διάμετρος ἡ ΜΝ. καὶ ἐστὶ κάθετος ἐπὶ τὴν ΜΝ ἡ ΚΛ, ἐπεὶ καὶ ἡ ΔΕ ἐπὶ τὴν ΒΓ· τὸ ἀρα ὑπὸ τῶν ΜΑΝ ἵσον ἐστὶ τῷ ἀπὸ τῆς ΚΛ. καὶ ἐπεὶ ἐστιν, ὡς τὸ ἀπὸ τῆς ΒΓ πρὸς τὸ ὑπὸ τῶν ΒΑΓ, οὕτως ἡ ΘΖ πρὸς ΖΑ, τὸ δὲ
ΔE perpendicular to BG, and let the section so made on the surface of the cone be ΔZE, and let ZH, the diameter of the section, be parallel to ΑΓ, one side of the axial triangle, and from the point Z let ZΘ be drawn perpendicular to ZH, and let BG² : BA . AG = ZΘ : ZA, and let any point K be taken at random on the section, and through K let KA be drawn parallel to ΔE. I say that KA² = ΘZ . ZA.

For let MN be drawn through A parallel to BG; but KA is parallel to ΔE; therefore the plane through

KL, MN is parallel to the plane through BG, ΔE [Eucl. xi. 15], that is to the base of the cone. Therefore the plane through KA, MN is a circle, whose diameter is MN [Prop. 4]. And KA is perpendicular to MN, since ΔE is perpendicular to BG [Eucl. xi. 10]; therefore

MA . AN = KA².

And since BG² : BA . AG = ΘZ : ZA,
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ἀπὸ τῆς ΒΓ πρὸς τὸ ὑπὸ τῶν ΒΑΓ λόγου ἐχει τὸν συγκείμενον ἐκ τε τοῦ, ὅπε ἐχει ἡ ΒΓ πρὸς ΓΑ καὶ ἡ ΒΓ πρὸς ΒΑ, ὅ ἀρα τῆς ΘΖ πρὸς ΖΑ λόγος σύγκειται ἐκ τοῦ τῆς ΒΓ πρὸς ΓΑ καὶ τοῦ τῆς ΓΒ πρὸς ΒΑ. ἀλλ’ ὡς μὲν ἡ ΒΓ πρὸς ΓΑ, οὕτως ἡ ΜΝ πρὸς ΝΑ, τουτέστων ἡ ΜΛ πρὸς ΛΖ, ὡς δὲ ἡ ΒΓ πρὸς BA, οὕτως ἡ MN πρὸς MA, τουτέστων ἡ ΛΜ πρὸς ΜΖ, καὶ λοιπὴ ἡ ΝΛ πρὸς ΖΑ. ὅ ἀρα τῆς ΘΖ πρὸς ΖΑ λόγος σύγκειται ἐκ τοῦ τῆς ΜΛ πρὸς ΛΖ καὶ τοῦ τῆς ΝΛ πρὸς ΖΑ. ὅ δὲ συγκείμενος λόγους ἐκ τοῦ τῆς ΜΛ πρὸς ΛΖ καὶ τοῦ τῆς ΛΝ πρὸς ΖΑ ὁ τοῦ ὑπὸ ΜΛΝ ἐστὶ πρὸς τὸ ὑπὸ ΛΖΑ. ὡς ἀρα ἡ ΘΖ πρὸς ΖΑ, οὕτως τὸ ὑπὸ ΜΛΝ πρὸς τὸ ὑπὸ ΛΖΑ. ὡς δὲ ἡ ΘΖ πρὸς ΖΑ, τῆς ΖΛ κοινοῦ ὑψους λαμβανομένης οὕτως τὸ ὑπὸ ΘΖΑ πρὸς τὸ ὑπὸ ΛΖΑ· ὡς ἀρα τὸ ὑπὸ ΜΛΝ πρὸς τὸ ὑπὸ ΛΖΑ, οὕτως τὸ ὑπὸ ΘΖΑ πρὸς τὸ ὑπὸ ΛΖΑ. ἦσον ἀρα ἐστὶ τὸ ὑπὸ ΜΛΝ τῷ ὑπὸ ΘΖΑ. τὸ δὲ ὑπὸ ΜΛΝ ἦσον ἐστὶ τῷ ἀπὸ τῆς ΚΛ· καὶ τὸ ἀπὸ τῆς ΚΛ ἀρα ἦσον ἐστὶ τῷ ὑπὸ τῶν ΘΖΑ.

Καλείσθω δὲ ἡ μὲν τουιαύτη τομὴ παραβολῆ, ἡ δὲ ΘΖ παρ’ ἦν δύνανται αἱ καταγόμενοι τεταγμένως ἐπὶ τὴν ΖΗ διάμετρον, καλείσθω δὲ καὶ ὀρθλα.

νβ'

Ἐὰν κώνος ἐπιπέδῳ τμηθῇ διὰ τοῦ ἀξονος, τμηθῆ δὲ καὶ ἐτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν

* A parabola (παραβολή) because the square on the ordinate ΚΛ is applied (παραβαλεῖν) to the parameter ΘΖ in the form 308
while \( \Gamma \Gamma^2 : \Gamma A \cdot \Lambda \Gamma = (\Gamma \Gamma : \Gamma A)(\Gamma \Gamma : \Gamma A) \),

therefore \( \Theta Z : ZA = (\Gamma \Gamma : \Gamma A)(\Gamma \Gamma : \Gamma A) \).

But \( \Gamma \Gamma : \Gamma A = MN : NA \)
\[ = MA : AZ, \quad [\text{Eucl. vi. 4} \]

and \( \Gamma \Gamma : \Gamma A = MN : MA \)
\[ = \Lambda M : MZ \quad [\text{ibid.} \]
\[ = NA : ZA. \quad [\text{Eucl. vi. 2} \]

Therefore \( \Theta Z : ZA = (MA : AZ)(NA : ZA) \).

But \( (MA : AZ)(AN : ZA) = MA \cdot AN : AZ \cdot ZA \).

Therefore \( \Theta Z : ZA = MA \cdot AN : AZ \cdot ZA \).

But \( \Theta Z : ZA = \Theta Z \cdot ZA : \Lambda Z \cdot ZA \),

by taking a common height \( ZA \); therefore \( MA \cdot AN : AZ \cdot ZA = \Theta Z \cdot ZA : \Lambda Z \cdot ZA \).

Therefore \( MA \cdot AN = \Theta Z \cdot ZA. \quad [\text{Eucl. v. 9} \]

But \( MA \cdot AN = K\Lambda^2 \); and therefore \( K\Lambda^2 = \Theta Z \cdot ZA \).

Let such a section be called a parabola, and let \( \Theta Z \) be called the parameter of the ordinates to the diameter \( ZH \), and let it also be called the erect side (latus rectum).\(^a\)

Prop. 12

Let a cone be cut by a plane through the axis, and let it be cut by another plane cutting the base of the cone in of the rectangle \( \Theta Z \cdot ZA \), and is exactly equal to this rectangle. It was Apollonius's most distinctive achievement to have based his treatment of the conic sections on the Pythagorean theory of the application of areas (\( \pi \alpha \rho \alpha \beta o \lambda \gamma \tau \omega \nu \chi o \rho \iota \lambda o \nu)\), for which \( v. \) vol. i. pp. 186-215. The explanation of the term latus rectum will become more obvious in the cases of the hyperbola and the ellipse; \( v. \) infra, p. 317 n. a.
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τοῦ κῶνου κατ' εὐθείαν πρὸς ὅρθας οὐδαν τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου, καὶ ἡ διάμετρος τῆς τομῆς ἐκβαλλομένη συμπίπτη μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου ἐκτὸς τῆς τοῦ κῶνου κορυφῆς, ἢτις ἀν ἀπὸ τῆς τομῆς ἀχθῇ παράλληλος τῇ κοινῇ τομῇ τοῦ τέμνοντος ἑπιπέδου καὶ τῆς βάσεως τοῦ κώνου, ἔως τῆς διαμέτρου τῆς τομῆς δυνησται τι χωρόν παρακείμενον παρὰ τίνα εὐθείαν, πρὸς ἦν λόγον ἔχει ἡ ἐπ' εὐθείας μὲν οὖσα τῇ διαμέτρῳ τῆς τομῆς, ὑποτείνουσα δὲ τὴν ἐκτὸς τοῦ τριγώνου γωνίαν, διὸ τὸ τετράγωνον τὸ ἀπὸ τῆς ἡγιμένης ἀπὸ τῆς κορυφῆς τοῦ κῶνου παρὰ τὴν διαμέτρου τῆς τομῆς ἔως τῆς βάσεως τοῦ τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν τῆς βάσεως τμημάτων, ὃν ποιεῖ ἡ ἀχθείσα, πλάτος ἔχον τὴν ἀπολαμβανομένην ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς, ὑπερβάλλον εἰδεί όμοιο τε καὶ ὁμοίως κειμένῳ τῷ περιεχομένῳ ὑπὸ τε τῆς ὑποτεινούσης τὴν ἐκτὸς γωνίαν τοῦ τριγώνου καὶ τῆς παρ' ἢν δύνανται αἱ καταγόμεναι· καλείσθω δὲ ἡ τοιαύτη τομὴ ὑπερβολή.

"Εστώ κῶνος, οὗ κορυφή μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμῆσθω ἑπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τριγώνον, τετμῆσθω δὲ καὶ ἑτέρῳ ἑπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κῶνου κατ' εὐθείαν τὴν ΔΕ πρὸς ὅρθας οὐδαν τῇ ΒΓ βάσει τοῦ ΑΒΓ τριγώνου, καὶ ποιεῖτο τομὴν ἐν τῇ ἑπιφανείᾳ τοῦ κώνου τὴν ΔΖΕ γραμμῆν, ἢ δὲ διαμέτρος τῆς τομῆς ἡ ΖΗ ἐκβαλλομένη συμπιπτέτω μιᾷ πλευρᾷ τοῦ ΑΒΓ τριγώνου τῇ ΑΓ ἐκτὸς τῆς τοῦ κώνου κορυφῆς κατὰ τὸ Θ, καὶ διὰ τοῦ Α τῇ διαμέτρῳ τῆς τομῆς 310
a straight line perpendicular to the base of the axial triangle, and let the diameter of the section, when produced, meet one side of the axial triangle beyond the vertex of the cone; then if any straight line be drawn from the section of the cone parallel to the common section of the cutting plane and the base of the cone as far as the diameter of the section, its square will be equal to the area applied to a certain straight line; this line is such that the straight line subtending the external angle of the triangle, lying in the same straight line with the diameter of the section, will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle bounded by the segments of the base made by the line so drawn; the breadth of the applied figure will be the intercept made by the ordinate on the diameter in the direction of the vertex of the section; and the applied figure will exceed by a figure similar and similarly situated to the rectangle bounded by the straight line subtending the external angle of the triangle and the parameter of the ordinates; and let such a section be called a hyperbola.

Let there be a cone whose vertex is the point \(A\) and whose base is the circle \(\Gamma\), and let it be cut by a plane through the axis, and let the section so made be the triangle \(\Delta A\Gamma\), and let it be cut by another plane cutting the base of the cone in the straight line \(\Delta E\) perpendicular to \(\Gamma\), the base of the triangle \(\Delta A\Gamma\), and let the section so made on the surface of the cone be the curve \(\Delta ZE\), and let \(ZH\), the diameter of the section, when produced, meet \(\Gamma\), one side of the triangle \(\Delta A\Gamma\), beyond the vertex of the cone at \(\Theta\), and through \(A\) let \(AK\) be drawn parallel to \(ZH\), the
τῇ ΖΗ παράλληλος ἡχθω ἢ ΑΚ, καὶ τεμνέτω τὴν
ΒΓ, καὶ ἀπὸ τοῦ Ζ τῇ ΖΗ πρὸς ὀρθάς ἡχθω ἢ

ΖΛ, καὶ πεποιήθω, ὡς τὸ ἀπὸ ΚΑ πρὸς τὸ ὑπὸ
ΒΚΓ, οὕτως ἢ ΖΘ πρὸς ΖΛ, καὶ εἰλήφθω τι
ςμεῖον ἐπὶ τῆς τομῆς τυχοῦ τὸ Μ, καὶ διὰ τοῦ
Μ τῇ ΔΕ παράλληλος ἡχθω ἢ ΜΝ, διὰ δὲ τοῦ Ν
τῇ ΖΑ παράλληλος ἢ ΝΟΞ, καὶ ἐπιζευκθείσα ἡ
ΘΛ ἐκβεβληθήσῳ ἐπὶ τὸ Ε, καὶ διὰ τῶν Λ, Ε τῇ
ΖΝ παράλληλοι ἡχθωσαν αἰ ΔΟ, ΞΠ. λέγω, ὅτι
ἡ ΜΝ δύναται τὸ ΖΕ, δὲ παράκειται παρὰ τὴν
ΖΛ, πλάτος ἔχον τὴν ΖΝ, ὑπερβάλλων εἰδεί τῷ
ΛΣ ὀμοίω ὀντι τῷ ὑπὸ τῶν ΘΖΛ.

"Ἡχθω γὰρ διὰ τοῦ Ν τῇ ΒΓ παράλληλος ἢ
ΡΝΣ. ἐστι δὲ καὶ ἡ ΝΜ τῇ ΔΕ παράλληλος. τὸ
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diameter of the section, and let it cut $\Gamma \Gamma$, and from $Z$ let $Z\Lambda$ be drawn perpendicular to $ZH$, and let $KA^2: BK \cdot K\Gamma = Z\Theta : Z\Lambda$, and let there be taken at random any point $M$ on the section, and through $M$ let $MN$ be drawn parallel to $\Delta \varepsilon$, and through $N$ let $NO\Xi$ be drawn parallel to $Z\Lambda$, and let $\Theta \Lambda$ be joined and produced to $\Xi$, and through $\Lambda, \Xi$, let $\Lambda \Omega, \Xi \Pi$ be drawn parallel to $ZN$. I say that the square on $MN$ is equal to $Z\Xi$, which is applied to the straight line $Z\Lambda$, having $ZN$ for its breadth, and exceeding by the figure $\Lambda \Xi$ which is similar to the rectangle contained by $\Theta Z, Z\Lambda$.

For let $PN\Xi$ be drawn through $N$ parallel to $\Gamma \Gamma$; but $NM$ is parallel to $\Delta \varepsilon$; therefore the plane through
ἀρα διὰ τῶν MN, ΡΣ ἐπίπεδον παράλληλόν ἐστὶ
τῷ διὰ τῶν ΒΓ, ΔΕ, τούτεστι τῇ βάσει τοῦ κώνου.
εάν ἀρα ἐκβληθῇ τὸ διὰ τῶν MN, ΡΣ ἐπίπεδον,
ἡ τομὴ κύκλος ἐσται, οὗ διάμετρος ἡ ΡΝΣ.
καὶ ἐστιν ἐπ' αὐτὴν κάθετος ἡ MN· τὸ ἀρα ὑπὸ τῶν
ΡΝΣ ἦσον ἦστι τῷ ἀπὸ τῆς MN. καὶ ἐπεὶ ἐστιν,
ὡς τὸ ἀπὸ ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ, οὕτως ἡ ΖΘ
πρὸς ΖΛ, οὗ δὲ τοῦ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ
λόγος σύγκειται ἐκ τε τοῦ, δόν ἐχει ἡ ΑΚ πρὸς ΚΓ
καὶ ἡ ΑΚ πρὸς ΚΒ, καὶ οὗ τῆς ΖΘ ἀρα πρὸς τὴν
ΖΛ λόγος σύγκειται ἐκ τοῦ, δόν ἐχει ἡ ΑΚ πρὸς
ΚΓ καὶ ἡ ΑΚ πρὸς ΚΒ. ἀλλ' ὡς μὲν ἡ ΑΚ
πρὸς ΚΓ, οὕτως ἡ ΘΗ πρὸς ΗΓ, τουτέστιν ἡ ΘΝ
πρὸς ΝΣ, ὡς δ' ἡ ΑΚ πρὸς ΚΒ, οὕτως ἡ ΖΗ πρὸς
ΗΒ, τουτέστιν ἡ ΖΝ πρὸς ΝΡ. ὁ ἀρα τῆς ΘΖ
πρὸς ΖΛ λόγος σύγκειται ἐκ τε τοῦ τῆς ΘΝ πρὸς
ΝΣ καὶ τοῦ τῆς ΖΝ πρὸς ΝΡ. ὁ δ' συγκείμενος
λόγος ἐκ τοῦ τῆς ΘΝ πρὸς ΝΣ καὶ τοῦ τῆς ΖΝ
πρὸς ΝΡ ὁ τοῦ ὑπὸ τῶν ΘΝΖ ἐστὶ πρὸς τὸ ὑπὸ
tῶν ΣΝΡ· καὶ ὡς ἄρα τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ
ὑπὸ τῶν ΣΝΡ, οὕτως ἡ ΘΖ πρὸς ΖΛ, τουτέστιν
ἡ ΘΝ πρὸς ΝΕ. ἀλλ' ὡς ἡ ΘΝ πρὸς ΝΕ, τῆς
ΖΝ κοινοῦ ὑψος λαμβανομένης οὕτως τὸ ὑπὸ
tῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΖΝΕ· καὶ ὡς ἄρα
τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΣΝΡ, οὕτως
τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΕΝΖ. τὸ
ἄρα ὑπὸ ΣΝΡ ἦσον ἦστι τῷ ὑπὸ ΕΝΖ. τὸ δὲ
ἀπὸ MN ἦσον ἐδείχθη τῷ ὑπὸ ΣΝΡ· καὶ τὸ ἀπὸ
tῆς MN ἀρα ἦσον ἦστι τῷ ὑπὸ τῶν ΕΝΖ. τὸ δὲ
ὑπὸ ΕΝΖ ἦστι τὸ ΕΖ παραλληλόγραμμον. ἡ ἄρα
MN, ΡΣ is parallel to the plane through BΓ, ΔΕ
[Eucl. xi. 15], that is to the base of the cone. If,
then, the plane through MN, ΡΣ be produced, the
section will be a circle with diameter PNΣ [Prop. 4].
And MN is perpendicular to it; therefore

\[ PN \cdot ΝΣ = MN^2. \]

And since \( AK^2 : BK \cdot KG = ZΩ : ZΛ, \)
while \( AK^2 : BK \cdot KG = (AK : KG)(AK : KB), \)
therefore \( ZΩ : ZΛ = (AK : KG)(AK : KB). \)
But \( AK : KG = ΘH : HG, \)
i.e., \( = ΘN : ΝΣ, \) [Eucl. vi. 4
and \( AK : KB = ZH : HB, \)
i.e., \( = ZN : NP. \) [ibid.
Therefore \( ΘZ : ZΛ = (ΘN : ΝΣ)(ZN : NP). \)
But \( (ΘN : ΝΣ)(ZN : NP) = ΘN \cdot NZ : ΣN \cdot NP; \)
and therefore

\[ ΘN \cdot NZ : ΣN \cdot NP = ΘZ : ZΛ \]

\[ = ΘN : ΝΞ. \] [ibid.
But \( ΘN : ΝΞ = ΘN \cdot NZ : ZN \cdot ΝΞ, \)
by taking a common height ZN.
And therefore

\[ ΘN \cdot NZ : ΣN \cdot NP = ΘN \cdot NZ : ΞN \cdot NZ. \]
Therefore \( ΣN \cdot NP = ΞN \cdot NZ. \) [Eucl. v. 9
But \( MN^2 = ΣN \cdot NP, \)
as was proved;
and therefore \( MN^2 = ΞN \cdot NZ. \)
But the rectangle ΞN \cdot NZ is the parallelogram ΞZ.
Mη δύναται τὸ ΖΛ, δ' παράκειται παρὰ τὴν ΖΛ, πλάτος ἔχου τὴν ΖΝ, ὑπερβάλλον τῷ ΔΞ ὁμοίω ὀντι τῷ ὑπὸ τῶν ὈΖΛ. καλείσθω δὲ ἡ μὲν τοιαύτη τομή ὑπερβολή, ἡ δὲ ΔΖ παρ' ἦν δύνανται αἱ ἐπὶ τὴν ΖΗ καταγόμεναι τεταγμένως· καλείσθω δὲ ἡ αὐτὴ καὶ ὀρθία, πλαγία δὲ ἡ ΖΘ.

'Εὰν κώνος ἐπιπέδω τμηθῇ διὰ τοῦ ἄξονος, τμηθῇ δὲ καὶ ἐτέρω ἐπιπέδῳ συμπίπτοντι μὲν ἐκατέρα πλευρά τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παρὰ τὴν βάσιν τοῦ κώνου ἡγεμένῳ μήτε ὑπεναντίως, τὸ δὲ ἐπιπέδον, ἐν ὦ ἐστιν ἡ βάσις τοῦ κώνου, καὶ τὸ τέμνον ἐπιπέδον συμπίπτη κατ' εὑθεῖαν πρὸς ὀρθᾶς ὀδύσαν ἦτοι τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου ἢ τῇ ἐπ' εὐθείας αὐτῇ, ἤτοι ἃν ἀπὸ τῆς τομῆς τοῦ κώνου παράλληλος ἄχθη τῇ κοινῇ τομῇ τῶν ἐπιπέδων ἐως τῆς διαμέτρου τῆς τομῆς, δυνήσεται τι χωρίον παρακείμενον παρά τινα εὐθεῖαν, πρὸς ἧν λόγον ἐχεῖ ἡ διάμετρος τῆς τομῆς, διὸ τὸ τετράγωνον τὸ ἀπὸ τῆς ἡγεμένης ἀπὸ τῆς κορυφῆς τοῦ κώνου παρὰ τὴν διαμέτρου τῆς τομῆς ἐως τῆς βάσεως τοῦ τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν ἀπολαμβανομένων ὑπ' αὐτῆς πρὸς ταῖς τοῦ τριγώνου εὐθείας, πλάτος ἔχου τὴν ἀπολαμβανομένην ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς, ἐλλεῖπον εἴδει ὀμοίω τε καὶ ὀμοίως κειμένω τῷ περιεχο- μένῳ ὑπὸ τε τῆς διαμέτρου καὶ τῆς παρ' ἦν δύνανται· καλείσθω δὲ ἡ τοιαύτη τομῇ ἐλλεῖψις.

"Εστω κώνος, οὗ κορυφῆ μὲν τὸ Α σημεῖον,
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Therefore the square on MN is equal to $EZ$, which is applied to $ZA$, having $ZN$ for its breadth, and exceeding by $AZ$ similar to the rectangle contained by $OZ$, $ZA$. Let such a section be called a hyperbola, let $AZ$ be called the parameter to the ordinates to $ZH$; and let this line be also called the erect side (latus rectum), and $Z\Theta$ the transverse side.\textsuperscript{a}

Prop. 13

Let a cone be cut by a plane through the axis, and let it be cut by another plane meeting each side of the axial triangle, being neither parallel to the base nor subcontrary, and let the plane containing the base of the cone meet the cutting plane in a straight line perpendicular either to the base of the axial triangle or to the base produced; then if a straight line be drawn from any point of the section of the cone parallel to the common section of the planes as far as the diameter of the section, its square will be equal to an area applied to a certain straight line; this line is such that the diameter of the section will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle contained by the intercepts made by it on the sides of the triangle; the breadth of the applied figure will be the intercept made by it on the diameter in the direction of the vertex of the section; and the applied figure will be deficient by a figure similar and similarly situated to the rectangle bounded by the diameter and the parameter; and let such a section be called an ellipse.

Let there be a cone, whose vertex is the point $A$

\textsuperscript{a} The erect and transverse side, that is to say, of the figure ($\ell\delta\sigma$) applied to the diameter. In the case of the parabola, the transverse side is infinite.
βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμῆσθω ἐπιπέδω διὰ τοῦ ἄξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τριγώνου, τετμῆσθω δὲ καὶ ἑτέρῳ ἐπιπέδῳ συμπληροῦντι μὲν ἐκατέρα πλευρά τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παραλλήλω τῇ βάσει τοῦ κώνου μήτε ὑπεναντίως ἡγμένῳ, καὶ ποιεῖτω τομὴν ἐν τῇ ἐπιφάνειᾳ τοῦ κώνου τὴν ΔΕ γραμμήν· κοινῇ

δὲ τομὴ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ, ἐν ὧν ἐστὶν ἡ βάσις τοῦ κώνου, ἔστω ἡ ΖΗ πρὸς ὀρθὰς οὕσα τῇ ΒΓ, ἡ δὲ διάμετρος τῆς τομῆς ἐστὶν ἡ ΕΔ, καὶ ἀπὸ τοῦ Ε τῇ ΕΔ πρὸς ὀρθὰς ἡχθων ἡ ΕΘ, καὶ διὰ τοῦ Λ τῇ ΕΔ παράλληλος ἡχθων ἡ ΑΚ, καὶ πεποιηθῶν ὡς τὸ ἀπὸ ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΘ, καὶ εἰλήφθων τι σημεῖον ἐπὶ τῆς τομῆς τὸ Λ, καὶ διὰ τοῦ Λ τῇ ΖΗ παράλληλος ἡχθων ἡ ΛΜ. λέγω, ὅτι ἡ ΛΜ δύναται τι χωρίον, δι' ἑκείνα παρὰ τὴν ΕΘ, πλάτος ἐχον τὴν ΕΜ, ἐλλείπον εἰδει ὁμοίω τῷ ὑπὸ τῶν ΔΕΘ.

Ἐπεζεύχθω γὰρ ἡ ΔΘ, καὶ διὰ μὲν τοῦ Μ τῇ
and whose base is the circle $BB\Gamma$, and let it be cut by a plane through the axis, and let the section so made be the triangle $AB\Gamma$, and let it be cut by another plane meeting either side of the axial triangle, being drawn neither parallel to the base nor subcontrary, and let the section made on the surface of the cone be the curve $\Delta E$; let the common section of the cutting plane and of that containing the base of the cone be $ZH$, perpendicular to $BB\Gamma$, and let the diameter of the section be $E\Delta$, and from $E$ let $E\Theta$ be drawn perpendicular to $E\Delta$, and through $A$ let $AK$ be drawn parallel to $E\Delta$, and let $AK^2 : BK \cdot K\Gamma = \Delta E : E\Theta$, and let any point $A$ be taken on the section, and through $A$ let $AM$ be drawn parallel to $ZH$. I say that the square on $AM$ is equal to an area applied to the straight line $E\Theta$, having $EM$ for its breadth, and being deficient by a figure similar to the rectangle contained by $\Delta E$, $E\Theta$.

For let $\Delta \Theta$ be joined, and through $M$ let $M\Xi N$ be
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ΘΕ παράλληλος ἡχωθ η ΜΕΝ, διά δε των Θ, Ε τη ΕΜ παράλληλοι ἡχωθσαν αι ΘΝ, ΞΟ, και διά του Μ τη ΒΓ παράλληλος ἡχωθ η ΠΜΡ. ετει ουν η ΠΡ τη ΒΓ παράλληλος ἑστιν, ἑστι δε και η ΛΜ τη ΖΗ παράλληλος, το ἀρα δια των ΛΜ, ΠΡ ἐπιπέδου παράλληλον ἑστι τῇ δια των ΖΗ, ΒΓ ἐπιπέδω, τοντέστι τῇ βάσει του κόσων. εὰν ἄρα ἐκβληθῇ δια των ΛΜ, ΠΡ ἐπιπέδου, ἡ τομὴ κύκλος ἑσται, οὗ διάμετρος ἡ ΠΡ. καὶ ἑστὶ κάθετος ἐπ’ αὐτὴν ἡ ΛΜ· το ἀρα ὑπὸ των ΠΜΡ ἰσον ἑστὶ τῷ ἀπὸ τῆς ΛΜ. καὶ ετει ἑστιν, ὡς το ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ τῶν ΒΚΓ, οὕτως ἡ ΕΔ πρὸς τῆν ΕΘ, ὃ δὲ τοῦ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ τῶν ΒΚΓ λόγος σύγκειται ἐκ τοῦ, διν ἐχει ἡ ΑΚ πρὸς ΚΒ, καὶ ἡ ΑΚ πρὸς ΚΓ, ἀλλ’ ὡς μὲν ἡ ΑΚ πρὸς ΚΒ, οὕτως ἡ ΕΗ πρὸς ΗΒ, τοντέστιν ἡ ΕΜ πρὸς ΜΠ, ὡς δὲ ἡ ΑΚ πρὸς ΚΓ, οὕτως ἡ ΔΗ πρὸς ΗΓ, τοντέστιν ἡ ΔΜ πρὸς ΜΡ, ὃ ἀρα τῆς ΔΕ πρὸς τῆν ΕΘ λόγος σύγκειται ἐκ τε τοῦ τῆς ΕΜ πρὸς ΜΠ καὶ τοῦ τῆς ΔΜ πρὸς ΜΡ. ὃ δὲ συγκείμενος λόγος ἐκ τε τοῦ, διν ἐχει ἡ ΕΜ πρὸς ΜΠ, καὶ ἡ ΔΜ πρὸς ΜΡ, ὃ τοῦ ὑπὸ τῶν ΕΜΔ ἑστὶ πρὸς τὸ ὑπὸ τῶν ΠΜΡ. ἑστιν ἁρα ὡς τὸ ὑπὸ τῶν ΕΜΔ πρὸς τὸ ὑπὸ τῶν ΠΜΡ, οὕτως ἡ ΔΕ πρὸς τῆν ΕΘ, τοντέστιν ἡ ΔΜ πρὸς τῆν ΜΕ. ὡς δὲ ἡ ΔΜ πρὸς ΜΕ, τῆς ΜΕ κοινοῦ ὑψους λαμβανομένης, οὕτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΕΜΕ. καὶ ὡς ἁρα τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΠΜΡ, οὕτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΕΜΕ. ἰσον ἁρα ἑστὶ τὸ ὑπὸ ΠΜΡ τῷ ὑπὸ ΕΜΕ. τὸ δὲ ὑπὸ ΠΜΡ ἰσον ἐδείχθη τῷ ἀπὸ τῆς ΛΜ· καὶ τὸ ὑπὸ ΕΜΕ ἁρα ἑστὶν ἰσον τῷ ἀπὸ τῆς ΛΜ. ἡ ΛΜ
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drawn parallel to ΘΕ, and through Θ, Ζ, let ΘΝ, ΖΟ be drawn parallel to EM, and through M let ΠΜ be drawn parallel to BG. Then since ΠΡ is parallel to BG, and ΑΜ is parallel to ZH, therefore the plane through ΑΜ, ΠΡ is parallel to the plane through ZH, BG [Eucl. xi. 15], that is to the base of the cone. If, therefore, the plane through ΑΜ, ΠΡ be produced, the section will be a circle with diameter ΠΡ [Prop. 4]. And ΑΜ is perpendicular to it; therefore

\[ ΠΜ \cdot MP = ΑΜ^2. \]

And since \[ ΑΚ^2 : BK \cdot KG = EΔ : EΘ, \]
and \[ ΑΚ^2 : BK \cdot KG = (AK : KB)(AK : KG), \]
while \[ AK : KB = EH : HB = EM : MΠ, \]  [Eucl. vi. 4]
and \[ AK : KG = ΔΗ : HG = ΔΜ : MP, \]  [ibid.]
therefore \[ ΔΕ : EΘ = (EM : MΠ)(ΔΜ : MP). \]
But \[ (EM : MΠ)(ΔΜ : MP) = EM \cdot ΔΔ : ΠΜ \cdot MP. \]
Therefore \[ EM \cdot ΔΔ : ΠΜ \cdot MP = ΔΕ : EΘ \]
\[ = ΔΜ : MΞ. \]  [ibid.]
But \[ ΔΜ : MΞ = ΔΜ \cdot ME : ΞΜ \cdot ME, \]
by taking a common height ME.
Therefore \[ ΔΜ \cdot ME : ΠΜ \cdot MP = ΔΜ \cdot ME : ΞΜ \cdot ME. \]
Therefore \[ ΠΜ \cdot MP = ΞΜ \cdot ME. \]  [Eucl. v. 9]
But \[ ΠΜ \cdot MP = ΑΜ^2, \]
as was proved; and therefore \[ ΞΜ \cdot ME = ΑΜ^2. \]
Let be the parameter of a conic section and the corresponding diameter, and let the diameter of the section and the tangent at its extremity be taken as axes of co-ordinates (in general oblique). Then Props. 11-13 are equivalent to the Cartesian equations.
Therefore the square on $AM$ is equal to $MO$, which is applied to $OE$, having $EM$ for its breadth, and being deficient by the figure $ON$ similar to the rectangle $AE \cdot EO$. Let such a section be called an eclipse, let $EO$ be called the parameter to the ordinates to $AE$, and let this line be called the erect side (latus rectum), and $EA$ the transverse side.$^a$

Prop. 14

*If the vertically opposite surfaces [of a double cone] be cut by a plane not through the vertex, there will be formed on each of the surfaces the section called a hyperbola, and the diameter of both sections will be the same, and the parameter to the ordinates drawn parallel to the straight line in the base of the cone will be equal, and the transverse side of the figure will be common, being the straight line between the vertices of the sections; and let such sections be called opposite.*

Let there be vertically opposite surfaces having the point $A$ for vertex, and let them be cut by a plane not through the vertex, and let the sections so made on the surface be $AEZ$, $H\Theta K$. I say that each of the sections $AEZ$, $H\Theta K$ is the so-called hyperbola.

$y^2 = px^2$ (the parabola),
and

$y^2 = px^2 \pm \frac{p}{d}x^2$ (the hyperbola and ellipse respectively).

It is the essence of Apollonius's treatment to express the fundamental properties of the conics as equations between *areas*, whereas Archimedes had given the fundamental properties of the central conics as *proportions*

$$y^2 : (a^2 + x^2) = a^2 : b^2.$$  

This form is, however, equivalent to the Cartesian equations referred to axes through the centre.
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"Εστώ γὰρ ὁ κύκλος, καθ’ οὐ φέρεται ἡ τὴν ἐπιφάνειαν γράφουσα εὐθεία, ὁ ΒΔΓΖ, καὶ ἥχθω μὲν τῇ κατὰ κορυφὴν ἐπιφανείαν παράλληλον αὐτῷ ἐπίπεδον τὸ ΞΗΟΚ. κοινῶ δὲ τομαὶ τῶν ΗΘΚ, ΖΕΔ τομῶν καὶ τῶν κύκλων αἱ ΖΔ, ΗΚ. ἐσονται δὴ παράλληλοι. ἄξων δὲ ἐστώ τῆς κωνικῆς ἐπι- φανείας ἡ ΔΑΥ εὐθεία, κέντρα δὲ τῶν κύκλων τὰ Δ, Υ, καὶ ἀπὸ τοῦ Δ ἐπὶ τὴν ΖΔ κάθετος ἀξθεία ἐκβεβλήσθω ἐπὶ τὰ Β, Γ σημεία, καὶ διὰ τῆς ΒΓ καὶ τοῦ ἄξονος ἐπίπεδον ἐκβεβλήσθω, ποιήσει δὴ τομὰς ἐν μὲν τοῖσ κύκλοις παράλληλον ἐπὶ τὰς ΞΟ, ΒΓ, ἐν δὲ τῇ ἐπιφάνειᾳ τὰς ΒΑΟ, ΓΑΣ. 324
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For let $BA\Gamma Z$ be the circle round which revolves the straight line describing the surface, and in the vertically opposite surface let there be drawn parallel to it a plane $\Xi HOK$; the common sections of the sections $H\Theta K$, $ZE\Delta$ and of the circles [Prop. 4] will be $Z\Delta$, $HK$; and they will be parallel [Eucl. xi. 16]. Let the axis of the conical surface be $\Lambda\Lambda Y$, let the centres of the circles be $\Lambda$, $Y$, and from $\Lambda$ let a perpendicular be drawn to $Z\Delta$ and produced to the points $B$, $\Gamma$, and let the plane through $B\Gamma$ and the axis be produced; it will make in the circles the parallel straight lines $\Xi O$, $B\Gamma$, and on the surface $BAO$, $\Gamma A\Xi$;
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έσται δὴ καὶ ἥ ΞΟ τῇ ΗΚ πρὸς ὀρθὰς, ἐπειδὴ καὶ ἥ ΒΓ τῇ ΖΔ ἐστὶ πρὸς ὀρθὰς, καὶ ἔστιν ἐκατέρα παράλληλος. καὶ ἔτει τὸ διὰ τοῦ ἄξονος ἐπίπεδον ταῖς τομαῖς συμβάλλει κατὰ τὰ Μ, Ν σημεῖα ἐντὸς τῶν γραμμῶν, δὴλον, ὡς καὶ τὰς γραμμὰς τέμνει τὸ ἐπίπεδον. τεμνέτω κατὰ τὰ Θ, Ε. τὰ ἄρα Μ, Ε, Θ, Ν σημεῖα ἐν τῷ διὰ τοῦ ἄξονος ἐστὶν ἐπίπεδῳ καὶ ἐν τῷ ἐπίπεδῳ, ἐν ὃ εἰσίν αἱ γραμμαί· εὐθεία ἄρα ἐστὶν ἡ ΜΕΘΝ γραμμή. καὶ φανερὸν, ὅτι τά τε Ε, Θ, Α, Γ ἐπ’ εὐθείας ἐστὶ καὶ τὰ Β, Ε, Α, Ο. ἐν τῇ γὰρ τῇ κωνικῇ ἐπιφάνειᾳ ἐστὶ καὶ ἐν τῷ διὰ τοῦ ἄξονος ἐπίπεδῳ. ἥχωσαν δὴ ἀπὸ μὲν τῶν Θ, Ε τῇ ΘΕ πρὸς ὀρθὰς αἱ ΘΡ, ΕΠ, διὰ δὲ τοῦ Α τῇ ΜΕΘΝ παράλληλος ἥχω ἡ ΣΑΤ, καὶ πεποιηθῶθη, ὡς μὲν τὸ ἀπὸ τῆς ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὗτως ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ ἀπὸ τῆς ΑΤ πρὸς τὸ ὑπὸ ΟΤΕ, οὗτως ἡ ΘΘ πρὸς ΘΡ. ἐτεί oὐν κῶνος, oὐ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, τετμηται ἐπίπεδῳ διὰ τοῦ ἄξονος, καὶ πεποίηκε τομὴν τὸ ΑΒΓ τριγώνων, τετμηται δὲ καὶ ἑτέρῳ ἐπίπεδῳ τέμνοντι τὴν βάσιν τοῦ κῶνου κατ’ εὐθείαν τὴν ΔΜΖ πρὸς ὀρθὰς οὔσαν τῇ ΒΓ, καὶ πεποίηκε τομὴν ἐν τῇ ἐπιφάνειᾳ τῆς ΔΕΖ, ἡ δὲ διάμετρος ἡ ΜΕ ἐκβαλλομένη συμ- πέπτυκε μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου ἐκτὸς τῆς κορυφῆς τοῦ κῶνου, καὶ διὰ τοῦ Α σημείου τῇ διαμέτρῳ τῆς τομῆς τῇ ΕΜ παράλληλος ἦκται ἡ ΑΣ, καὶ ἀπὸ τοῦ Ε τῇ ΕΜ πρὸς ὀρθὰς ἦκται ἡ ΕΠ, καὶ ἔστιν ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὗτος ἡ ΘΘ πρὸς ΕΠ, ὡς μὲν ΔΕΖ ἀρα τομὴ ὑπερβολὴ ἐστὶν, ἡ δὲ ΕΠ παρ’ ἧν δύναται αἱ ἐπὶ τὴν ΕΜ καταγόμεναι τεταγμένως, πλαγιὰ 326.
now $\Xi O$ will be perpendicular to $HK$, since $BG$ is perpendicular to $Z\Delta$, and each is parallel [Eucl. xi. 10]. And since the plane through the axis meets the sections at the points $M, N$ within the curves, it is clear that the plane cuts the curves. Let it cut them at the points $\Theta, E$; then the points $M, E, \Theta, N$ are both in the plane through the axis and in the plane containing the curves; therefore the line $ME\Theta N$ is a straight line [Eucl. xi. 3]. And it is clear that $\Xi, \Theta, A, \Gamma$ are on a straight line, and also $B, E, A, O$; for they are both on the conical surface and in the plane through the axis. Now let $\Theta P, E\Pi$ be drawn from $\Theta, E$ perpendicular to $\Theta E$, and through $A$ let $\Sigma A T$ be drawn parallel to $ME\Theta N$, and let

$$A\Sigma^2 : B\Sigma \cdot \Sigma \Gamma = \Theta E : E\Pi,$$

and

$$AT^2 : OT \cdot T\Xi = \Theta E : \Theta P.$$ 

Then since the cone, whose vertex is the point $A$ and whose base is the circle $BG$, is cut by a plane through the axis, and the section so made is the triangle $AB\Gamma$, and it is cut by another plane cutting the base of the cone in the straight line $\Delta MZ$ perpendicular to $BG$, and the section so made on the surface is $\Delta EZ$, and the diameter $ME$ produced meets one side of the axial triangle beyond the vertex of the cone, and $A\Sigma$ is drawn through the point $A$ parallel to the diameter of the section $EM$, and $E\Pi$ is drawn from $E$ perpendicular to $EM$, and $A\Sigma^2 : B\Sigma \cdot \Sigma \Gamma = \Theta E : E\Pi$, therefore the section $\Delta EZ$ is a hyperbola, in which $E\Pi$ is the parameter to the ordinates to $EM$, and $\Theta E$ is the
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dè τού εἰδους πλευρά ἡ ΘΕ. ὁμοίως δὲ καὶ ἡ ἩΘΚ ύπερβολή ἐστιν, ἦς διάμετρος μὲν ἡ ΘΝ, ἢ δὲ ΘΡ παρ’ ἦν δύνανται αἱ ἐπὶ τὴν ΘΝ καταγό-
μεναὶ τεταγμένως, πλαγία δὲ τοῦ εἰδους πλευρά ἡ ΘΕ.

Λέγω, ὅτι ἵση ἐστὶν ἡ ΘΡ τῇ ΕΠ. ἐπεὶ γὰρ

παράλληλος ἐστιν ἡ ΒΓ τῇ ΞΟ, ἐστὶν ὡς ἡ ΑΣ πρὸς ΣΓ, οὔτως ἡ ΑΤ πρὸς ΤΕ, καὶ ὡς ἡ ΑΣ πρὸς ΣΒ, οὔτως ἡ AT πρὸς TO. ἀλλ’ ὁ τῆς

ΑΣ πρὸς ΣΓ λόγος μετὰ τοῦ τῆς ΑΣ πρὸς ΣΒ ὁ τοῦ ἀπὸ ΑΣ ἐστὶ πρὸς τὸ ὑπὸ ΒΣΓ, ὁ δὲ τῆς

ΑΤ πρὸς ΤΕ μετὰ τοῦ τῆς ΑΤ πρὸς TO ὁ τοῦ

ἀπὸ AT πρὸς τὸ ὑπὸ ΕΤΟ. ἐστὶν ἄρα ὡς τὸ ἀπὸ

ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὔτως τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΕΤΟ. καὶ ἐστὶν ὡς μὲν τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ

ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΕΤΟ, ἡ ΘΕ πρὸς ΘΡ. καὶ ὡς ἄρα ἡ ΘΕ πρὸς ΕΠ, ἡ ΕΘ πρὸς ΘΡ. ἵση ἄρα ἐστὶν ἡ ΕΠ τῇ ΘΡ.

(vi.) Transition to New Diameter

Ibid., Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17-154. 8

Ἐὰν ύπερβολὴς ἡ ἐλλειψεως ἡ κύκλου περι-

φερείας εὐθεία ἐπιφανεύοσα συμπίπτη τῇ διαμέτρῳ,

καὶ διὰ τῆς ἀφῆς καὶ τοῦ κέντρου εὐθεία ἐκβληθῇ,

ἀπὸ δὲ τῆς κορυφῆς ἀναχθείσα εὐθεία παρὰ τεταγ-

μένως κατηγμένην συμπίπτη τῇ διὰ τῆς ἀφῆς καὶ

* Apollonius is the first person known to have recognized

the opposite branches of a hyperbola as portions of the same

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transverse side of the figure [Prop. 12]. Similarly $H\Theta K$ is a hyperbola, in which $\Theta N$ is a diameter, $\Theta P$ is the parameter to the ordinates to $\Theta N$, and $\Theta E$ is the transverse side of the figure.

I say that $\Theta P = \varepsilon \Pi$. For since $B\Gamma$ is parallel to $\Xi O$,

$$A\Sigma : \Sigma \Gamma = AT : T\Xi,$$

and

$$A\Sigma : \Sigma B = AT : TO.$$

But

$$(A\Sigma : \Sigma \Gamma)(A\Sigma : \Sigma B) = A\Sigma^2 : BS \cdot \Sigma \Gamma,$$

and

$$(AT : T\Xi)(AT : TO) = AT^2 : \Xi T \cdot TO.$$

Therefore

$$A\Sigma^2 : BS \cdot \Sigma \Gamma = AT^2 : \Xi T \cdot TO.$$

But

$$A\Sigma^2 : BS \cdot \Sigma \Gamma = \Theta E : \varepsilon \Pi,$$

while

$$AT^2 : \Xi T \cdot TO = \Theta E : \Theta P;$$

therefore

$$\Theta E : \varepsilon \Pi = E\Theta : \Theta P.$$

Therefore

$$\varepsilon \Pi = \Theta P.$$ [Eucl. v. 9

(vi.) Transition to New Diameter

_Ibid._, Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17–154. 8

Prop. 50

In a hyperbola, ellipse or circumference of a circle let a straight line be drawn to touch [the curve] and meet the diameter, and let the straight line through the point of contact and the centre be produced, and from the vertex let a straight line be drawn parallel to a straight line drawn ordinate-wise so as to meet the straight line drawn curve. It is his practice, however, where possible to discuss the single-branch hyperbola (or the hyperbola _simpliciter_ as he would call it) together with the ellipse and circle, and to deal with the opposite branches separately. But occasionally, as in i. 30, the double-branch hyperbola and the ellipse are included in one enunciation.
to the κέντρου ἡμείνη εὐθεία, καὶ ποιηθή, ὡς τὸ τμῆμα τῆς ἐφαπτομένης τὸ μεταξὺ τῆς ἀφῆς καὶ τῆς ἀνηγμένης πρὸς τὸ τμῆμα τῆς ἡμείνης διὰ τῆς ἀφῆς καὶ τοῦ κέντρου τὸ μεταξὺ τῆς ἀφῆς καὶ τῆς ἀνηγμένης, εὐθεία τὸς πρὸς τὴν διπλασίαν τῆς ἐφαπτομένης, ἢτις ἂν ἀπὸ τῆς τομῆς ἀχθῇ ἐπὶ τὴν διὰ τῆς ἀφῆς καὶ τοῦ κέντρου ἡμείνην εὐθείαν παράλληλος τῇ ἐφαπτομένη, δυνηθεῖται τι χωρίον ὀρθογώνιον παρακείμενον παρὰ τὴν πορισθείσων, πλάτος ἔχουν τὴν ἀπολαμβανομένην ύπναυτῆς πρὸς τῇ ἁφῇ, ἐπὶ μὲν τῆς ὑπερβολῆς ὑπερβάλλον εἶδει ὁμοίῳ τῷ περιεχομένῳ ύπὸ τῆς διπλασίας τῆς μεταξὺ τοῦ κέντρου καὶ τῆς ἁφῆς καὶ τῆς πορισθείσῃς εὐθείας, ἐπὶ δὲ τῆς ἐλλεύψεως καὶ τοῦ κύκλου ἐλλείπον.

"Εστω ὑπερβολή ἡ ἐλλεύψεις ἡ κύκλου περιφέρεια, ᾳ διάμετρος ἡ ΑΒ, κέντρον δὲ τὸ Γ, ἐφαπτομένη δὲ ἡ ΔΕ, καὶ ἐπιζευγθείσα ἡ ΓΕ ἐκβεβλήσθω ἐφ' ἐκάτερα, καὶ κείσθω τῇ ΕΓ ἵση ἡ ΓΚ, καὶ διὰ τοῦ Β τεταγμένως ἀνήχθω ἡ BZH, διὰ δὲ τοῦ Ε τῇ ΕΓ πρὸς ὀρθὰς ἡχθω ἡ ΕΘ, καὶ γινέσθω, ὥς ἡ ΖΕ πρὸς ΕΗ, οὕτως ἡ ΕΘ πρὸς τὴν διπλασίαν τῆς ΕΔ, καὶ ἐπιζευγθείσα ἡ ΘΚ ἐκβεβλήσθω, καὶ εἰληφθῶ τι ἐπὶ τῆς τομῆς σημεῖον τὸ Λ, καὶ διὰ αὐτοῦ τῇ ΕΔ παράλληλος ἡχθω ἡ ΛΜΘ, τῇ δὲ

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* To save space, the figure is here given for the hyperbola only; in the mss. there are figures for the ellipse and circle as well.

The general enunciation is not easy to follow, but the particular enunciation will make it easier to understand. The 330
through the point of contact and the centre, and let the
segment of the tangent between the point of contact and
the line drawn ordinate-wise bear to the segment of the
line drawn through the point of contact and the centre
between the point of contact and the line drawn ordinate-
wise the same ratio as a certain straight line bears to
double the tangent; then if any straight line be drawn
from the section parallel to the tangent so as to meet the
straight line drawn through the point of contact and
the centre, its square will be equal to a certain rectilineal
area applied to the postulated straight line, having for its
breadth the intercept between it and the point of contact,
in the case of the hyperbola exceeding by a figure similar
to the rectangle bounded by double the straight line between
the centre and the point of contact and the postulated
straight line, in the case of the ellipse and circle falling
short.\(^\text{a}\)

In a hyperbola, ellipse or circumference of a circle,
with diameter \(AB\) and centre \(\Gamma\), let \(\Delta E\) be a tangent,
and let \(\Gamma E\) be joined and produced in either direction,
and let \(\Gamma K\) be placed equal to \(E\Gamma\), and through
\(B\) let \(BZH\) be drawn ordinate-wise, and through
\(E\) let \(E\Theta\) be drawn perpendicular to \(E\Gamma\), and let
\(ZE : EH = E\Theta : 2E\Delta\), and let \(\Theta K\) be joined and pro-
duced, and let any point \(\Lambda\) be taken on the section,
and through it let \(\Lambda M\Xi\) be drawn parallel to \(E\Delta\) and

purpose of this important proposition is to show that, if any
other diameter be taken, the ordinate-property of the conic
with reference to this diameter has the same form as the
ordinate-property with reference to the original diameter.
The theorem amounts to a transformation of co-ordinates from
the original diameter and the tangent at its extremity to any
diameter and the tangent at its extremity. In succeeding
propositions, showing how to construct conics from certain
data, Apollonius introduces the axes for the first time as
special cases of diameters.
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BH ᾧ ΔΡΝ, τῇ δὲ ΕΘ ᾧ ΜΠ. λέγω, ὅτι τὸ ἀπὸ ΛΜ ἴσον ἐστὶ τῷ ὑπὸ ΕΜΠ.

"Ἡχθω γὰρ διὰ τοῦ Γ τῇ ΚΠ παράλληλος ᾧ ΓΣΟ. καὶ ἐπεὶ ἴση ἐστὶν ᾧ ΕΓ τῇ ΓΚ, ὡς δὲ ᾧ

ΕΓ πρὸς ΚΓ, ᾧ ΕΣ πρὸς ΣΘ, ἴση ἄρα καὶ ᾧ ΕΣ τῇ ΣΘ. καὶ ἐπεὶ ἐστὶν, ὡς ᾧ ΖΕ πρὸς ΕΗ, ᾧ
ΘΕ πρὸς τὴν διπλασίαν τῆς ΕΔ, καὶ ἐστὶ τῆς ΕΘ ἴμισεια ᾧ ΕΣ, ἐστὶν ἄρα, ὡς ᾧ ΖΕ πρὸς ΕΗ, ᾧ
ΣΕ πρὸς ΕΔ. ὡς δὲ ᾧ ΖΕ πρὸς ΕΗ, ᾧ ΛΜ πρὸς
ΜΡ. ὡς ἄρα ᾧ ΛΜ πρὸς ΜΡ, ᾧ ΣΕ πρὸς ΕΔ.
καὶ ἐπεὶ τὸ ΡΝΓ τριγώνον τοῦ ΗΒΓ τριγώνου,
tουτέστι τοῦ ΓΔΕ, ἐπὶ μὲν τῆς ὑπερβολῆς μείζον
ἐδείχθη, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου
ἐλάσσον τῷ ΔΝΕ, κοινῶν ἀφαιρεθέντων ἐπὶ μὲν
tῆς ὑπερβολῆς τοῦ τε ΕΓΔ τριγώνου καὶ τοῦ
ΝΡΜΣ τετραπλεύρου, ἐπὶ δὲ τῆς ἐλλείψεως καὶ
tοῦ κύκλου τοῦ ΜΕΓ τριγώνου, τοῦ ΔΜΡ τριγώνου
tῷ ΜΕΔΣ τετραπλεύρῳ ἐστὶν ἴσον. καὶ ἐστι
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ΔPN parallel to BI, and let MΠ be drawn parallel to EΘ. I say that ΔM² = EM · MΠ.

For through Γ let ΓΣO be drawn parallel to KΠ. Then since

\[ \frac{EG}{FK} = \frac{EG}{FK} \]

and

\[ \frac{EG}{FK} = \frac{ES}{ΣΘ}, \]  \hspace{1cm} \text{[Eucl. vi. 2]}

therefore

\[ ES = ΣΘ. \]

And since

\[ \frac{ZE}{EH} = ΘE : 2ΕΔ, \]

and

\[ ES = \frac{1}{2}ΕΘ, \]

therefore

\[ \frac{ZE}{EH} = \frac{ΣE}{ΕΔ}. \]

But

\[ \frac{ZE}{EH} = \frac{AM}{MP}; \]  \hspace{1cm} \text{[Eucl. vi. 4]}

therefore

\[ \frac{AM}{MP} = \frac{ΣE}{ΕΔ}. \]

And since it has been proved [Prop. 43] that in the hyperbola

triangle PNT = triangle HBG + triangle ΔΝΞ,

i.e., triangle PNT = triangle ΓΔΕ + triangle ΔΝΞ, a

while in the ellipse and the circle

triangle PNT

= triangle HBG - triangle ΔΝΞ,

i.e., triangle PNT + triangle ΔΝΞ = triangle ΓΔΕ, b

therefore by taking away the common elements—in the hyperbola the triangle EΓΔ and the quadrilateral NPMΞ, in the ellipse and the circle the triangle MΞΓ, triangle ΛMP = quadrilateral MΕΔΞ.

a For this step v. Eutocius’s comment on Prop. 43.

b See Eutocius.
παράλληλος ἢ ΜΞ τῇ ΔΕ, ἢ δὲ ὕπο ΛΜΡ τῇ ὕπο ΕΜΞ ἐστιν ἵση ἵσον ἁρα ἐστὶ τὸ ὕπο ΛΜΡ τῷ ὕπο τῆς ΕΜ καὶ συναμφοτέρου τῆς ΕΔ, ΜΞ. καὶ ἐπεὶ ἐστιν, ὡς ἢ ΜΓ πρὸς ΓΕ, ἢ τε ΜΞ πρὸς ΕΔ καὶ ἢ ΜΟ πρὸς ΕΣ, ὡς ἁρα ἢ ΜΟ πρὸς ΕΣ, ἢ ΜΞ πρὸς ΔΕ. καὶ συνθέντι, ὡς συναμφότερος ἢ ΜΟ, ΣΕ πρὸς ΕΣ, οὗτος συναμφότερος ἢ ΜΞ, ΕΔ πρὸς ΕΔ· ἐναλλάξ, ὡς συναμφότερος ἢ ΜΟ, ΣΕ πρὸς συναμφότερον τῆν ΕΜ, ΕΔ ἢ ΣΕ πρὸς ΕΔ. ἀλλ' ὡς μὲν συναμφότερος ἢ ΜΟ, ΕΣ πρὸς συναμφότερον τῆς ΜΞ, ΔΕ, τὸ ὕπο συναμφότερου τῆς ΜΟ, ΕΣ καὶ τῆς ΕΜ πρὸς τὸ ὕπο συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΕΜ, ὡς δὲ ἢ ΣΕ πρὸς ΕΔ, ἢ ΖΕ πρὸς ΕΗ, τούτεστιν ἢ ΛΜ πρὸς ΜΡ, τούτεστι τὸ ἀπὸ ΛΜ πρὸς τὸ ὕπο ΛΜΡ· ὡς ἁρα τὸ ὕπο συναμφοτέρου τῆς ΜΟ, ΕΣ καὶ τῆς ΜΕ πρὸς τὸ ὕπο συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΕΜ, τὸ ἀπὸ ΛΜ πρὸς τὸ ὕπο ΛΜΡ. καὶ ἐναλλάξ, ὡς τὸ ὕπο συναμφοτέρου τῆς ΜΟ, ΕΣ καὶ τῆς ΜΕ πρὸς τὸ ἀπὸ ΜΛ, οὗτος τὸ ὕπο συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΜΕ πρὸς τὸ ὕπο ΛΜΡ. ἵσον δὲ τὸ ὕπο ΛΜΡ τῷ ὕπο τῆς ΜΕ καὶ συναμφοτέρου τῆς ΜΞ, ΕΔ· ἵσον ἁρα καὶ τὸ ἀπὸ ΛΜ τῷ ὕπο ΕΜ καὶ συναμφοτέρου τῆς ΜΟ, ΕΣ. καὶ ἐστὶν ἢ μὲν ΣΕ τῇ ΣΘ ἵση, ἢ δὲ ΣΘ τῇ ὑπὸ ΟΠ· ἵσον ἁρα τὸ ἀπὸ ΛΜ τῷ ὕπο ΕΜΠ.
But $M\Xi$ is parallel to $\Delta E$ and angle $\angle AMP = \angle EM\Xi$ (Eucl. i. 15);

therefore $\angle AM \cdot MP = EM \cdot (E\Delta + M\Xi)$.

And since $MG : GE = M\Xi : E\Delta$,

and $MG : GE = MO : ES$,  

[Eucl. vi. 4]

therefore $MO : ES = M\Xi : \Delta E$.

*Componendo*, $MO + SE : ES = M\Xi + E\Delta : E\Delta$;

and *permutando*

$MO + SE : EM + E\Delta = SE : E\Delta$.

But $MO + SE : EM + E\Delta = (MO + ES) \cdot EM : (M\Xi + E\Delta) \cdot EM$,

and $SE : E\Delta = ZE : EH$

$= AM : MP$  

[Eucl. vi. 4]

$= AM^2 : AM \cdot MP$;

therefore

$(MO + ES \cdot ME : (M\Xi + E\Delta) \cdot EM = AM^2 : AM \cdot MP$.

And *permutando*

$(MO + ES) \cdot ME : MA^2 = (M\Xi + E\Delta) \cdot ME : AM \cdot MP$.

But $AM \cdot MP = ME \cdot (M\Xi + E\Delta)$;

therefore $AM^2 = EM \cdot (MO + ES)$.

And $SE = \Sigma \Theta$, while $\Sigma \Theta = O\Pi$ [Eucl. i. 34];

therefore $AM^2 = EM \cdot MP$.  

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(b) Other Works

(i.) General

Papp. Coll. vii. 3, ed. Hultsch 636. 18-23

Τῶν δὲ προειρημένων τοῦ Ἀναλυομένου βιβλίων ἡ τάξις ἔστιν τοιαύτη: Εὐκλείδου Δεδομένων βιβλίων ᾧ, Ἀπollωνίου Δόγου ἀποτομῆς β, Χωρίου ἀποτομῆς β, Διωρισμένης τομῆς δύο, Ἔπαφών δύο, Εὐκλείδου Πορισμάτων τρία, Ὁ Ἀπollωνίου Νεύσεων δύο, τοῦ αὐτοῦ Τόπων ἑπιπέδων δύο, Κωνικῶν ἡ.

(ii.) On the Cutting-off of a Ratio

Ibid. vii. 5-6, ed. Hultsch 640. 4-22

Τῆς δὲ Ἀποτομῆς τοῦ λόγου βιβλίων ὅντων β πρότασις ἔστιν μία ὑποδημημένη, διὸ καὶ μίαν πρότασιν οὕτως γράφω. διὰ τοῦ δοθέντος σημείου εὐθείαν γραμμὴν ἀγαγεῖν τέμνουσαν ἀπὸ τῶν τῇ θέσει δοθεισῶν δύο εὐθείῶν πρὸς τοὺς ἐπ’ αὐτῶν δοθεῖσι σημείοις λόγου ἐχούσας τὸν αὐτὸν τῷ δοθέντι. τὰς δὲ γραφὰς διαφόρους γενέσθαι καὶ πλῆθος λαβεῖν συμβεβηκεν ὑποδιαιρέσεως γενομένης ἑνεκα τῆς τε πρὸς ἀλλήλας θέσεως τῶν διδομένων εὐθείῶν καὶ τῶν διαφόρων πτώσεων τοῦ διδομένου σημείου καὶ διὰ τᾶς ἀναλύσεως καὶ συνθέσεις αὐτῶν τε καὶ τῶν διορισμῶν. ἔχει γὰρ τὸ μὲν πρῶτον βιβλίον τῶν Δόγου ἀποτομῆς

* Unhappily the only work by Apollonius which has survived, in addition to the Conics, is On the Cutting-off of a
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(b) OTHER WORKS

(i.) General

Pappus, Collection vii. 3, ed. Hultsch 636. 18-23

The order of the aforesaid books in the Treasury of Analysis is as follows: the one book of Euclid's Data, the two books of Apollonius's On the Cutting-off of a Ratio, his two books On the Cutting-off of an Area, his two books On Determinate Section, his two books On Tangencies, the three books of Euclid's Porisms, the two books of Apollonius's On Vergings, the two books of the same writer On Plane Loci, his eight books of Conics.

(ii.) On the Cutting-off of a Ratio

Ibid. vii. 5-6, ed. Hultsch 640. 4-22

In the two books On the Cutting-off of a Ratio there is one enunciation which is subdivided, for which reason I state one enunciation thus: Through a given point to draw a straight line cutting off from two straight lines given in position intercepts, measured from two given points on them, which shall have a given ratio. When the subdivision is made, this leads to many different figures according to the position of the given straight lines in relation one to another and according to the different cases of the given point, and owing to the analysis and the synthesis both of these cases and of the propositions determining the limits of possibility. The first book of those On the Cutting-off of a Ratio, and that only in Arabic. Halley published a Latin translation in 1706. But the contents of the other works are indicated fairly closely by Pappus's references.
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toπονς ζ, πτώσεως καθε, διορισμοῦς δὲ ἕ, ὅν τρεῖς
μέν εἰσιν μέγιστοι, δύο δὲ ἐλάχιστοι. . . . τὸ δὲ
dεύτερον βιβλίον Λόγου ἀποτομῆς ἔχει τόπους οὐδὲ
πτώσεως δὲ ἐγ', διορισμοῦς δὲ τοὺς ἕκ τοῦ πρώτου.
ἀπάγεται γὰρ ὅλον εἰς τὸ πρῶτον.

(iii.) On the Cutting-off of an Area

Ibid. vii. 7, ed. Hultsch 640. 26–642. 5

Τῆς δ' Ἀποτομῆς τοῦ χωρίου βιβλία μέν ἐστιν
dύο, πρόβλημα δὲ κἀν τούτος ἐν ύποδιαιρούμενον
dίοι, καὶ τούτων μία πρὸτασις ἐστιν τὰ μὲν ἄλλα
ὀμοίως ἔχουσα τῇ προτέρα, μόνω δὲ τούτῳ δια-
φέροντα τῷ δεῖν τὰς ἀποτελομομένας δύο εὐθείας
ἐν ἐκείνῃ μὲν λόγον ἔχουσας δοθέντα ποιεῖν, ἐν δὲ
ταύτῃ χωρίον περιεχούσας δοθέν.

(iv.) On Determinate Section

Ibid. vii. 9, ed. Hultsch 642. 19–644. 16

'Εξῆς τοῦτοι ἀναδεδονται τῆς Διωρισμένης
τομῆς βιβλία β', ὅν ὀμοίως τοῖς πρῶτον μίαν
πρότασιν πάρεστιν λέγειν, διεξευγμένην δὲ ταύτην.

* The Arabic text shows that Apollonius first discussed the cases in which the lines are parallel, then the cases in which the lines intersect but one of the given points is at the point of intersection; in the second book he proceeds to the general case, but shows that it can be reduced to the case where one
**APOLLONIUS OF PERGA**

*Ratio* contains seven loci, twenty-four cases and five determinations of the limits of possibility, of which three are maxima and two are minima. . . . The second book *On the Cutting-off of a Ratio* contains fourteen loci, sixty-three cases and the same determinations of the limits of possibility as the first; for they are all reduced to those in the first book.

(iii.) *On the Cutting-off of an Area*


In the work *On the Cutting-off of an Area* there are two books, but in them there is only one problem, twice subdivided, and the one enunciation is similar in other respects to the preceding, differing only in this, that in the former work the intercepts on the two given lines were required to have a given ratio, in this to comprehend a given area.

(iv.) *On Determinate Section*


Next in order after these are published the two books *On Determinate Section*, of which, as in the previous cases, it is possible to state one comprehen-

of the given points is at the intersection of the two lines. By this means the problem is reduced to the application of a rectangle. In all cases Apollonius works by analysis and synthesis.

*Halley* attempted to restore this work in his edition of the *De sectione rationis*. As in that treatise, the general case can be reduced to the case where one of the given points is at the intersection of the two lines, and the problem is reduced to the application of a certain rectangle.
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τὴν δοθεῖσαν ἀπειρον εὐθείαν ἐνὶ σημεῖω τεμεῖν, ὥστε τῶν ἀπολαμβανομένων εὐθειῶν πρὸς τοῖς ἐπ’ αὐτῆς δοθεῖσι σημεῖοι ἦτοι τὸ ἀπὸ μᾶς τετράγωνον ἢ τὸ ὑπὸ δύο ἀπολαμβανομένων περιεχόμενον ὀρθογώνιον δοθέντα λόγον ἔχειν ἦτοι πρὸς τὸ ἀπὸ μᾶς τετράγωνον ἢ πρὸς τὸ ὑπὸ μᾶς ἀπολαμβανομένης καὶ τῆς ἔξω δοθείσης ἢ πρὸς τὸ ὑπὸ δύο ἀπολαμβανομένων περιεχόμενον ὀρθογώνιον, ἐφ’ ὑπότερα χρή τῶν δοθέντων σημείων.

(γ.) On Tangencies

_Ibid._ vii. 11, ed. Hultsch 644. 23–646. 19

‘Εξῆς δὲ τούτως τῶν Ἐπαφῶν ἐστὶν βιβλία δύο. προτάσεις δὲ ἐν αὐτοῖς δοκούσιν εἶναι πλεῖον, ἀλλὰ καὶ τούτων μίαν τίθημεν οὕτως ἔχουσαν ἔξῆς: σημείων καὶ εὐθειῶν καὶ κύκλων τριῶν ὀποιωνοῦν θέσει δοθέντων κύκλων ἄγαγεῖν δὴ ἐκάστον τῶν δοθέντων σημείων, εἰ δοθείη, ἢ ἐφαπτόμενον ἐκάστης τῶν δοθεισῶν γραμμῶν. ταύτης διὰ

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*a* As the Greeks never grasped the conception of one point being two coincident points, it was not possible to enunciate this problem so concisely as we can do: _Given four points_ $A, B, C, D$ _on a straight line, of which_ $A$ _may coincide with_ $C$ _and_ $B$ _with_ $D$, _to find another point_ $P$ _on the same straight line such that_ $AP \cdot CP = BP \cdot DP$, _if_ $AP \cdot CP = \lambda \cdot BP \cdot DP$, _where_ $A, B, C, D, \lambda$ _are given, the determination of_ $P$ _is equivalent to the solution of a quadratic equation, which the Greeks could achieve by means of the_ 340
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evive enunciation thus: To cut a given infinite straight line in a point so that the intercepts between this point and given points on the line shall furnish a given ratio, the ratio being that of the square on one intercept, or the rectangle contained by two, towards the square on the remaining intercept, or the rectangle contained by the remaining intercept and a given independent straight line, or the rectangle contained by two remaining intercepts, whichever way the given points [are situated]. . . . The first book contains six problems, sixteen subdivisions and five limits of possibility, of which four are maxima and one is a minimum. . . . The second book On Determinate Section contains three problems, nine subdivisions, and three limits of possibility.\(^a\)

(v.) On Tangencies


Next in order are the two books On Tangencies. Their enunciations are more numerous, but we may bring these also under one enunciation thus stated: Given three entities, of which any one may be a point or a straight line or a circle, to draw a circle which shall pass through each of the given points, so far as it is points which are given, or to touch each of the given lines.\(^b\) In application of areas. But the fact that limits of possibility, and maxima and minima were discussed leads Heath (*H.G.M.* ii. 180-181) to conjecture that Apollonius investigated the series of point-pairs determined by the equation for different values of \(\lambda\), and that “the treatise contained what amounts to a complete Theory of Involution.” The importance of the work is shown by the large number of lemmas which Pappus collected.

\(^b\) The word “lines” here covers both the straight lines and the circles.

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πλήθη τῶν ἐν ταῖς ὑποθέσεσι δεδομένων ὁμοίων ἢ ἀνομοίων κατὰ μέρος διαφόρους προτάσεις ἀναγκαίων γίνεσθαι δέκα: ἐκ τῶν τριῶν γὰρ ἀνομοίων γενῶν τριάδες διάφοροι ἀτακτοὶ γίνονται ἵ. ἦτοι γὰρ τὰ διδόμενα τρία σημεῖα ἢ τρεῖς εὐθείαι ἢ δύο σημεῖα καὶ εὐθεία ἢ δύο εὐθείαι καί σημείον ἢ δύο σημεία καί κύκλος ἢ δύο κύκλοι καί σημείον ἢ δύο εὐθείαι καί κύκλος ἢ δύο κύκλοι καί εὐθεία ἢ σημείον καί εὐθεία καί κύκλος ἢ τρεῖς κύκλοι. τούτων δύο μὲν τὰ πρῶτα δἐδεκταὶ ἐν τῷ δ' βιβλίῳ τῶν πρῶτων Στοιχείων, διὸ παρεῖ μὴ γράφων· τὸ μὲν γὰρ τριῶν δοθέντων σημείων μὴ ἐπ' εὐθείας ὅντων τὸ αὐτὸ ἐστιν τῷ περὶ τὸ δοθὲν τρίγωνον κύκλον περιγράψαι, τὸ δὲ ἢ δοθεισῶν εὐθειῶν μὴ παραλλήλων οὐσῶν, ἀλλὰ τῶν τριῶν συμπιπτουσῶν, τὸ αὐτὸ ἐστιν τῷ εἰς τὸ δοθὲν τρίγωνον κύκλον ἐγγράψαι· τὸ δὲ δύο παραλλήλων οὐσῶν καὶ μιᾶς ἐμπιπτούσης χως μέρος ὅν τῆς β' ὑποδιαιρέσεως προγράφεται ἐν τούτοις πάντων. καὶ τὰ ἔξής ἦν ἐν τῷ πρῶτῳ βιβλίῳ τὰ δὲ λειπόμενα δύο, τὸ δύο δοθεισῶν εὐθειῶν καί κύκλου ἢ τριῶν δοθέντων κύκλων μόνον ἐν τῷ δευτέρῳ βιβλίῳ διὰ τὰς πρὸς ἀλλήλους θέσεις τῶν κύκλων τε καὶ εὐθειῶν πλειόνας οὐσάς καὶ πλειόνων διορισμῶν δεομένας.

* Eucl. iv. 5 and 4.
* The last problem, to describe a circle touching three
this problem, according to the number of like or unlike entities in the hypotheses, there are bound to be, when the problem is subdivided, ten enunciations. For the number of different ways in which three entities can be taken out of the three unlike sets is ten. For the given entities must be (1) three points or (2) three straight lines or (3) two points and a straight line or (4) two straight lines and a point or (5) two points and a circle or (6) two circles and a point or (7) two straight lines and a circle or (8) two circles and a straight line or (9) a point and a straight line and a circle or (10) three circles. Of these, the first two cases are proved in the fourth book of the first Elements, for which reason they will not be described; for to describe a circle through three points, not being in a straight line, is the same thing as to circumscribe a given triangle, and to describe a circle to touch three given straight lines, not being parallel but meeting each other, is the same thing as to inscribe a circle in a given triangle; the case where two of the lines are parallel and one meets them is a subdivision of the second problem but is here given first place. The next six problems in order are investigated in the first book, while the remaining two, the case of two given straight lines and a circle and the case of three circles, are the sole subjects of the second book on account of the manifold positions of the circles and straight lines with respect one to another and the need for numerous investigations of the limits of possibility.

given circles, has been investigated by many famous geometers, including Newton (Arithmetica Universalis, Prob. 47). The lemmas given by Pappus enable Heath (H.G.M. ii. 182-185) to restore Apollonius's solution—a "plane" solution depending only on the straight line and circle.
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(vi.) On Plane Loci

Ibid. vii. 23, ed. Hultsch 662. 19–664. 7

Οἱ μὲν οὖν ἂρχαίοι εἰς τὴν τῶν ἐπιπέδων τούτων τόπων τάξιν ἀποβλέποντες ἠστοιχείωσαν· ἂς ἀμελή-
σαντες οἱ μετ’ αὐτοὺς προσέθηκαν ἔτερους, ὥς
οὐκ ἀπείρων τὸ πλῆθος οντων, εἴ θέλοι τις προσ-
γράφειν οὔ τῆς τάξεως ἐκείνης ἔχομενα. θήσω
οὖν τὰ μὲν προσκείμενα ὑστερα, τὰ δ’ ἐκ τῆς
tάξεως πρότερα μιὰ περιλαβῶν προτάσει ταύτην.

'Εὰν δύο εὐθείαι ἀχθῶσι ήτοι ἀπὸ ἑνὸς δεδο-
μένου σημείου ἡ ἀπὸ δύο καὶ ἢτοι ἐπ’ εὐθείας ἡ
παράλληλοι ἡ δεδομένην περιέχουσαι γωνίαν καὶ
ἡτοι λόγον ἔχουσαι πρὸς ἀλλήλας ἡ χωρίον περι-
έχουσαι δεδομένον, ἀπτηται δὲ τὸ τῆς μιᾶς πέρας
ἐπιπέδου τόπου θέσει δεδομένου, ἁπευταὶ καὶ τὸ
tῆς ἐτέρας πέρας ἐπιπέδου τόπου θέσει δεδομένου
ὁτὲ μὲν τοῦ ὀμογενοῦς, ὡτὲ δὲ τοῦ ἔτερου, καὶ ὡτὲ
μὲν ὀμοίως κειμένον πρὸς τὴν εὐθείαν, ὡτὲ δὲ
ἐναντίως. ταῦτα δὲ γίνεται παρὰ τὰς διαφορὰς
τῶν ὑποκειμένων.

(vii.) On Vergings

Ibid. vii. 27-28, ed. Hultsch 670. 4–672. 3

ΝΕὰν λέγεται γραμμὴ ἐπὶ σημείων, εὰν
ἐπεκβαλλομένη ἐπ’ αὐτὸ παραγίνηται [ . . . ]

1 τούτων is attributed by Hultsch to dittography.

a These words follow the passage (quoted supra, pp. 262-
265) wherein Pappus divides loci into ἐφεκτικοὶ, διεξοδικοὶ and
ἀναστροφικοὶ.

b It is not clear what straight line is meant—probably the
most obvious straight line in each figure.

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The ancients had regard to the arrangement of these plane loci with a view to instruction in the elements; heedless of this consideration, their successors have added others, as though the number could not be infinitely increased if one were to make additions from outside that arrangement. Accordingly I shall set out the additions later, giving first those in the arrangement, and including them in this single enunciation:

If two straight lines be drawn, from one given point or from two, which are in a straight line or parallel or include a given angle, and either bear a given ratio one towards the other or contain a given rectangle, then, if the locus of the extremity of one of the lines be a plane locus given in position, the locus of the extremity of the other will also be a plane locus given in position, which will sometimes be of the same kind as the former, sometimes of a different kind, and will sometimes be similarly situated with respect to the straight line, sometimes contrariwise. These different cases arise according to the differences in the suppositions.

A line is said to verge to a point if, when produced, it passes through the point. [ . . . ] The general

Pappus proceeds to give seven other enunciations from the first book and eight from the second book. These have enabled reconstructions of the work to be made by Fermat, van Schooten and Robert Simson.

Examples of vergings have already been encountered several times; v. pp. 186-189 and vol. i. p. 244 n. a.
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προβλήματος δὲ οὗτος καθολικοῦ τούτου: δύο δοθεισῶν γραμμῶν θέσει θείας μεταξὺ τούτων ευθείαν τῷ μεγέθει δεδομένην νεύοσαν ἐπὶ δοθέν σημείον, ἐπὶ τούτου τῶν ἐπὶ μέρους διάφορα τὰ ὑποκείμενα ἐχόντων, ἢ μὲν ἢν ἐπίπεδα, ἢ δὲ στερεά, ἢ δὲ γραμμικά, τῶν δὲ ἐπιπέδων ἀποκληρώσαντες τὰ πρὸς πολλὰ χρησιμώτερα ἔδειξαν τὰ προβλήματα ταῦτα.

Θέσει δεδομένων ἡμικυκλίου τε καὶ εὐθείας πρὸς ὀρθὰς τῇ βάσει ἢ δύο ἡμικυκλίων ἐπὶ εὐθείας ἐχόντων τάς βάσεις θείαν δοθεῖσαν τῷ μεγέθει εὐθείαν μεταξὺ τῶν δύο γραμμῶν νεύοσαν ἐπὶ γωνίαν ἡμικυκλίου.

Καὶ ρόμβου δοθέντος καὶ ἐπεκβεβλημένης μᾶς πλευρᾶς ἁρμόσαυ ὑπὸ τὴν ἐκτὸς γωνίαν δεδομένην τῷ μεγέθει εὐθείαν νεύοσαν ἐπὶ τὴν ἀντικρυ γωνίαν.

Καὶ θέσει δοθέντος κύκλου ἐναρμόσαι εὐθείαν μεγέθει δεδομένην νεύοσαν ἐπὶ δοθέν.

Τούτων δὲ ἐν μὲν τῷ πρῶτῳ τεύχει δέδεκται τὸ ἐπὶ τοῦ ἐνὸς ἡμικυκλίου καὶ εὐθείας ἔχουν πτώσεις δὲ καὶ τὸ ἐπὶ τοῦ κύκλου ἔχον πτώσεις δύο καὶ τὸ ἐπὶ τοῦ ρόμβου πτώσεις ἔχου ὑ, ἐν δὲ τῷ δεύτερῳ τεύχει τὸ ἐπὶ τῶν δύο ἡμικυκλίων τῆς ὑποθέσεως πτώσεις ἔχουσας ἢ, ἐν δὲ ταῦται ὑποδιαίρεσεις πλεῖονς διοριστικαὶ ἔνεκα τοῦ δεδομένου μεγέθους τῆς εὐθείας.
problem is: Two straight lines being given in position, to place between them a straight line of given length so as to verge to a given point. When it is subdivided the subordinate problems are, according to differences in the suppositions, sometimes plane, sometimes solid, sometimes linear. Among the plane problems, a selection was made of those more generally useful, and these problems have been proved:

Given a semicircle and a straight line perpendicular to the base, or two semicircles with their bases in a straight line, to place a straight line of given length between the two lines and verging to an angle of the semicircle [or of one of the semicircles];

Given a rhombus with one side produced, to insert a straight line of given length in the external angle so that it verges to the opposite angle;

Given a circle, to insert a chord of given length verging to a given point.

Of these, there are proved in the first book four cases of the problem of one semicircle and a straight line, two cases of the circle, and two cases of the rhombus; in the second book there are proved ten cases of the problem in which two semicircles are assumed, and in these there are numerous subdivisions concerned with limits of possibility according to the given length of the straight line.a

A restoration of Apollonius’s work On Vergings has been attempted by several writers, most completely by Samuel Horsley (Oxford, 1770). A lemma by Pappus enables Apollonius’s construction in the case of the rhombus to be restored with certainty; v. Heath, H.G.M. ii. 190-192.
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(viii.) *On the Dodecahedron and the Icosahedron*

Hypsicl. [Eucl. Elem. xiv.], Eucl. ed. Heiberg
v. 6. 19–8. 5

'Ο αὐτὸς κύκλος περιλαμβάνει τὸ τε τοῦ δωδεκαέδρου πεντάγωνον καὶ τὸ τοῦ εἴκοσαέδρου τρίγωνον τῶν εἰς τὴν αὐτὴν σφαῖραν ἐγγραφομένων. τούτο δὲ γράφεται ὑπὸ μὲν Ἱ'Αρισταίου ἐν τῷ ἐπιγραφομένῳ Τῶν ἐ σχημάτων συγκρίσει, ὑπὸ δὲ Ἱ'Απολλώνιου ἐν τῇ δευτέρᾳ ἐκδόσει τῆς Συγκρίσεως τοῦ δωδεκαέδρου πρὸς τὸ εἴκοσαέδρον, ὅτι ἑστίν, ὡς ἡ τοῦ δωδεκαέδρου ἐπιφάνεια πρὸς τὴν τοῦ εἴκοσαέδρου ἐπιφάνειαν, οὕτως καὶ αὐτὸ τὸ δωδεκαέδρον πρὸς τὸ εἴκοσαέδρον διὰ τὸ τῆς αὐτῆς εἶναι κάθετον ἀπὸ τοῦ κέντρου τῆς σφαίρας ἐπὶ τὸ τοῦ δωδεκαέδρου πεντάγωνον καὶ τὸ τοῦ εἴκοσαέδρου τρίγωνον.

(ix.) *Principles of Mathematics*

Marin. in Eucl. Dat., Eucl. ed. Heiberg vi. 234. 13-17

Διὸ τῶν ἀπλούστερον¹ καὶ μιᾷ τῶν διαφορὰ περιγράφειν τὸ δεδομένον προθεμένων οἱ µὲν τεταγμένοι, ὡς Ἱ'Απολλώνιος ἐν τῇ Περὶ νεύσεων καὶ

¹ ἀπλούστερον Heiberg, ἀπλουστέρων cod.
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(viii.) On the Dodecahedron and the Icosahedron


The pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere can be included in the same circle. For this is proved by Aristaeus in the work which he wrote *On the Comparison of the Five Figures*, and it is proved by Apollonius in the second edition of his work *On the Comparison of the Dodecahedron and the Icosahedron* that the surface of the dodecahedron bears to the surface of the icosahedron the same ratio as the volume of the dodecahedron bears to the volume of the icosahedron, by reason of there being a common perpendicular from the centre of the sphere to the pentagon of the dodecahedron and the triangle of the icosahedron.

(ix.) Principles of Mathematics

Marinus, *Commentary on Euclid's Data*, Eucl. ed. Heiberg vi. 234. 13-17

Therefore, among those who made it their aim to define the datum more simply and with a single *differentia*, some called it the *assigned*, such as Apollonius in his book *On Vergings* and in his

* The so-called fourteenth book of Euclid's *Elements* is really the work of Hypsicles, for whom *infra*, pp. 394-397.
* For the regular solids *vol. i. pp. 216-225*. The face of the dodecahedron is a pentagon and the face of the icosahedron a triangle.
* A proof is given by Hypsicles as Prop. 2 of his book. Whether the Aristaeus is the same person as the author of the *Solid Loci* is not known.
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ἐν τῇ Καθόλου πραγματείᾳ, οἱ δὲ γνώριμοι, ὡς Διόδορος.

(x.) On the Cochlias

Procl. in Eucl. i., ed. Friedlein 105. 1-6

Τὴν περὶ τῶν κύλινδρον ἐλικα γραφομένην, ὅταν εὐθεῖας κινούμενης περὶ τὴν ἐπιφάνειαν τοῦ κυλίνδρου σημεῖον ὀμοσαχῶς ἐπ’ αὐτῆς κινήται. γίνεται γὰρ ἔλιξ, ἢ ὀμοιομερῶς πάντα τὰ μέρη πᾶσιν ἐφαρμόζει, καθάπερ Ἄπολλώνιος ἐν τῷ Περὶ τοῦ κοχλίου γράμματι δείκνυσιν.

(xi.) On Unordered Irrationals

Procl. in Eucl. i., ed. Friedlein 74. 23-24

Τὰ Περὶ τῶν ἀτάκτων ἀλόγων, ἢ ὄ Ἄπολλώνιος ἐπὶ πλέον ἐξειργάσατο.

Schol. i. in Eucl. Elem. x., Eucl. ed. Heiberg v. 414. 10-16

Ἐν μὲν οὖν τοῖς πρώτοις περὶ συμμετρων καὶ ἀσυμμετρων διαλαμβάνει πρὸς τὴν φύσιν αὐτῶν αὐτὰ ἔξετάζων, ἐν δὲ τοῖς ἔξησι περὶ ῥητῶν καὶ ἀλόγων οὐ πασῶν· τινὲς γὰρ αὐτῷ ὡς ἐνυπάρχουσι ἐγκαλοῦσιν· ἀλλὰ τῶν ἀπλουστάτων εἰδῶν, δὲν

* Heath (H.G.M. ii. 192-193) conjectures that this work must have dealt with the fundamental principles of mathematics, and to it he assigns various remarks on such subjects attributed to Apollonius by Proclus, and in particular his attempts to prove the axioms. The different ways in which entities are said to be given are stated in the definitions quoted from Euclid’s Data in vol. i. pp. 478-479.

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General Treatise,a others the known, such as Dio-
dorus.b

(x.) On the Cochlias

Proclus, On Euclid i., ed. Friedlein 105. 1-6

The cylindrical helix is described when a point moves uniformly along a straight line which itself moves round the surface of a cylinder. For in this way there is generated a helix which is homoeomeric, any part being such that it will coincide with any other part, as is shown by Apollonius in his work On the Cochlias.

(xi.) On Unordered Irrationals

Proclus, On Euclid i., ed. Friedlein 74. 23-24

The theory of unordered irrationals, which Apollonius fully investigated.

Euclid, Elements x., Scholium i.,c ed. Heiberg

v. 414. 10-16

Therefore in the first [theorems of the tenth book] he treats of symmetrical and asymmetrical magni-
tudes, investigating them according to their nature, and in the succeeding theorems he deals with rational and irrational quantities, but not all, which is held up against him by certain detractors; for he dealt only with the simplest kinds, by the combination of which

a Possibly Diodorus of Alexandria, for whom v. vol. i. p. 300 and p. 301 n. b.
b In Studien über Euklid, p. 170, Heiberg conjectured that this scholium was extracted from Pappus's commentary, and he has established his conjecture in Videnskabernes Selskabs Skrifter, 6 Raekke, hist.-philos. Afd. ii. p. 236 seq. (1888).
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συντθεμένων γίνονται ἀπειροί ἄλογοι, ὡν τινας καὶ ὁ Ἄπολλώνιος ἀναγράφει.

(xii.) Measurement of a Circle


'Ἰστέον δὲ, ὦτι καὶ Ἄπολλώνιος ὁ Περγαῖος ἐν τῷ Ὀκυτοκίῳ ἀπέδειξεν αὐτὸ δι’ ἀριθμῶν ἐτέρων ἐπὶ τὸ σύνεγγυς μᾶλλον ἄγαγων. τούτῳ δὲ ἀκριβέστερον μὲν εἶναι δοκεῖ, ὥστε ὁ χρήσμον δὲ πρὸς τὸν Ἀρχιμήδους σκοπόν ἔφαμεν γὰρ αὐτὸν σκοπὸν ἐχειν ἐν τῷ δὲ τῷ βιβλίῳ τὸ σύνεγγυς εὑρεῖν διὰ τὰς ἐν τῷ βιῳ χρείας.

(xiii.) Continued Multiplications


Τούτου δὲ προτεθεωρημένου πρόδηλον, πῶς ἐστιν τὸν δοθέντα στίχον πολλαπλασιάσαι καὶ εἰπεὶν τὸν γενόμενον ἀριθμὸν ἐκ τοῦ τὸν πρῶτον ἀριθμὸν ἄν εἰληφε τὸ πρῶτον τῶν γραμμάτων ἐπὶ τὸν δεύτερον ἀριθμὸν ἄν εἰληφε τὸ δεύτερον τῶν γραμμάτων πολλαπλασιασθῆναι καὶ τὸν γενόμενον ἐπὶ τὸν τρίτον ἀριθμὸν ἄν εἰληφε τὸ τρίτον γράμμα

1 The extensive interpolations are omitted.

* Pappus’s commentary on Eucl. Elem. x. was discovered in an Arabic translation by Woepcke (Mémoires présentées par divers savans à l’Académie des sciences, 1856, xiv.). It contains several references to Apollonius’s work, of which one is thus translated by Woepcke (p. 693): “Enfin, Apollonius distinguia les espèces des irrationnelles ordonnées, et 352
an infinite number of irrationals are formed, of which latter Apollonius also describes some." 

(xii.) Measurement of a Circle

Eutocius, Commentary on Archimedes' Measurement of a Circle, Archim. ed. Heiberg iii. 258. 16-22

It should be noticed, however, that Apollonius of Perga proved the same thing (sc. the ratio of the circumference of a circle to the diameter) in the Quick-deliverer by a different calculation leading to a closer approximation. This appears to be more accurate, but it is of no use for Archimedes' purpose; for we have stated that his purpose in this book was to find an approximation suitable for the everyday needs of life.

(xiii.) Continued Multiplications

Pappus, Collection ii. 17-21, ed. Hultsch 18. 23-24. 20

This theorem having first been proved, it is clear how to multiply together a given verse and to tell the number which results when the number represented by the first letter is multiplied into the number represented by the second letter and the product is multiplied into the number represented by the third découvrit la science des quantités appelées (irrationnelles) inordonnées, dont il produisit un très-grand nombre par des méthodes exactes."

b We do not know what the approximation was.

* Heiberg (Apollon. Perg. ed. Heiberg ii. 124, n. 1) suggests that these calculations were contained in the 'Ωκτόκιον', but there is no definite evidence.

c The passages, chiefly detailed calculations, adjudged by Hultsch to be interpolations are omitted.
Apollonius, it is clear from Pappus, had a system of tetrads for calculations involving big numbers, the unit being the myriad or fourth power of 10. The tetrads are called μυριάδες ἀπλαί, μυριάδες διπλαί, μυριάδες τριπλαί, simple myriads, double myriads, triple myriads and so on, by which are meant 10000, 10000, 10000 and so on. In the text of 354
letter and so on in order until the end of the verse which Apollonius gave in the beginning, that is

Ἀρτέμιδος κλείτε κράτος ἐξοχον ἐννέα κοῦραν

(where he says κλείτε for ὑπομνήσατε, recall to mind).

Since there are thirty-eight letters in the verse, of which ten, namely ρ ρ σ ρ ρ χ ν ρ (=100, 300, 200 300, 100, 300, 200, 600, 400, 100), represent numbers less than 1000 and divisible by 100, and seventeen, namely μ ι ο θ ι ι ο ε ι ι ι ι ι ι ι ι ι ι ι ι ι ι ι (=40, 10, 70, 20, 30, 10, 20, 70, 60, 70, 70, 50, 50, 50, 20, 70, 10), represent numbers less than 100 and divisible by 10, while the remaining eleven, namely, α ε δ ε ε α ε ε ε α α (=-1, 5, 4, 5, 5, 1, 5, 5, 1, 1), represent numbers less than 10, then if for those ten numbers we substitute an equal number of hundreds, and if for the seventeen numbers we similarly substitute seventeen tens, it is clear from the above arithmetical theorem, the twelfth, that the ten hundreds together with the seventeen tens make \(10^{10}000^9\).

And since the bases of the numbers divisible by 100 and those divisible by 10 are the following twenty-seven

\[
1, 3, 2, 3, 1, 3, 2, 6, 4, 1 \\
4, 1, 7, 2, 3, 1, 2, 7, 6, 7, 7, 5, 5, 5, 2, 7, 1,
\]

Pappus they are sometimes abbreviated to \(μ^a, μ^b, μ^γ\) and so on.

From Pappus, though the text is defective, Apollonius's procedure in multiplying together powers of 10 can be seen to be equivalent to adding the indices of the separate powers of 10, and then dividing by 4 to obtain the power of the myriad which the product contains. If the division is exact, the number is the \(n\)-myriad, say, meaning \(10000^n\). If there is a remainder, \(3, 2\) or \(1\), the number is \(1000, 100\) or 10 times the \(n\)-myriad as the case may be.
On the Burning Mirror

Fragmentum mathematicum Bobiense 113. 28-33, ed. Belger, Hermes, xvi., 1881, 279-280

Oī mēn oūn palaioi ὑπέλαβον τὴν ἔξαψιν ποιεῖσθαι peri τὸ κέντρον τοῦ κατόπτρου, τούτῳ δὲ ψεῦδος Ἀπολλώνιος μάλα δεόντως . . . (ἐν τῷ) πρὸς τοὺς κατοπτρικοὺς ἔδειξεν, καὶ περὶ τινὰ δὲ τόπον ἡ ἐκπύρωσις ἦσται, διασεσάφηκεν ἐν τῷ Περὶ τοῦ πυρίου.

1 As amended by Heiberg, Zeitschrift für Mathematik und Physik, xxviii., 1883, hist. Abth. 124-125.
while there are eleven less than ten, that is the numbers

1, 5, 4, 5, 5, 1, 5, 5, 1, 1,

if we multiply together the solid number formed by these eleven with the solid number formed by the twenty-seven the result will be the solid number

\[ 19 \cdot 10000^4 + 6036 \cdot 10000^3 + 8480 \cdot 10000^2. \]

When these numbers are multiplied into the solid number formed by the hundreds and the tens, that is with \(10 \cdot 10000^9\) as calculated above, the result is

\[ 196 \cdot 10000^{13} + 368 \cdot 10000^{12} + 4800 \cdot 10000^{11}. \]

(xiv.) On the Burning Mirror

*Fragmentum mathematicum Bobiense 113. 28-33,* ed. Belger, Herae, xvi., 1881, 279-280

The older geometers thought that the burning took place at the centre of the mirror, but Apollonius very suitably showed this to be false . . . in his work on mirrors, and he explained clearly where the kindling takes place in his works On the Burning Mirror.

* This fragment is attributed to Anthemius by Heiberg, but its antiquated terminology leads Heath (H.G.M. ii. 194) to suppose that it is much earlier.

* Of Apollonius's other achievements, his solution of the problem of finding two mean proportionals has already been mentioned (vol. i. p. 267 n. b) and sufficiently indicated; for his astronomical work the reader is referred to Heath, H.G.M. ii. 195-196.
XX. LATER DEVELOPMENTS IN GEOMETRY
XX. LATER DEVELOPMENTS IN GEOMETRY

(a) Classification of Curves

Procl. in Eucl. i., ed. Friedlein 111. 1-112. 11

Διαιρεῖ δ' αὖ τὴν γραμμήν ὁ Γέμινος¹ πρῶτον μὲν εἰς τὴν ἀσύνθετον καὶ τὴν σύνθετον—καλεῖ δὲ σύνθετον τὴν κεκλασμένην καὶ γωνίαν ποιοῦσαν—ἐπειτα τὴν ἀσύνθετον² εἰς τε τὴν σχηματοποιοῦσαν καὶ τὴν ἐπ’ ἀπειρον ἐκβάλλομένην, σχήμα λέγων ποιεῖν τὴν κυκλικήν, τὴν τοῦ θυρεοῦ, τὴν κυκτοειδῆ, μὴ ποιεῖν δὲ τὴν τοῦ ὀρθογωνίου κώνου τομήν, τὴν τοῦ ἀμβλυγωνίου, τὴν κογχοειδῆ, τὴν εὐθείαν, πάσας τὰς τουαύτας. καὶ πάλιν κατ’ ἄλλον τρόπον τῆς ἀσύνθετον γραμμῆς τὴν μὲν ἀπλῆν εἶναι, τὴν δὲ μικτῆν, καὶ τῆς ἀπλῆς τὴν μὲν σχήμα ποιεῖν ὡς τὴν κυκλικήν, τὴν δὲ ἀόριστον εἶναι ὡς τὴν εὐθείαν, τῆς δὲ μικτῆς τὴν μὲν ἐν τοῖς ἐπιτέδεοις εἶναι, τὴν δὲ ἐν τοῖς στερεοῖς, καὶ τῆς ἐν ἐπιτέδεοις τὴν μὲν ἐν αὐτῇ συμπλήτεω ὡς τὴν κυκτοειδῆ, τὴν δ’ ἐπ’ ἀπειρον ἐκβάλλεσθαι, τῆς δὲ ἐν στερεοῖς

¹ Γέμινος Tittel, Γεμίνος Friedlein.
² σύνθετον codd., correxii.

* No great new developments in geometry were made by the Greeks after the death of Apollonius, probably through 360
XX. LATER DEVELOPMENTS IN GEOMETRY

(a) Classification of Curves

Proclus, *On Euclid i.*, ed. Friedlein 111. 1–112. 11

Geminus first divides lines into the *incomposite* and the *composite*, meaning by composite the broken line forming an angle; and then he divides the incomposite into those forming a figure and those extending without limit, including among those forming a figure the circle, the ellipse and the cissoid, and among those not forming a figure the parabola, the hyperbola, the conchoid, the straight line, and all such lines. Again, in another manner he says that some incomposite lines are simple, others mixed, and among the simple are some forming a figure, such as the circle, and others indeterminate, such as the straight line, while the mixed include both lines on planes and lines on solids, and among the lines on planes are lines meeting themselves, such as the cissoid, and others extending without limit, and among lines on solids are the limits imposed by their methods, and the recorded additions to the corpus of Greek mathematics may be described as reflections upon existing work or "stock-taking." On the basis of geometry, however, the new sciences of trigonometry and mensuration were founded, as will be described, and the revival of geometry by Pappus will also be reserved for separate treatment.
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τὴν μὲν κατὰ τὰς τομὰς ἐπινοεῖσθαι τῶν στερεῶν, τὴν δὲ περὶ τὰ στερεὰ υφίστασθαι. τὴν μὲν γὰρ ἐλίκα τὴν περὶ σφαῖραν ἢ κῶνον περὶ τὰ στερεὰ υφεστάναι, τὰς δὲ κωνικὰς τομὰς ἢ τὰς στερεικὰς ἀπὸ τοιᾶς τοιῆς γεννάσθαι τῶν στερεῶν. ἐπινοηθήναι δὲ ταύτας τὰς τομὰς τὰς μὲν ὑπὸ Μεναιχμοῦ τὰς κωνικὰς, δὲ καὶ Ἐρατοσθένης ἱστορῶν λέγει: “μὴ δὲ Μεναιχμίους κωνοτομεῖν τριάδας”. τὰς δὲ ὑπὸ Περσέως, δὲ καὶ τὸ ἐπίγραμμα ἐποίησεν ἐπὶ τῇ εὑρέσει—

Τρεῖς γραμμὰς ἐπὶ πέντε τομαῖς εὑρὼν ἐλικώδεις ¹
Περσεὺς τῶν δ’ ἐνεκεν δαίμονας ἠλάσατο.

αἱ μὲν δὴ τρεῖς τομαῖ τῶν κώνων εἰσὶν παραβολὴ καὶ ὑπερβολὴ καὶ ἔλλευψις, τῶν δὲ στερεικῶν τομῶν ἡ μὲν ἐστὶν ἐμπεπλεγμένη, ἐοικυπα τῇ τοῦ ἱπποῦ πέδη, ἡ δὲ κατὰ τὰ μέσα πλατύνεται, εξ ἐκατέρου δὲ ἀπολύγει μέρους, ἡ δὲ παραμήκης οὐσά τῷ μὲν μέσῳ διαστήματι ἔλαττον χρηταί, εὐφύνεται δὲ ἐφ’ ἐκάτερα. τῶν δὲ ἄλλων μίξεων τὸ πλήθος ἀπέραντόν ἐστιν· καὶ γὰρ στερεῶν σχημάτων πλήθος ἐστὶν ἀπεριον καὶ τομαί αὐτῶν συνιστανται πολυεἰδεῖς.

Ibid., ed. Friedlein 356. 8-12

Καὶ γὰρ Ἀπολλώνιος ἐφ’ ἐκάστης τῶν κωνικῶν γραμμῶν τί τὸ σύμπτωμα δείκνυοι, καὶ ὁ Νικομήδης ἐπὶ τῶν κοχχοειδῶν, καὶ ὁ Ἰππίας ἐπὶ

¹ Ελικώδεις Knoche, εὑρὼν τὰς στερεικὰς λέγων codd.

¹ v. vol. i. pp. 296-297.
² For Perseus, v. p. 364 n. a and p. 365 n. b.
LATER DEVELOPMENTS IN GEOMETRY

lines conceived as *formed by sections* of the solids and lines *formed round the solids*. The helix round the sphere or cone is an example of the lines formed round solids, and the conic sections or the spiric curves are generated by various sections of solids. Of these sections, the conic sections were discovered by Menaechmus, and Eratosthenes in his account says: "Cut not the cone in the triads of Menaechmus"; and the others were discovered by Perseus, who wrote an epigram on the discovery—

Three spiric lines upon five sections finding,
Perseus thanked the gods therefor.

Now the three conic sections are the parabola, the hyperbola and the ellipse, while of the spiric sections one is *interlaced*, resembling the horse-fetter, another is widened out in the middle and contracts on each side, a third is elongated and is narrower in the middle, broadening out on either side. The number of the other mixed lines is unlimited; for the number of solid figures is infinite and there are many different kinds of section of them.


For Apollonius shows for each of the conic curves what is its property, as does Nicomedes for the
Obviously the work of Perseus was on a substantial scale to be associated with these names, but nothing is known of him beyond these two references. He presumably flourished after Euclid (since the conic sections were probably well developed before the spiric sections were tackled) and before Geminus (since Proclus relies on Geminus for his knowledge of the spiric curves). He may therefore be placed between 300 and 75 B.C.

Nicomedes appears to have flourished between Eratosthenes and Apollonius. He is known only as the inventor of the conchoid, which has already been fully described (vol. i. pp. 298-309).

It is convenient to recall here that about a century later flourished Diocles, whose discovery of the cissoid has already been sufficiently noted (vol. i. pp. 270-279). He has also been referred to as the author of a brilliant solution of the problem of dividing a cone in a given ratio, which is equivalent to the solution of a cubic equation (supra, p. 162 n. a). The Dionysodorus who solved the same problem (ibid.) may have been the Dionysodorus of Caunus mentioned in the Herculaneum Roll, No. 1044 (so W. Schmidt in Bibliotheca mathematica, iv. pp. 321-325), a younger contemporary of Apollonius; he is presumably the same person as the 364.
conchoid and Hippias for the quadratrices and Perseus for the spiric curves.

*Ibid.*, ed. Friedlein 119. 8-17

We say that this is the case with the spiric surface; for it is conceived as generated by the revolution of a circle remaining perpendicular [to a given plane] and turning about a fixed point which is not its centre. Hence there are three forms of spire according as the centre is on the circumference, or within it, or without. If the centre is on the circumference, the spire generated is said to be continuous, if within interlaced, and if without open. And there are three spiric sections according to these three differences.

Dionysodorus mentioned by Heron, *Metrica* ii. 13 (cited *infra*, p. 481), as the author of a book *On the Spire*.

This last sentence is believed to be a slip, perhaps due to too hurried transcription from Geminus. At any rate, no satisfactory meaning can be obtained from the sentence as it stands. Tannery (*Mémoires scientifiques* ii. pp. 24-28) interprets Perseus’ epigram as meaning “three curves in addition to five sections.” He explains the passages thus: Let \(a\) be the radius of the generating circle, \(c\) the distance of the centre of the generating circle from the axis of revolution, \(d\) the perpendicular distance of the plane of section (assumed to be parallel to the axis of revolution) from the axis of revolution. Then in the open spire, in which \(c>a\), there are five different cases:

1. \(c+a>d>c\). The curve is an oval.
2. \(d=c\). Transition to (3).
3. \(c>d>c-a\). The curve is a closed curve narrowest in the middle.
4. \(d=c-a\). The curve is the *hippopede* (horse-fetter), which is shaped like the figure of 8 (*v. vol. i. pp. 414-415* for the use of this curve by Eudoxus).
5. \(c-a>d>0\). The section consists of two symmetrical ovals.

Tannery identifies the “five sections” of Perseus with these five types of section of the open spire; the three curves
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(b) ATTEMPTS TO PROVE THE PARALLEL POSTULATE

(i.) General

Procl. in Eucl. i., ed. Friedlein 191. 16–193. 9

"Καὶ ἐὰν εἰς δύο εὐθείας εὐθεία ἐμπλήττοσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἑλάττωνας ποιῆ, ἐκβαλλομένας τὰς εὐθείας ἐπ’ ἀπειρον συμπλίττειν, ἐφ’ ἀ μέρη εἰσὶν αἰ τῶν δύο ὀρθῶν ἑλάττωνες."

Τοῦτο καὶ παντελῶς διαγράφειν χρή τῶν αὐτημάτων· θεώρημα γάρ ἐστι, πολλὰς μὲν ἀπορίας ἐπιδεχόμενον, ὡς καὶ ὁ Πτολεμαῖος ἐν τοῖς βιβλίοις διαλύσει προὔθετο, πολλῶν δὲ εἰς ἀπόδειξιν δεόμενον καὶ ὅρων καὶ θεωρημάτων. καὶ τὸ γε ἀντιστρέφον καὶ ὁ Εὐκλείδης ὡς θεώρημα δείκνυσιν. ὥσε δὲ ἀν тινες ἀπατώμενοι καὶ τούτο τάττειν ἐν τοῖς αὐτῆμασιν ἄξιωσειν, ὡς διὰ τὴν ἑλάττωσιν τῶν δύο ὀρθῶν αὐτόθεν τὴν πίστιν παρεχόμενον
described by Proclus are (1), (3) and (4). When the spire is continuous or closed, \( c = a \) and there are only three sections corresponding to (1), (2) and (3); (4) and (5) reduce to two equal circles touching one another. But the interlaced spire, in which \( c < a \), gives three new types of section, and in these Tannery sees his "three curves in addition to five sections." There are difficulties in the way of accepting this interpretation, but no better has been proposed.

Further passages on the spire by Heron, including a formula for its volume, are given infra, pp. 476-483.

a Eucl. i. Post. 5, for which κ. vol. i. pp. 442-443, especially n. c.

Aristotle (Anal. Prior. ii. 16, 65 a 4) alludes to a petitio principii current in his day among those who "think they establish the theory of parallels"—τὰς παράλληλους γράφειν. As Heath notes (The Thirteen Books of Euclid's Elements, 366
(b) Attempts to Prove the Parallel Postulate

(i.) General

Proclus, *On Euclid i.*, ed. Friedlein 191. 16-193. 9

"If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles." 

This ought to be struck right out of the Postulates; for it is a theorem, and one involving many difficulties, which Ptolemy set himself to resolve in one of his books, and for its proof it needs a number of definitions as well as theorems. Euclid actually proves its converse as a theorem. Possibly some would erroneously consider it right to place this assumption among the Postulates, arguing that, as the angles are less than two right angles, there is

vol. i. pp. 191-192), Philoponous’s comment on this passage suggests that the *petitio principii* lay in a *direction* theory of parallels. Euclid appears to have admitted the validity of the criticism and, by *assuming* his famous postulate once and for all, to have countered any logical objections.

Nevertheless, as the extracts here given will show, ancient geometers were not prepared to accept the undemonstrable character of the postulate. Attempts to prove it continued to be made until recent times, and are summarized by R. Bonola, “Sulla teoria delle parallele e sulle geometrica non-euclidee” in *Questioni riguardanti la geometria elementare*, and by Heath, *loc. cit.*, pp. 204-219. The chapter on the subject in W. Rouse Ball’s *Mathematical Essays and Recreations*, pp. 307-326, may also be read with profit. Attempts to prove the postulate were abandoned only when it was shown that, by not conceding it, alternative geometries could be built.
GREEK MATHEMATICS

τῆς τῶν εὐθείων συνενώσεως καὶ συμπτώσεως. πρὸς οὖς ὁ Γεμῖνος ὁρθῶς ἀπήντησε λέγων ὅτι παρ’ αὐτῶν ἐμάθομεν τῶν τῆς ἐπιστήμης ταύτης ἡγεμόνων μὴ πάνω προσέχειν τὸν νοῦν ταῖς πιθαναῖς φαντασίαις εἰς τὴν τῶν λόγων τῶν ἐν γεωμετρίᾳ παραδοχῆν. ὅμως γὰρ φησὶ καὶ Ἀριστοτέλης ῥητορικὸν ἀποδείξεις ἀπαιτεῖν καὶ γεωμετρῶν πιθανολογοῦντος ἀνέχεσθαι, καὶ ὁ παρὰ τῷ Πλάτωνι Συμμίας, ὅτι "τοὺς ἐκ τῶν εἰκότων τὰς ἀποδείξεις ποιουμένους σύνοιδα οὖσιν ἀλαζόσι." καντάθα τοῖς τὸ μὲν ἡλαττωμένων τῶν ὁρθῶν συνενῶς τὰς εὐθείας ἀληθεῖς καὶ ἀναγκαῖον, τὸ δὲ συνενῶσας ἐπὶ πλέον ἐν τῷ ἐκβάλλεσθαι συμπεσεῖσθαι ποτὲ πιθανόν, ἀλλ’ οὐκ ἀναγκαῖον, εἰ μὴ τις ἀποδείξειαν λόγος, ὅτι ἐπὶ τῶν εὐθείων τούτο ἀληθεῖς. τὸ γὰρ εἶναι τινας γραμμὰς συνιστώσας μὲν ἐν ἀπειρον, ἀσυμπτῶτους δὲ ὑπαρχοῦσας, καὶ τοι δοκοῦν ἀπίθανον εἶναι καὶ παράδοξον, ἀπὸς ἀληθεῖς ἐστὶ καὶ πεφώραται ἐν ἀλλων εἰδῶν τῆς γραμμῆς. μήποτε οὖν τούτῳ καὶ ἐν τῶν εὐθείων δυνατῷ, ὅπερ ἐπὶ ἐκείνων τῶν γραμμῶν; ἐως γὰρ ἄν δι’ ἀποδείξεως αὐτὸ καταδησόμεθα, περισσᾶ τὴν φαντασίαν τὰ ἐν ἀλλων δεικνύμενα γραμμῶν. εἰ δὲ καὶ οἱ διαμφισβητοῦντες λόγοι πρὸς τὴν σύμπτωσιν πολὺ τὸ πληκτικὸν ἔχουν, πῶς οὐχὶ πολλῶ πλέον ἄν τὸ πιθανόν τούτῳ καὶ τὸ ἀλογὸν ἐκβάλλουμεν τῆς ἱμετέρας παραδοχῆς;

Ἀλλ’ ὅτι μὲν ἀπόδειξιν χρὴ ἵπτειν τοῦ πρω-κειμένου θεωρήματος δὴλον ἐκ τούτων, καὶ ὅτι

* For Geminus, v. infra, p. 370 n. c.*
immediate reason for believing that the straight lines converge and meet. To such, Geminus a rightly rejoined that we have learnt from the pioneers of this science not to incline our mind to mere plausible imaginings when it is a question of the arguments to be used in geometry. For Aristotle b says it is as reasonable to demand scientific proof from a rhetorician as to accept mere plausibilities from a geometer, and Simmias is made to say by Plato c that he "recognizes as quacks those who base their proofs on probabilities." In this case the convergence of the straight lines by reason of the lessening of the right angles is true and necessary, but the statement that, since they converge more and more as they are produced, they will some time meet is plausible but not necessary, unless some argument is produced to show that this is true in the case of straight lines. For the fact that there are certain lines which converge indefinitely but remain non-secant, although it seems improbable and paradoxical, is nevertheless true and well-established in the case of other species of lines. May not this same thing be possible in the case of straight lines as happens in the case of those other lines? For until it is established by rigid proof, the facts shown in the case of other lines may turn our minds the other way. And though the controversial arguments against the meeting of the two lines should contain much that is surprising, is that not all the more reason for expelling this merely plausible and irrational assumption from our accepted teaching?

It is clear that a proof of the theorem in question must be sought, and that it is alien to the special

a Eth. Nic. i. 3. 4, 1094 b 25-27.  
b Phaedo 92 d.
GREEK MATHEMATICS

Posidonius and Geminus

Ibid., ed. Friedlein 176. 5-10

Kai δὲ Εὐκλείδης τοῦτον ὁρίζεται τὸν τρόπον τὰς παραλλήλους εὐθείας, ὁ δὲ Ποσειδώνος, παράλληλοι, φησίν, εἴσον αἱ μῆτε συνεύονσαι μῆτε ἀπονεύονσαι ἐν ἐνὶ ἐπιπέδῳ, ἀλλ’ ἵσας ἔχουσαι

*a i.e., Eucl. i. 28.

b Posidonius was a Stoic and the teacher of Cicero; he was born at Apamea and taught at Rhodes, flourishing 151–135 B.C. He contributed a number of definitions to elementary geometry, as we know from Proclus, but is more famous for a geographical work On the Ocean (lost but copiously quoted by Strabo) and for an astronomical work Περὶ μετεώρων. In this he estimated the circumference of the earth (v. supra, p. 267) and he also wrote a separate work on the size of the sun.

c As with so many of the great mathematicians of antiquity, we know practically nothing about Geminus’s life, not even his date, birthplace or the correct spelling of his name. As he wrote a commentary on Posidonius’s Περὶ μετεώρων, we have an upper limit for his date, and “the view most generally accepted is that he was a Stoic philosopher, born probably in the island of Rhodes, and a pupil of Posidonius, and that he wrote about 73–67 B.C. ” (Heath, H.G.M. ii. 223). Further details may be found in Manitius’s edition of the so-called Gemini elementa astronomiae.

Geminus wrote an encyclopaedic work on mathematics
character of the Postulates. But how it should be proved, and by what sort of arguments the objections made against it may be removed, must be stated at the point where the writer of the *Elements* is about to recall it and to use it as obvious. Then it will be necessary to prove that its obvious character does not appear independently of proof, but by proof is made a matter of knowledge.

(ii.) *Posidonius* and *Geminus*

*Ibid.*, ed. Friedlein 176. 5-10

Such is the manner in which Euclid defines parallel straight lines, but Posidonius says that parallels are lines in one plane which neither converge nor diverge which is referred to by ancient writers under various names, but that used by Eutocius (*Ṭow maθημάτων thewria*, *v. supra*, pp. 280-281) was most probably the actual title. It is unfortunately no longer extant, but frequent references are made to it by Proclus, and long extracts are preserved in an Arabic commentary by an-Nairizi.

It is from this commentary that Geminus is known to have attempted to prove the parallel-postulate by a definition of parallels similar to that of Posidonius. The method is reproduced in Heath, *H.G.M.* ii. 228-230. It tacitly assumes “Playfair’s axiom,” that through a given point only one parallel can be drawn to a given straight line; this axiom—which was explicitly stated by Proclus in his commentary on Eucl. i. 30 (Procl. *in Eucl. 1.*, ed. Friedlein 374. 18–375. 3)—is, in fact, equivalent to Euclid’s Postulate 5. Saccheri noted an even more fundamental objection, that, before Geminus’s definition of parallels can be used, it has to be proved that the locus of points equidistant from a straight line is a straight line; and this cannot be done without some equivalent postulate. Nevertheless, Geminus deserves to be held in honour as the author of the first known attempt to prove the parallel-postulate, a worthy predecessor to Lobachewsky and Riemann.
πάσας τὰς καθέτους τὰς ἀγομένας ἀπὸ τῶν τῆς ἐτέρας σημείων ἐπὶ τὴν λοιπὴν.

(iii.) Ptolemy

Ibid., ed. Friedlein 362. 12–363. 18

'Αλλ' οὕτως μὲν ὁ Στοιχειωτής δείκνυσιν ὅτι δύο ὀρθαῖς ἴσων οὐσῶν τῶν ἐντὸς αἱ εὐθείαι παράλληλοί εἰσιν, φανερὸν ἐκ τῶν γεγραμμένων. Πτολεμαίως δὲ ἐν οἷς ἀποδείκται προέθετο τὰς ἀπ' ἐλαττόνων ἡ δύο ὀρθῶν ἐκβαλλομένας συμπίπτειν, ἐφ' ἡ μέρη εἰσιν αἱ τῶν δύο ὀρθῶν ἐλάσσονες, τούτῳ πρὸ πάντων δεικνὺς τὸ θέωρημα τὸ δεύτερον ὀρθαῖς ἴσων ὑπορχοῦσῶν τῶν ἐντὸς παράλληλοις εἶναι τὰς εὐθείας σωτῷ πώς δεικνύσιν.

"Εστώσαν δύο εὐθείας αἱ AB, ΓΔ, καὶ τεμνῶν τις αὐτὰς εὐθεία ἡ EZΗΘ, ὥστε τὰς ὑπὸ ΒΖΗ καὶ ὑπὸ ΖΗΔ γωνίας δύο ὀρθαῖς ἴσας ποιεῖν. λέγω ὅτι παράλληλοί εἰσιν αἱ εὐθείαι, τούτῳστῳ 372
but the perpendiculars drawn from points on one of the lines to the other are all equal.

(iii.) Ptolemy


How the writer of the *Elements* proves that, if the interior angles be equal to two right angles, the straight lines are parallel is clear from what has been written. But Ptolemy, in the work in which he attempted to prove that straight lines produced from angles less than two right angles will meet on the side on which the angles are less than two right angles, first proved this theorem, that if the interior angles be equal to two right angles the lines are parallel, and he proves it somewhat after this fashion.

Let the two straight lines be AB, ΓΔ, and let any straight line EZΗΘ cut them so as to make the angles BZH and ZHΔ equal to two right angles. I say that the straight lines are parallel, that is they are non-

* For the few details known about Ptolemy, *v. infra*, p. 408 and n. b.
* This work is not otherwise known.
There is a Common Notion to this effect interpolated in the text of Euclid; v. vol. i. pp. 444 and 445 n. a.

The argument would have been clearer if it had been proved that the two interior angles on one side of $ZH$ were severally equal to the two interior angles on the other side, that is $BZH = \Gamma HZ$ and $\Delta HZ = AZH$; whence, if $ZA$, $HG$ meet at $\Lambda$, the triangle $ZHA$ can be rotated about the mid-

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*"Hdη μὲν οὖν καὶ ἄλλω τινὲς ὡς θεώρημα προ-
tάξαντες τούτο αὕτημα παρά τῷ Στοιχειώτη ἁρφθεν ἀποδείξαμεν ἥξιωσεν. Δοκεῖ δὲ καὶ ὁ Πτολεμαῖος*
secant. For, if it be possible, let BZ, HΔ, when produced, meet at K. Then since the straight line HZ stands on AB, it makes the angles AZH, BZH equal to two right angles [Eucl. i. 13]. Similarly, since HZ stands on ΓΔ, it makes the angles ΓHZ, ΔHZ equal to two right angles [ibid.]. Therefore the four angles AZH, BZH, ΓHZ, ΔHZ are equal to four right angles, and of them two, BZH, ZHΔ, are by hypothesis equal to two right angles. Therefore the remaining angles AZH, ΓHZ are also themselves equal to two right angles. If then, the interior angles being equal to two right angles, ZB, HΔ meet at K when produced, ZA, HΓ will also meet when produced. For the angles AZH, ΓHZ are also equal to two right angles. Therefore the straight lines will either meet on both sides or on neither, since these angles also are equal to two right angles. Let ZA, HΓ meet, then, at Λ. Then the straight lines ΔABK, ΔΓΔK enclose a space, which is impossible. Therefore it is not possible that, if the interior angles be equal to two right angles, the straight lines should meet. Therefore they are parallel.*

Ibid., ed. Friedlein 365. 5–367. 27

Therefore certain others already classed as a theorem this postulate assumed by the writer of the Elements and demanded a proof. Ptolemy appears point of ZH so that ZH lies where HZ is in the figure, while ZK, HK lie along the sides HΓ, ZA respectively: and therefore HΓ, ZA must meet at the point where K falls.

The proof is based on the assumption that two straight lines cannot enclose a space. But Riemann devised a geometry in which this assumption does not hold good, for all straight lines having a common point have another point common also.
Αυτό δεικνύει εν τῷ περὶ τού τάς ἀπ’ ἐλαττόνων ἡ δύο ὀρθῶν ἐκβαλλομένας συμπίπτειν, καὶ δείκνυσι πολλὰ προλαβῶν τῶν μέχρι τοῦθε ν θεωρήματος ὑπὸ τοῦ Στοιχειωτοῦ προσποδειγμένων. καὶ ὑποκείσθω πάντα εἰναι ἀληθῆ, ἵνα μὴ καὶ ἡμεῖς ὁχλὸν ἐπεισάγωμεν ἄλλον, καὶ ὡς λημμάτιον τοῦτο δείκνυσθαι διὰ τῶν προειρημένων. εὖ δὲ καὶ τοῖτο τῶν προειρημένων τὸ τάς ἀπὸ δυεῖν ὀρθαῖς ἰσων ἐκβαλλομένας μηδαμῶς συμπίπτειν. λέγω τοῖνυν ὅτι καὶ τὸ ἀνάπαλω ἀληθὲς, καὶ τὸ παραλλήλων οὐσῶν τῶν εὐθείων καὶ τεμνομένων ὑπὸ μᾶς εὐθείας τᾶς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθαίς ἰσας εἰναι. ἀνάγκη γὰρ τὴν τέμνουσαν τᾶς παραλλήλους ἡ δύο ὀρθαῖς ἰσας ποιεῖν τᾶς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας ἡ δύο ὀρθῶν ἐλάσσους ἡ μείζους. ἐστωσαν οὖν παράλληλοι αἱ AB, ΓΔ, καὶ ἐμπιπτέτως εἰς αὐτὰς ἡ ΗΖ; λέγω ὅτι οὐ ποιεῖ δύο ὀρθῶν μείζους τᾶς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ. εἰ γὰρ αἱ ὑπὸ ΑΖΗ, ΓΗΖ δύο ὀρθῶν μείζους, αἱ λοιπαὶ αἱ ὑπὸ ΒΖΗ, ΔΗΖ δύο ὀρθῶν ἐλάσσους. ἀλλὰ καὶ δύο ὀρθῶν μειζους αἱ αὐταὶ. οὐδὲν γὰρ μᾶλλον αἱ AZ, ΓΗ παράλληλοι ἡ ZB, ΗΔ, ὡστε εἰ ἡ ἐμπεσοῦσα εἰς τὰς AZ, ΓΗ δύο ὀρθῶν μείζους ποιεῖ τὰς ἐντὸς, καὶ 876
to have proved it in his book on the proposition that straight lines drawn from angles less than two right angles meet if produced, and he uses in the proof many of the propositions proved by the writer of the *Elements* before this theorem. Let all these be taken as true, in order that we may not introduce another mass of propositions, and by means of the aforesaid propositions this theorem is proved as a lemma, that straight lines drawn from two angles together equal to two right angles do not meet when produced—a—for this is common to both sets of preparatory theorems. I say then that the converse is also true, that if parallel straight lines be cut by one straight line the interior angles on the same side are equal to two right angles. For the straight line cutting the parallel straight lines must make the interior angles on the same side equal to two right angles or less or greater. Let AB, ΓΔ be parallel straight lines, and let HZ cut them; I say that it does not make the interior angles on the same side greater than two right angles. For if the angles AZH, ΓHZ are greater than two right angles, the remaining angles BZH, ΔHZ are less than two right angles. But these same angles are greater than two right angles; for AZ, ΓH are not more parallel than ZB, HΔ, so that if the straight line falling on AZ, ΓH make the interior angles greater than two right angles, the same straight line falling

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*a* This is equivalent to Eucl. i. 28.

*b* This is equivalent to Eucl. i. 29.

*c* By Eucl. i. 13, for the angles AZH, BZH are together equal to two right angles and so are the angles ΓHZ, ΔHZ.
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η εἰς τὰς ΖΒ, ΗΔ ἐμπίπτουσα δύο ὀρθῶν ποιῆσει μείζους τὰς ἐντὸς· ἀλλ' αἱ αὐταὶ καὶ δύο ὀρθῶν ἐλάσσους· αἱ γὰρ τέσσαρες αἱ ὑπὸ AZH, ΓΗΖ, ΒΖΗ, ΔΗΖ τέτρασιν ὀρθαῖς ἵσαι· ὀπερ ἀδύνατον. ὁμοίως δὴ δείξομεν ὅτι εἰς τὰς παραλλήλους ἐμπίπτουσα οὐ ποιεῖ δύο ὀρθῶν ἐλάσσους τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας. εἰ δὲ μῆτε μείζους μὴτε ἐλάσσους ποιεῖ τῶν δύο ὀρθῶν, λείπεται τὴν ἐμπίπτουσαν δύο ὀρθαῖς ὑσας ποιεῖν τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας.

Τούτον δὴ οὖν προδεδειγμένον τὸ προκείμενον ἀναμφίσβηττως ἀποδείκνυται. λέγω γὰρ ὅτι ἐὰν εἰς δύο εὐθείας εὐθεία ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσους ποιῆ, συμπεσοῦνται αἱ εὐθείαι ἐκβαλλόμεναι, ἐφ' ἀ μέρη εἰῶν αἱ τῶν δύο ὀρθῶν ἐλάσσους. μὴ γὰρ συμπιπτέτωσαν. ἀλλ' εἰ ἀσύμπτωτοί εἰσιν, ἐφ' ἀ μέρη αἱ τῶν δύο ὀρθῶν ἐλάσσους, πολλῷ μᾶλλον ἔσονται ἀσύμπτωτοι ἐπὶ θάτερα, ἐφ' ἀ τῶν δύο εἰῶν ὀρθῶν αἱ μείζους, ὥστε ἐφ' ἐκάτερα ἀν εἰς ἀσύμπτωτοι αἱ εὐθείαι. εἰ δὲ τούτο, παράλληλοι εἰσιν. ἀλλὰ δεδεικται ὅτι ἡ εἰς τὰς παραλλήλους ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δύο ὀρθαῖς ἵσαι ποιῆσει γωνίας. αἱ αὐταὶ ἅρα καὶ δύο ὀρθαὶς ἵσαι καὶ δύο ὀρθῶν ἐλάσσους, ὀπερ ἀδύνατον.

Ταῦτα προδεδειχθος ὦ Πτολεμαῖος καὶ καταν-

* See note c on p. 377.
* The fallacy lies in the assumption that "AZ, ΓΗ are not more parallel than ΖΒ, ΗΔ," so that the angles ΒΖΗ, ΔΗΖ must also be greater than two right angles. This assump-

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on ZB, $H\Delta$ also makes the interior angles greater than two right angles; but these same angles are less than two right angles, for the four angles $AZH, THZ, BZH, AHZ$ are equal to four right angles $^{a}$; which is impossible. Similarly we may prove that a straight line falling on parallel straight lines does not make the interior angles on the same side less than two right angles. But if it make them neither greater nor less than two right angles, the only conclusion left is that the transversal makes the interior angles on the same side equal to two right angles.$^{b}$

With this preliminary proof, the theorem in question is proved beyond dispute. I mean that if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced, will meet on that side on which are the angles less than two right angles. For [, if possible] let them not meet. But if they are non-secant on the side on which are the angles less than two right angles, by much more will they be non-secant on the other side, on which are the angles greater than two right angles, so that the straight lines would be non-secant on both sides. Now if this should be so, they are parallel. But it has been proved that a straight line falling on parallel straight lines makes the interior angles on the same side equal to two right angles. Therefore the same angles are both equal to and less than two right angles, which is impossible.

Having first proved these things and squarely faced

$^{a}$

$^{b}$
τήσας εἰς τὸ προκείμενον ἀκριβέστερον τι προσ-
θείναι βούλεται καὶ δεῖξαι ὅτι, ἐὰν εἰς δύο ἑυθείας
ἑυθεία ἐμπίπτουσα τὰς ἑντὸς καὶ ἐπὶ τὰ αὐτὰ
μέρη δύο ὀρθῶν ποιῆ ἑλάσσονας, οὐ μόνον οὐκ
εἰσὶν ἀσύμπτωτοι αἱ ἑυθεῖαι, ὡς δέδεικται, ἀλλὰ
καὶ ἡ σύμπτωσις αὐτῶν κατ’ ἐκείνα γίνεται τὰ
μέρη, ἕφ’ ἀι τῶν δύο ὀρθῶν ἑλάσσονες, οὐκ ἐφ’
ἀι ἑλάσσονες. ἐστωσαν γὰρ δύο ἑυθεῖαι αἱ ΑΒ,
ΓΔ καὶ ἐμπίπτουσα εἰς αὐτὰς ἡ ΕΖΗΘ ποιεῖτω
τὰς ὑπὸ ΑΖΗ καὶ ὑπὸ ΓΗΖ δύο ὀρθῶν ἑλάσσους.

αἱ λοιπαὶ ἄρα ἑλάσσον δύο ὀρθῶν. ὅτι μὲν [οὐν]¹
οὐκ ἀσύμπτωτοι αἱ ἑυθεῖαι δέδεικται. εἰ δὲ συμ-
πίπτουσιν, ἡ ἐπὶ τὰ Α, Γ συμπεσοῦνται, ἡ ἐπὶ
tὰ Β, Δ. συμπιπτέτωσαν ἐπὶ τὰ Β, Δ κατὰ τὸ
Κ. ἐπεὶ οὖν αἱ μὲν ὑπὸ ΑΖΗ καὶ ΓΗΖ δύο
ὀρθῶν εἰσὶν ἑλάσσοι, αἱ δὲ ὑπὸ ΑΖΗ, ΒΖΗ δύο
ὀρθαῖς ἰσαί, κοινῆς ἀφαιρεθείσης τῆς ὑπὸ ΑΖΗ,
the theorem in question, Ptolemy tries to make a more precise addition and to prove that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, not only are the straight lines not non-secant, as has been proved, but their meeting takes place on that side on which the angles are less than two right angles, and not on the side on which they are greater. For let $AB, \Gamma\Delta$ be two straight lines and let $EZH\Theta$ fall on them and make the angles $AZH, \GammaHZ$ less than two right angles. Then the remaining angles are greater than two right angles [Eucl. i. 13]. Now it has been proved that the straight lines are not non-secant. If they meet, they will meet either on the side of $A, \Gamma$ or on the side of $B, \Delta$. Let them meet on the side of $B, \Delta$ at $K$. Then since the angles $AZH, \GammaHZ$ are less than two right angles, while the angles $AZH, BZH$ are equal to two right angles, when the common angle $AZH$ is taken away, the angle $\GammaHZ$ will be less

1 $\tilde{o}v$ is clearly out of place.
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η υπὸ ΓΗΖ ἐλάσσων ἦσται τῆς ὑπὸ ΒΖΗ.

τριγώνον ἀρα τοῦ ΚΖΗ ἡ ἐκτὸς τῆς ἐντὸς καὶ ἀπεναντίον ἐλάσσων, ὅπερ ἀδύνατον. οὐκ ἀρα
κατὰ ταύτα συμπίπτουσιν. ἀλλὰ μὴν συμπίπτουσι.
κατὰ άλλα ἀρα ἡ σύμπτωσις αὐτῶν ἦσται, καθ' ἀ αἰ τῶν δύο ὅρθων εἰσιν ἐλάσσονες.

(iv.) Proclus

Ibid., ed. Friedlein 371. 23-373. 2

Τουτον δὴ προσποτεθέντος λέγω ὅτι, εὰν παραλλήλων εὐθειῶν τὴν ἔτεραν τέμνει τις εὐθεία, τεμεῖ καὶ τὴν λοιπὴν.

"Εστωσαν γὰρ παράλληλοι αἱ ΑΒ, ΓΔ, καὶ τεμνέτω τὴν ΑΒ ἡ ΕΖΗ. λέγω ὅτι τὴν ΓΔ τεμεὶ.

'Επεὶ γὰρ δύο εὐθειαί εἰσιν ἄφ' ἐνὸς σημείου τοῦ Ζ, εἰς ἀπειρὸν ἐκβαλλόμενα αἱ ΒΖ, ΖΗ, παντὸς μεγέθους μείζονα ἔχουσι διάστασιν, ὡστε καὶ τούτου, ὅσον ἐστὶ τὸ μεταξὺ τῶν παραλλήλων.

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than the angle BZH. Therefore the exterior angle of the triangle KZH will be less than the interior and opposite angle, which is impossible [Eucl. i. 16]. Therefore they will not meet on this side. But they do meet. Therefore their meeting will be on the other side, on which the angles are less than two right angles.

(iv.) Proclus

Ibid., ed. Friedlein 371. 23–373. 2

This having first been assumed, I say that, if any straight line cut one of parallel straight lines, it will cut the other also.

For let AB, ΓΔ be parallel straight lines, and let EZH cut AB. I say that it will cut ΓΔ.

For since BZ, ZH are two straight lines drawn from one point Z, they have, when produced indefinitely, a distance greater than any magnitude, so that it will also be greater than that between the parallels.
The method is ingenious, but Clavius detected the flaw, which lies in the initial assumption, taken from Aristotle, that two divergent straight lines will eventually be so far apart that a perpendicular drawn from a point on one to the other will be greater than any assigned distance; Clavius draws attention to the conchoid of Nicomedes (v. vol. i. pp. 298-301), which continually approaches its asymptote, and therefore continually gets farther away from the tangent at the vertex; but the perpendicular from any point on the curve to that tangent will always be less than the distance between the tangent and the asymptote.
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Whenever, therefore, they are at a distance from one another greater than the distance between the parallels, \(ZH\) will cut \(\Gamma\Delta\). If, therefore, any straight line cuts one of parallels, it will cut the other also.

This having first been established, we shall prove in turn the theorem in question. For let \(AB, \Gamma\Delta\) be two straight lines, and let \(EZ\) fall on them so as to

\[
\begin{align*}
A & \quad \Gamma \\
K & \quad E \\
\Gamma & \quad Z \\
& \quad \Delta
\end{align*}
\]

make the angles \(BEZ, \DeltaZE\) less than two right angles. I say that the straight lines will meet on that side on which are the angles less than two right angles.

For since the angles \(BEZ, \DeltaZE\) are less than two right angles, let the angle \(\Theta EB\) be equal to the excess of the two right angles. And let \(\Theta E\) be produced to \(K\). Then since \(EZ\) falls on \(K\Theta, \Gamma\Delta\) and makes the interior angles \(\Theta EZ, \DeltaZE\) equal to two right angles, the straight lines \(\Theta K, \Gamma\Delta\) are parallel. And \(AB\) cuts \(K\Theta\); therefore, by what was before shown, it will also cut \(\Gamma\Delta\). Therefore \(AB, \Gamma\Delta\) will meet on that side on which are the angles less than two right angles, so that the theorem in question is proved.\(^a\)
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(c) ISOPERIMETRIC FIGURES


"Ωσαύτως δ' οτι, των ίσην περίμετρον ἐχόντων σχημάτων διαφόρων, ἐπειδή μεῖζονα ἐστὶν τὰ πολυγωνότερα, τῶν μὲν ἐπιτεδῶν ὁ κύκλος γίνεται μεῖζων, τῶν δὲ στερεῶν ἡ σφαῖρα."

Πουνσόμεθα δὴ τὴν τούτων ἀπόδειξιν ἐν ἐπιτομῇ ἐκ τῶν Ζηνοδώρῳ δεδειγμένων ἐν τῷ Περὶ ἰσοπεριμέτρων σχημάτων.

Τῶν ίσην περίμετρον ἐχόντων τεταγμένων εὐ-

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* Ptolemy, Math. Syn. i. 3, ed. Heiberg i. pars i. 13. 16-19.
* Zenodorus, as will shortly be seen, cites a proposition by Archimedes, and therefore must be later in date than Archimedes; as he follows the style of Archimedes closely, he is generally put not much later. Zenodorus's work is not
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(c) ISOPERIMENTRIC FIGURES

Theon of Alexandria, Commentary on Ptolemy's Syntaxis
1. 3, ed. Rome, Studi e Testi, lxxii. (1936), 354. 19-357. 22

"In the same way, since the greatest of the various figures having an equal perimeter is that which has most angles, the circle is the greatest among plane figures and the sphere among solid."

We shall give the proof of these propositions in a summary taken from the proofs by Zenodorus in his book On Isoperimetric Figures.

Of all rectilinear figures having an equal perimeter—

extant, but Pappus also quotes from it extensively (Coll. v. ad init.), and so does the passage edited by Hultsch (Papp. Coll., ed. Hultsch 1138-1165) which is extracted from an introduction to Ptolemy's Syntaxis of uncertain authorship (v. Rome, Studi e Testi, liv., 1931, pp. xiii-xvii). It is disputed which of these versions is the most faithful.

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θυγράμμων σχημάτων, λέγω δὴ ἱσοπλεύρων τε καὶ ἱσογωνίων, τὸ πολυγωνότερον μείζον ἔστιν.

"Εστώ γὰρ ἱσοπερίμετρα ἱσόπλευρα τε καὶ ἱσο-
γώνια τὰ ΑΒΓ, ΔΕΖ, πολυγωνότερον δὲ ἕστω
τὸ ΑΒΓ. λέγω, δὲ μείζον ἔστιν τὸ ΑΒΓ.

Εἰλήφθω γὰρ τὰ κέντρα τῶν περὶ τὰ ΑΒΓ, ΔΕΖ πολύγωνα περιγραφομένων κύκλων τὰ Ἡ, Ṣ, καὶ ἐπεζεύχθωσαν αἱ ἩΒ, ἩΓ, ṢΕ, ṢΩ. καὶ ἔστι ἀπὸ τῶν Ἡ, Ṣ ἐπὶ τὰς ΒΓ, ΔΕΖ κάθετο ηκθωσαν
αἱ ΗΚ, ṢΛ. ἐπεὶ οὖν πολυγωνότερον ἔστιν τὸ
ΑΒΓ τοῦ ΔΕΖ, πλεονάκις ἡ ΒΓ τῆς τοῦ ΑΒΓ
περίμετρον καταμετρεῖ ἦπερ ἡ ΔΕΖ τῆς τοῦ ΔΕΖ.
καὶ εἰσὶν ἵσαι αἱ περίμετροι. μείζον ἁρὰ ἡ ΔΕΖ
tῆς ΒΓ. ῥοτε καὶ ἡ ΔΛ τῆς ΒΚ. κεῖσθω τῇ
ΒΚ ῥη ΛΜ, καὶ ἐπεζευχθω ἡ ὉΜ. καὶ ἐπεὶ
ἔστω ὡς ἡ ΔΕΖ εὑθείᾳ πρὸς τὴν τοῦ ΔΕΖ πολυ-
γώνου περίμετρον οὕτως ἡ ὑπὸ ΘΩΖ πρὸς ἰ ὀρθῶς,
διὰ τὸ ἱσοπλευρὸν εἶναι τὸ πολύγωνον καὶ ἵσαι
ἀπολαμβάνειν περιφερεῖας τοῦ περιγραφομένου
κύκλου καὶ τὰς πρὸς τῷ κέντρῳ γωνίας τὸν αὐτὸν
ἐχειν λόγου ταῖς περιφερεῖαις ἐφ’ ὡν βεβηκασιν,
ὡς δὲ ἡ τοῦ ΔΕΖ περίμετρος, τουτέστων ἡ τοῦ
ΑΒΓ, πρὸς τὴν ΒΓ οὕτως αἱ ὀρθὰ πρὸς τὴν
ὑπὸ ΒΗΓ, δι’ ἱσου ἁρὰ ὡς ἡ ΔΕΖ πρὸς ΒΓ, του-
τέστων ἡ ΔΛ πρὸς ΛΜ, οὕτως καὶ ἡ ὑπὸ ΘΩΖ
γωνία πρὸς τὴν ὑπὸ ΒΗΓ, τουτέστων ἡ ὑπὸ ΘΔΛ
πρὸς τὴν ὑπὸ ΒΗΚ. καὶ ἐπεὶ ἡ ΔΛ πρὸς ΛΜ
μείζονα λόγου ἔχει ἦπερ ἡ ὑπὸ ΘΩΖ γωνία πρὸς
τὴν ὑπὸ ΜΘΛ, ὡς ἔξησ δείξομεν, ὡς δὲ ἡ ΔΛ

* ΘΩΖ is not, in fact, joined in the ms. figures.
* This is proved in a lemma immediately following the
  proposition by drawing an arc of a circle with Θ as centre

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I mean equilateral and equiangular figures—the greatest is that which has most angles.

For let $\triangle AB\Gamma$, $\triangle \Delta EZ$ be equilateral and equiangular figures having equal perimeters, and let $\triangle AB\Gamma$ have the more angles. I say that $\triangle AB\Gamma$ is the greater.

For let $H$, $\Theta$ be the centres of the circles circumscribed about the polygons $\triangle AB\Gamma$, $\triangle \Delta EZ$, and let $HB$, $HT\Gamma$, $\Theta E$, $\Theta Z$ be joined. And from $H$, $\Theta$ let $HK$, $\Theta \Lambda$ be drawn perpendicular to $T\Gamma$, $EZ$. Then since $\triangle AB\Gamma$ has more angles than $\triangle \Delta EZ$, $T\Gamma$ is contained more often in the perimeter of $\triangle AB\Gamma$ than $EZ$ is contained in the perimeter of $\triangle \Delta EZ$. And the perimeters are equal. Therefore $EZ > T\Gamma$; and therefore $\Theta \Lambda > BK$. Let $\Lambda M$ be placed equal to $BK$, and let $\Theta M$ be joined. Then since the straight line $EZ$ bears to the perimeter of the polygon $\triangle \Delta EZ$ the same ratio as the angle $\Theta OZ$ bears to four right angles—owing to the fact that the polygon is equilateral and the sides cut off equal arcs from the circumscribing circle, while the angles at the centre are in the same ratio as the arcs on which they stand [Eucl. iii. 26]—and the perimeter of $\triangle \Delta EZ$, that is the perimeter of $\triangle AB\Gamma$, bears to $T\Gamma$ the same ratio as four right angles bears to the angle $BHT\Gamma$, therefore $ex \ aequali$ [Eucl. v. 17]

$$EZ : T\Gamma = \text{angle } \Theta OZ : \text{angle } BHT\Gamma,$$

i.e.,

$$\Theta \Lambda : \Lambda M = \text{angle } \Theta OZ : \text{angle } BHT\Gamma,$$

i.e.,

$$\Theta \Lambda : \Lambda M = \text{angle } \Theta \Lambda : \text{angle } BHK.$$

And since $\Theta \Lambda > \text{angle } \Theta \Lambda : \text{angle } M\Theta \Lambda$, as we shall prove in due course,$^b$

and $\Theta M$ as radius cutting $\Theta E$ and $\Theta \Lambda$ produced, as in Eucl. Optic. 8 (v. vol. i. pp. 502-505); the proposition is equivalent to the formula $\tan a : \tan \beta > a : \beta$ if $\frac{1}{4}\pi > a > \beta$. 

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πρὸς ΛΜ ἢ ὑπὸ ΘΩΛ πρὸς τὴν ὑπὸ ΒΗΚ, ἢ ὑπὸ ΘΩΛ πρὸς τὴν ὑπὸ ΒΗΚ μεῖζονα λόγον ἔχει ἢπερ πρὸς τὴν ὑπὸ ΜΘΛ. μεῖζων ἄρα ἢ ὑπὸ ΜΘΛ γωνία τῆς ὑπὸ ΒΗΚ. ἐστὶν δὲ καὶ ὁρθή ἢ πρὸς τῷ Δ ὁρθῇ τῇ πρὸς τῷ Κ ἵση. λουτὴ ἄρα ἢ ὑπὸ ΗΒΚ μεῖζων ἐσται τῆς ὑπὸ ΘΗΛ. κείσθω τῇ ὑπὸ ΗΒΚ ἱση ἢ ὑπὸ ΛΜΝ καὶ διήκθω ἢ ΛΘ ἐπὶ τὸ Ν. καὶ ἐπεὶ ἱση ἐστίν ἢ ὑπὸ ΗΒΚ τῇ ὑπὸ ΝΜΛ, ἀλλὰ καὶ ἢ πρὸς τῷ Δ ἱση τῇ πρὸς τῷ Κ, ἐστὶ δὲ καὶ ἢ ΒΚ πλευρὰ τῇ ΜΛ ἱση, ἱση ἄρα καὶ ἢ ΗΚ τῇ ΝΛ. μεῖζων ἄρα ἢ ΗΚ τῆς ΘΛ. μεῖζον ἄρα καὶ τὸ ὑπὸ τῆς ΑΒΓ περιμέτρου καὶ τῆς ΗΚ τοῦ ὑπὸ τῆς ΔΕΖ περιμέτρου καὶ τῆς ΘΛ. καὶ ἐστιν τὸ μὲν ὑπὸ τῆς ΑΒΓ περιμέτρου καὶ τῆς ΗΚ διπλάσιον τοῦ ΑΒΓ πολυγώνου, ἐπεἰ καὶ τὸ ὑπὸ τῆς ΒΓ καὶ τῆς ΗΚ διπλάσιον ἐστὶν τοῦ ΗΒΓ τριγώνου. τὸ δὲ ὑπὸ τῆς ΔΕΖ περιμέτρου καὶ τῆς ΘΛ διπλάσιον τοῦ ΔΕΖ πολυγώνου. μεῖζον ἄρα τὸ ΑΒΓ πολύγωνον τοῦ ΔΕΖ.

Ibid. 358. 12-360. 3

Τούτου δεδειγμένου λέγω, ὅτι ἐὰν κύκλος εὐθυγράμμως ἰσοπλεύρῳ τε καὶ ἰσογωνίῳ ἰσοπερίμετρος ἢ, μεῖζων ἐσται ὁ κύκλος.

Κύκλος γὰρ ὁ ΑΒΓ ἰσοπλεύρῳ τε καὶ ἰσογωνίῳ τῷ ΔΕΖ εὐθυγράμμῳ ἰσοπερίμετρος ἐστω: λέγω, ὅτι μεῖζων ἐστὶν ὁ κύκλος.

Εἰλήφθω τοῦ μὲν ΑΒΓ κύκλου κέντρον τὸ Η, τοῦ δὲ περὶ τὸ ΔΕΖ πολύγωνον περιγραφομένου τὸ Θ, καὶ περιγεγράφθω περὶ τὸν ΑΒΓ κύκλον 390
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and \( \angle E\Lambda : \angle M = \angle E\Omega \Lambda : \angle BHK \),

\[ \therefore \angle E\Omega \Lambda : \angle BHK > \angle E\Omega \Lambda : M\Omega \Lambda. \]

\[ \therefore \angle M\Omega \Lambda > \angle BHK. \]

Now the right angle at \( \Lambda \) is equal to the right angle at \( K \). Therefore the remaining angle \( HBK \) is greater than the angle \( \Omega M\Lambda \) [by Eucl. i. 32]. Let the angle \( \Lambda MN \) be placed equal to the angle \( HBK \), and let \( \Lambda \Omega \) be produced to \( N \). Then since the angle \( HBK \) is equal to the angle \( NMA \), and the angle at \( \Lambda \) is equal to the angle at \( K \), while \( BK \) is equal to the side \( MA \), therefore \( HK \) is equal to \( NA \) [Eucl. i. 26]. Therefore \( HK > \Omega \Lambda \). Therefore the rectangle contained by the perimeter of \( AB\Gamma \) and \( HK \) is greater than the rectangle contained by the perimeter of \( DEZ \) and \( \Theta \Lambda \). But the rectangle contained by the perimeter of \( AB\Gamma \) and \( HK \) is double of the polygon \( AB\Gamma \), since the rectangle contained by \( B\Gamma \) and \( HK \) is double of the triangle \( HB\Gamma \) [Eucl. i. 41]; and the rectangle contained by the perimeter of \( DEZ \) and \( \Theta \Lambda \) is double of the polygon \( DEZ \). Therefore the polygon \( AB\Gamma \) is greater than \( DEZ \).

**Ibid.** 358. 12–360. 3

This having been proved, I say that *if a circle have an equal perimeter with an equilateral and equiangular rectilineal figure, the circle shall be the greater.*

For let \( AB\Gamma \) be a circle having an equal perimeter with the equilateral and equiangular rectilineal figure \( DEZ \). I say that the circle is the greater.

Let \( H \) be the centre of the circle \( AB\Gamma \), \( \Theta \) the centre of the circle circumscribing the polygon \( DEZ \); and let there be circumscribed about the circle \( AB\Gamma \) the
πολύγωνον ὁμοίον τῷ ΔΕΖ τῷ ΚΛΜ, καὶ ἐπεζεύχθω ἡ HB, καὶ κάθετος ἀπὸ τοῦ Θ ἐπὶ τὴν EZ ἤχθω ἡ ΘΝ, καὶ ἐπεζεύχθωσαν αἱ ΗΛ, ΘΕ.

ἐπεὶ οὖν ἡ τοῦ ΚΛΜ πολυγώνου περίμετρος μείζων ἐστὶν τῆς τοῦ ΑΒΓ κύκλου περιμέτρου ὡς ἐν τῷ Περὶ σφαιρας καὶ κυλίνδρου Ἀρχιμήδης, ἵνα δὲ ἡ τοῦ ΑΒΓ κύκλου περίμετρος τῆς τοῦ ΔΕΖ πολυγώνου περιμέτρων, μείζων ἀρα καὶ ἡ τοῦ ΚΛΜ πολυγώνου περίμετρος τῆς τοῦ ΔΕΖ πολυγώνου περιμέτρου. καὶ εἰσὶν ὁμοία τὰ πολύγωνα μείζων ἀρα ἡ ΒΛ τῆς ΝΕ. καὶ ὁμοίον τῷ ΗΛΒ τρίγωνον τῷ ΘΕΝ τριγώνω, ἐπεὶ καὶ τὰ 392
polygon $K\Lambda M$ similar to $\Delta EZ$, and let $HB$ be joined, and from $\Theta$ let $\Theta N$ be drawn perpendicular to $EZ$, and let $HA, OE$ be joined. Then since the perimeter of the polygon $K\Lambda M$ is greater than the perimeter of the circle $AB\Gamma$, as Archimedes proves in his work *On the Sphere and Cylinder*, while the perimeter of the circle $AB\Gamma$ is equal to the perimeter of the polygon $\Delta EZ$, therefore the perimeter of the polygon $K\Lambda M$ is greater than the perimeter of the polygon $\Delta EZ$. And the polygons are similar; therefore $BA > NE$. And the triangle $HAB$ is similar to the triangle $\Theta EN$, 

* Prop. 1, *v. supra*, pp. 48-49.
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ολα πολύγωνα. μείζων ἃρα καὶ ἡ HB τῆς ΘΝ. καὶ ἐστὶν ἵση ἡ τοῦ ΑΒΓ κύκλου περιμέτρος τῇ τοῦ ΔΕΖ πολυγώνου περιμέτρῳ. τὸ ἄρα ὑπὸ τῆς περιμέτρου τοῦ ΑΒΓ κύκλου καὶ τῆς HB μείζων ἐστὶν τοῦ ὑπὸ τῆς περιμέτρου τοῦ ΔΕΖ πολυγώνου καὶ τῆς ΘΝ. ἀλλὰ τὸ μὲν ὑπὸ τῆς περιμέτρου τοῦ ΑΒΓ κύκλου καὶ τῆς HB διπλάσιον τοῦ ΑΒΓ κύκλου Ἀρχιμήδης ἐδείξευ, οὗ καὶ τὴν δείξειν ἔξις ἐκθησάμεθα. τὸ δὲ ὑπὸ τῆς περιμέτρου τοῦ ΔΕΖ πολυγώνου καὶ τῆς ΘΝ διπλάσιον τοῦ ΔΕΖ πολυγώνου. μείζων ἃρα ὁ ΑΒΓ κύκλος τοῦ ΔΕΖ πολυγώνου, ὃπερ ἔδει δείξαι.

Ibid. 364. 12-14

Δέγω δὴ καὶ ὅτι τῶν ἱσοπεριμέτρων εὐθυγράμμων σχημάτων καὶ τὰς πλευρὰς ἱσοπληθεῖσα ἔχοντων τὸ μέγιστον ἱσόπλευρον τῇ ἑστὶν καὶ ἱσογώνιον.

Ibid. 374. 12-14

Δέγω δὴ ὅτι καὶ ἡ σφαῖρα μείζων ἐστὶν πάντων τῶν ἵσην ἐπιφάνειαν ἔχοντων στερεῶν σχημάτων, προσχρησάμενος τοῖς ὑπὸ Ἀρχιμήδους δεδειγμένους ἐν τῷ Περὶ σφαίρας καὶ κυλίνδρου.

(d) DIVISION OF ZODIAC CIRCLE INTO 360 PARTS:

HYPSCICLES

Hypsicl. Anaph., ed. Manitius 5. 25-31

Τοῦ τῶν Ζωδίων κύκλων εἰς τῇ περιφερείας ἴσαι


* The proofs of these two last propositions are worked out by similar methods.

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since the whole polygons are similar; therefore $HB > \Theta N$. And the perimeter of the circle $AB\Gamma$ is equal to the perimeter of the polygon $\Delta EZ$. Therefore the rectangle contained by the perimeter of the circle $AB\Gamma$ and $HB$ is greater than the rectangle contained by the perimeter of the polygon $\Delta EZ$ and $\Theta N$. But the rectangle contained by the perimeter of the circle $AB\Gamma$ and $HB$ is double of the circle $AB\Gamma$ as was proved by Archimedes, whose proof we shall set out next; and the rectangle contained by the perimeter of the polygon $\Delta EZ$ and $\Theta N$ is double of the polygon $\Delta EZ$ [by Eucl. i. 41]. Therefore the circle $AB\Gamma$ is greater than the polygon $\Delta EZ$, which was to be proved.

Ibid. 364. 12-14

Now I say that, of all rectilineal figures having an equal number of sides and equal perimeter, the greatest is that which is equilateral and equiangular.

Ibid. 374. 12-14

Now I say that, of all solid figures having an equal surface, the sphere is the greatest; and I shall use the theorems proved by Archimedes in his work On the Sphere and Cylinder.

(d) Division of Zodiac Circle into 360 Parts: Hypsicles

Hypsicles, On Risings, ed. Manitius * 5. 25-31

The circumference of the zodiac circle having been

* Des Hypsikles Schrift Anaphorikos nach Überlieferung und Inhalt kritisch behandelt, in Programm des Gymnasiums zum Heiligen Kreuz in Dresden (Dresden, 1888), 1. Abt. 395
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διηρημένου, ἐκάστῃ τῶν περιφερειῶν μοίρα τοπικὴ καλεῖσθω. ὅμως δὴ καὶ τοῦ χρόνου, ἐν ὦ ὁ ξωδιάκος ἀφ' ὦ ἐτυχε σημεῖον ἐπὶ τὸ αὐτὸ σημεῖον παραγίγνεται, εἰς τξ χρόνους ίσους διηρημένου, ἐκαστὸς τῶν χρόνων μοίρα χρονικὴ καλεῖσθω.

(e) Handbooks

(i.) Cleomedes

Cleom. De motu circ. ii. 6, ed. Ziegler 218. 8–224. 8

Τουσίτων δὲ τῶν περὶ τὴν ἐκλεψιν τῆς σελήνης εἶναι ἐπιδεδευμένων δοκεῖ ἐναντιοῦσθαι τῷ λόγῳ τῷ κατασκευάζοντι ἐκλείπειν τὴν σελήνην εἰς τὴν σκιὰν ἐμπίπτουσαν τῆς γῆς τὰ λεγόμενα κατὰ τὰς παραδόξους τῶν ἐκλεψεων. φασὶ γὰρ τινες, ὅτι γίνεται σελήνης ἐκλεψις καὶ ἄμφοτέρων τῶν φωτῶν ὑπὲρ τὸν ὀρίζοντα θεωρομένων. τούτων δὲ δῆλον ποιεί, διὸτι μὴ ἐκλείπει η σελήνη τῇ σκιᾷ

* Hypsicles, who flourished in the second half of the second century B.C., is the author of the continuation of Euclid’s Elements known as Book xiv. Diophantus attributed to him a definition of a polygonal number which is equivalent to the formula \[ \frac{1}{2} n(2 + (n - 1)(a - 2)) \] for the nth a-gonal number.

The passage here cited is the earliest known reference in Greek to the division of the ecliptic into 360 degrees. This number appears to have been adopted by the Greeks from the Chaldaeans, among whom the zodiac was divided into twelve signs and each sign into thirty parts according to one system, sixty according to another (v. Tannery, Mémoires scientifiques, ii. pp. 256–268). The Chaldaeans do not, however, seem to have applied this system to other circles; Hipparchus is believed to have been the first to divide the 396
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divided into 360 equal arcs, let each of the arcs be called a degree in space, and similarly, if the time in which the zodiac circle returns to any position it has left be divided into 360 equal times, let each of the times be called a degree in time.a

(e) Handbooks

(i.) Cleomedes b

Cleomedes, On the Circular Motion of the Heavenly Bodies ii. 6, cd. Ziegler 218. 8–224. 8

Although these facts have been proved with regard to the eclipse of the moon, the argument that the moon suffers eclipse by falling into the shadow of the earth seems to be refuted by the stories told about paradoxical eclipses. For some say that an eclipse of the moon may take place even when both luminaries are seen above the horizon. This should make it clear that the moon does not suffer eclipse by circle in general into 360 degrees. The problem which Hypsicles sets himself in his book is: Given the ratio between the length of the longest day and the length of the shortest day at any given place, to find how many time-degrees it takes any given sign to rise. A number of arithmetical lemmas are proved.

b Cleomedes is known only as the author of the two books Κυκλικὴ θεωρίᾳ μετεώρων. This work is almost wholly based on Posidonius. He must therefore have lived after Posidonius and presumably before Ptolemy, as he appears to know nothing of Ptolemy’s works. In default of better evidence, he is generally assigned to the middle of the first century B.C.

The passage explaining the measurement of the earth by Eratosthenes has already been cited (supra, pp. 266-273). This is the only other passage calling for notice.
GREEK MATHEMATICS

τής γῆς περιπτώσεως, ἀλλ’ ἔτερον τρόπον.... οἱ παλαιότεροι τῶν μαθηματικῶν οὕτως ἐπεχείρουν λύειν τὴν ἀπορίαν ταύτην. ἔφασαν γὰρ, ὅτι.... οἱ δ’ ἐπὶ γῆς ἐστῶτες οὐδὲν ἂν κωλύσωσι ὁδὸν ἀμφοτέρους αὐτοὺς ἐπὶ τοὺς κυρτῶμασι τῆς γῆς ἐστῶτες.... τουαύτην μὲν οὖν οἱ παλαιότεροι τῶν μαθηματικῶν τὴν τῆς προσαγομένης ἀπορίας λύσιν ἐποιήσαντο. μὴ ποτε δ’ οὐχ ἵππος εἰσών ἐνηγεμένοι. ἐφ’ ὑποῖας μὲν γὰρ ἢ ὁπίς ἡμῶν γενομένη δύνατ’ ἂν τούτῳ παθεῖν, κωνοεῖδος τοῦ ὀρίζοντος γενομένου πολὺ ἀπὸ τῆς γῆς ἐκ τοῦ ἀέρα ἡμῶν ἐξαρθέντων, ἐπὶ δὲ τῆς γῆς ἐστῶτων οὕδαμως. εἰ γὰρ καὶ κύρτωμα ἔστων, ἐφ’ οὖν βεβήκαμεν, ἀφανίζεται ἡμῶν ἢ ὁπίς ἕπο τοῦ μεγέθους τῆς γῆς.... ἀλλὰ πρῶτον μὲν ἀπ’ αντιτέου λέγοντας, ὅτι τέπλασται ὁ λόγος οὗτος ὑπὸ τῶν ἀπορίαν βουλομένων ἐμποίησα τοῖς περὶ ταύτα καταγινομένοις τῶν ἀστρολόγων καὶ φιλοσόφων.... πολλῶν δὲ καὶ παντοδαπῶν περὶ τὸν ἀέρα παθῶν συνίστασθαι περικότως οὐκ ἂν εἰσὶν ἀδύνατον, ἢ δὴ καταδεδυκτός τοῦ ἱλίου καὶ ὑπὸ τὸν ὀρίζοντα ὄντος φαντασίαν ἡμῖν προσπεσεῖν ὥς μηδέποτε καταδεδυκτός αὐτοῦ, ἢ νέφους παχυτέρον πρὸς τῇ δύσει ὄντος καὶ λαμπρονομένου ὑπὸ τῶν ἡλιακῶν ἀκτίνων καὶ ἱλίου ἡμῖν φαντασίαν ἀποπέμποντος ἢ ἀνθηλίου γενομένου. καὶ γὰρ

* i.e., the horizon would form the base of a cone whose vertex would be at the eye of the observer. He could thus look down on both the sun and moon as along the generators of a cone, even though they were diametrically opposite each other.

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falling into the shadow of the earth, but in some other way. . . . The more ancient of the mathematicians tried to explain this difficulty after this fashion. They said that persons standing on the earth would not be prevented from seeing them both because they would be standing on the convexities of the earth. . . . Such is the solution of the alleged difficulty given by the more ancient of the mathematicians. But its soundness may be doubted. For, if our eye were situated on a height, the phenomenon in question might take place, the horizon becoming conical if we were raised sufficiently far above the earth, but it could in no wise happen if we stood on the earth. For though there might be some convexity where we stood, our sight itself becomes evanescent owing to the size of the earth. . . . The fundamental objection must first be made, that this story has been invented by certain persons wishing to make difficulty for the astronomers and philosophers who busy themselves with such matters. . . . Nevertheless, as the conditions which naturally arise in the air are many and various, it would not be impossible that, when the sun has just set and is below the horizon, we should receive the impression of its not having yet set, if there were a cloud of considerable density at the place of setting and if it were illumined by the solar rays and transmitted to us an image of the sun, or if there were a mock sun. For such images are often

b Lit. "anthelion," defined in the Oxford English Dictionary as "a luminous ring or nimbus seen (chiefly in alpine or polar regions) surrounding the shadow of the observer's head projected on a cloud or fog bank opposite the sun." The explanation here tentatively put forward by Cleomedes is, of course, the true one.
τοιαύτα πολλὰ φαντάζεται ἐν τῷ ἀέρι, καὶ μάλιστα περὶ τῶν Πόντων.

(ii.) Theon of Smyrna

Ptol. Math. Syn. x. 1, ed. Heiberg i. pars ii. 296. 14-16

Ἐν μὲν γὰρ ταῖς παρὰ Θέωνος τοῦ μαθηματικοῦ δοθείσαις ἦμῖν εὑρομεν ἀναγεγραμμένην τήρησιν τῷ ἰε' έτει Ἄδριανοῦ.

Theon Smyr., ed. Hiller 1. 1-2. 2

Ὅτι μὲν οὐχ οἶδο τοὺς μαθηματικῶς λεγομένων παρὰ Πλάτωνι μή καὶ αὐτὸν ἡσκη-μένων ἐν τῇ θεωρίᾳ ταύτῃ, πάς ἂν που διμολο-γήσεις· ώσ δὲ οὐδὲ τὰ ἀλλα ἀνωφελῆς οὐδὲ ἀνόνητος ἢ περὶ ταύτα ἐμπειρία, διὰ πολλῶν αὐτῶς ἐμφανίζειν έοικε. τὸ μὲν οὖν συμπάσης γεωμετρίας καὶ συμπάσης μουσικῆς καὶ ἀστρονομίας ἐμπειρον γενόμενον τοῖς Πλάτωνοις συγγράμμασιν ἐντυγ-χάνειν μακαριστὸν μὲν εἰ τῷ γένοιτο, οὐ μὴν εὑπορον οὐδὲ βάδιον ἀλλὰ πάνυ πολλοῦ τοῦ ἐκ παίδων πόνου δεόμενον. ὡςτε δὲ τοὺς διημαρτη-κότας τοῦ ἐν τοῖς μαθήμασιν ἀσκηθήναι, ὄρεγο-μένους δὲ τῆς γνώσεως τῶν συγγραμμάτων αὐτοῦ μὴ παντάπασιν ὃν ποθούσι διαμαρτεῖν, κεφαλαίωδη καὶ σύντομον ποιησόμεθα τῶν ἀναγκαίων καὶ ὃν δεῖ μάλιστα τοῖς ἐντευξομένοις Πλάτωνι μαθηματικῶν θεωρημάτων παράδοσιν, ἀριθμητικῶν τε καὶ μουσικῶν καὶ γεωμετρικῶν τῶν τε κατὰ στερεομετρίαν καὶ ἀστρονομίαν, ὃν χωρίς οὐχ
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seen in the air, and especially in the neighbourhood of Pontus.

(ii.) Theon of Smyrna

Ptolemy, Syntaxis x. 1, ed. Heiberg i. pars ii. 296. 14-16

For in the account given to us by Theon the mathematician we find recorded an observation made in the sixteenth year of Hadrian.⁹

Theon of Smyrna, ed. Hiller 1. 1-2. 2

Everyone would agree that he could not understand the mathematical arguments used by Plato unless he were practised in this science; and that the study of these matters is neither unintelligent nor unprofitable in other respects Plato himself would seem to make plain in many ways. One who had become skilled in all geometry and all music and astronomy would be reckoned most happy on making acquaintance with the writings of Plato, but this cannot be come by easily or readily, for it calls for a very great deal of application from youth upwards. In order that those who have failed to become practised in these studies, but aim at a knowledge of his writings, should not wholly fail in their desires, I shall make a summary and concise sketch of the mathematical theorems which are specially necessary for readers of Plato, covering not only arithmetic and music and geometry, but also their application to stereometry and astronomy, for

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οἷόν τε εἰναι φησι τυχεῖν τοῦ ἀριστοῦ βίου, διὰ πολλῶν πάνυ δηλώσας ὡς οὐ χρή τῶν μαθημάτων ἁμελεῖν.

* By way of example, Theon proceeds to relate Plato's reply to the craftsmen about the doubling of the cube (v.
without these studies, as he says, it is not possible to attain the best life, and in many ways he makes clear that mathematics should not be ignored.\(^a\)

vol. i. p. 257), and also the *Epinomis*. Theon's work, which has often been cited in these volumes, is a curious hotchpotch, containing little of real value to the study of Plato and no original work.
XXI. TRIGONOMETRY
XXI. TRIGONOMETRY

1. HIPPARCHUS AND MENELAUS


The beginnings of Greek trigonometry may be found in the science of *sphaeric*, the geometry of the sphere, for which v. vol. i. p. 5 n. b. It reached its culminating point in the *Sphaerica* of Theodosius.

Trigonometry in the strict sense was founded, so far as we know, by Hipparchus, the great astronomer, who was born at Nicaea in Bithymia and is recorded by Ptolemy to have made observations between 161 and 126 B.C., the most important of them at Rhodes. His greatest achievement was the discovery of the precession of the equinoxes, and he made a calculation of the mean lunar month which differs by less than a second from the present accepted figure. Unfortunately the only work of his which has survived is his early *Commentary on the Phenomena of Eudoxus and Aratus*. It

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1. HIPPARCUS AND MENELAUS

Theon of Alexandria, Commentary on Ptolemy's Syntaxis
i. 10, ed. Rome, Studi e Testi, Ixxii. (1936), 451. 4-5

An investigation of the chords in a circle is made by
Hipparchus in twelve books and again by Menelaus
in six.\(^a\)

Heron, Metrics i. 22, ed. H. Schöne (Heron iii.) 58. 13-20

Let \(AB\Gamma\Delta\varepsilon ZH\Theta K\) be an equilateral and equi-
angular enneagon,\(^b\) whose sides are each equal to 10.

To find its area. Let there be described about it a
circle with centre \(\Lambda\), and let \(E\Lambda\) be joined and pro-
is clear, however, from the passage here cited, that he drew
up, as did Ptolemy, a table of chords, or, as we should say,
a table of sines; and Heron may have used this table (v. the
next passage cited and the accompanying note).

Menelaus, who also drew up a table of chords, is recorded
by Ptolemy to have made an observation in the first year of
Trajan's reign (A.D. 98). He has already been encountered
(vol. i, pp. 348-349 and n. c) as the discoverer of a curve
called "paradoxical." His trigonometrical work Sphaerica
has fortunately been preserved, but only in Arabic, which
will prevent citation here. A proof of the famous theorem
in spherical trigonometry bearing his name can, however, be
given in the Greek of Ptolemy (infra, pp. 458-463); and a
summary from the Arabic is provided by Heath, H.G.M. ii.
262-273.

\(^a\) i.e., a figure of nine sides.
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ζεύχων ἦ ΕΔ καὶ ἐκβεβλήσθω ἐπὶ τὸ Μ, καὶ ἐπεζεύχω ἦ ΜΖ. τὸ ἄρα ΕΖΜ τρίγωνον δοθέν ἑστιν τοῦ ἐνναγώνου. δέδεικται δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθείων, ὅτι ἦ ΖΕ τῆς ΕΜ τρίτον μέρος ἑστιν ὡς ἐγγιστα.

2. PTOLEMY

(a) General

Suidas, s.v. Πτολεμαῖος

Πτολεμαῖος, ὁ Κλαύδιος χρηματίσας, Ἀλεξανδρεὺς, φιλόσοφος, γεγονός ἐπὶ τῶν χρόνων Μάρκου τοῦ βασιλέως. οὐτος ἐγραφεῖ Μηχανικὰ βιβλία ἃ, Περὶ φάσεων καὶ ἐπισημασίων ἀστερῶν ἀπλανῶν βιβλία β, Ἄπλωσιν ἐπιφανελας σφαίρας, Κανόνα πρόχειρον, τὸν Μέγαν ἀστρονόμον ἦτοι Σύνταξιν καὶ ἄλλα.

* A similar passage (i. 24, ed. H. Schöne 62. 11-20) asserts that the ratio of the side of a regular hendecagon to the diameter of the circumscribing circle is approximately \( \frac{7}{25} \); and of this assertion also it is said δέδεικται δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθείων. These are presumably the works of Hipparchus and Menelaus, though this opinion is controverted by A. Rome, “Premiers essais de trigonométtrie rectiligne chez les Grecs” in L’Antiquité classique, t. 2 (1933), pp. 177-192. The assertions are equivalent to saying that sin 20° is approximately 0.333... and sin 16° 21' 49" is approximately 0.28.

* Nothing else is certainly known of the life of Ptolemy except, as can be gleaned from his own works, that he made observations between A.D. 125 and 141 (or perhaps 151). Arabian traditions add details on which too much reliance should not be placed. Suidas’s statement that he was born 408
duced to M, and let MZ be joined. Then the triangle EZM is given in the enneagon. But it has been proved in the works on chords in a circle that ZE : EM is approximately $\frac{1}{3}$.

2. PTOLEMY

(a) General

Suidas, s.v. Ptolemaeus

Ptolemy, called Claudius, an Alexandrian, a philosopher, born in the time of the Emperor Marcus. He wrote Mechanics, three books, On the Phases and Seasons of the Fixed Stars, two books, Explanation of the Surface of a Sphere, A Ready Reckoner, the Great Astronomy or Syntaxis; and others.

In the time of the Emperor Marcus [Aurelius] is not accurate as Marcus reigned from A.D. 161 to 180.

Ptolemy’s Mechanics has not survived in any form; but the books On Balancings and On the Elements mentioned by Simplicius may have been contained in it. The lesser astronomical works of Ptolemy published in the second volume of Heiberg’s edition of Ptolemy include, in Greek, Φάσεις ἀπλανῶν ἀστέρων και συναγωγή ἑπιστημασίων and Προχέρων κανόνων διάταξις καὶ ψηφοφορία, which can be identified with two titles in Suidas’s notice. In the same edition is the Planisphaerium, a Latin translation from the Arabic, which can be identified with the "Ἀπλωσὶς ἑπιφάνειας οὐάλρας of Suidas; it is an explanation of the stereographic system of projection by which points on the heavenly sphere are represented on the equatorial plane by projection from a pole—circles are projected into circles, as Ptolemy notes, except great circles through the poles, which are projected into straight lines.

Allied to this, but not mentioned by Suidas, is Ptolemy’s Analemma, which explains how points on the heavenly sphere can be represented as points on a plane by means of orthogonal projection upon three planes mutually at right angles—
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Simpl. in De caelo iv. 4 (Aristot. 311 b 1), ed. Heiberg 710. 14-19

Πτολεμαίος δὲ ὁ μαθηματικὸς ἐν τῷ Περὶ ῥοτῶν τὴν ἐναντίαν ἑχων τῷ 'Αριστοτέλει δόξαν πειράται κατασκευάζειν καὶ αὐτός, ὅτι ἐν τῇ ἑαυτῶν χώρα ὁὔτε τὸ ὕδωρ ὁὔτε ὁ ἄηρ ἔχει βάρος. καὶ ὅτι μὲν τὸ ὕδωρ ὁὐκ ἔχει, δείκνυσιν ἐκ τοῦ τούς καταδύοντας μὴ αἰσθάνεσθαι βάρους τοῦ ἐπικεκμένου ὕδατος, καὶ τοι τινὰς εἰς πολὺ καταδύοντας βάθος.

Ibid. i. 2, 269 a 9, ed. Heiberg 20. 11

Πτολεμαίος ἐν τῷ Περὶ τῶν στοιχείων βιβλίω καὶ ἐν τοῖς Ὅπτικοις ....

Ibid. i. 1, 268 a 6, ed. Heiberg 9. 21-27

'Ὁ δὲ θαυμαστὸς Πτολεμαῖος ἐν τῷ Περὶ διαστάσεως μονοβίβλω ἀπέδειξεν, ὅτι οὐκ εἰσὶ

the meridian, the horizontal and the "prime vertical." Only fragments of the Greek and a Latin version from the Arabic have survived; they are given in Heiberg's second volume.

Among the "other works" mentioned by Suidas are presumably the Inscription in Canobus (a record of some of Ptolemy's discoveries), which exists in Greek; the Ἐπιστήμη τῶν πλανωμένων, of which the first book is extant in Greek and the second in Arabic; and the Optics and the book On Dimension mentioned by Simplicius.

But Ptolemy's fame rests most securely on his Great Astronomy or Syntaxis as it is called by Suidas. Ptolemy himself called this majestic astronomical work in thirteen books the Μαθηματικὴ σύνταξις or Mathematical Collection. In due course the lesser astronomical works came to be called the Μικρὸς ἀστρονομοῦμενος (τόπος), the Little Astronomy, and the Syntaxis came to be called the Μεγάλη σύνταξις, or Great Collection. Later still the Arabs, combining their article Αl
Ptolemy the mathematician in his work *On Balancings* maintains an opinion contrary to that of Aristotle and tries to show that in its own place neither water nor air has weight. And he proves that water has not weight from the fact that divers do not feel the weight of the water above them, even though some of them dive into considerable depths.

*Ibid.* i. 2, 269 a 9, ed. Heiberg 20. 11

Ptolemy in his book *On the Elements* and in his *Optics* . . .


The gifted Ptolemy in his book *On Dimension* showed that there are not more than three dimen-

with the Greek superlative μέγιστος, called it Al-majisti; corrupted into Almagest, this has since been the favourite name for the work.

The *Syntaxis* was the subject of commentaries by Pappus and Theon of Alexandria. The trigonometry in it appears to have been abstracted from earlier treatises, but condensed and arranged more systematically.

Ptolemy’s attempt to prove the parallel-postulate has already been noticed (*supra*, pp. 372-383).

*a* Ptolemy’s *Optics* exists in an Arabic version, which was translated into Latin in the twelfth century by Admiral Eugenius Siculus (*v. G. Govi, L’ottica di Claudio Tolomeo di Eugenio Ammiraglio di Sicilia*); but of the five books the first and the end of the last are missing. Until the Arabic text was discovered, Ptolemy’s *Optics* was commonly supposed to be identical with the Latin work known as *De Speculis*; but this is now thought to be a translation of Heron’s *Catoptrica* by William of Moerbeke (*v. infra, p. 502 n. a*).
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πλείονες τῶν τριῶν διαστάσεις, ἐκ τοῦ δεῖν μὲν τὰς διαστάσεις ὑποσμένας εἶναι, τὰς δὲ ὑποσμένας
diastásēs kat' eúdeías lambránesthai kathétous,
τρεῖς δὲ μόνας πρὸς ὅρθας ἀλλήλαις eúdeías dúnatón
eînai lábein, δύο μὲν καθ’ ἂς τὸ ἐπίπεδον ὀρίζεται,
τρίτην δὲ τὴν τὸ βάθος μετροῦσαν ὡστε, εἰ τις
eîn metà tîn trikhî diástasou állh, āmetros ἀν
eîn pantelōs kai àðristos.

(b) TABLE OF SINES

(i.) Introduction

Ptol. Math. Syn. i. 10, ed. Heiberg i. pars i. 31. 7–32. 9

i’. Περὶ τῆς πηλικότητος τῶν ἐν τῷ κύκλῳ
eúdeiōn

Πρὸς μὲν οὖν τὴν ἐξ ἐτοίμου χρήσων κανονικήν
tiwa metà taúta ēkthesin pousoúmeba tîs pēli-
kóttíous autów tîn μὲν περίμετρον eis tîz
μήμαta dieλóntes, paratìthentes δὲ tâs ὑπὸ tâs kath'
ëmìmoirîon parauξhíseis tîn perifereiôn ὑπo-
teiwoménavas eúdeías, toutêstîi pîsōn eîsîn
mēmá-
tîn òs tîs diámêtrou dià tî ex autów tîn ēpi-
logiümôn fanhsoúmevon en toîs àrîthmouì eûkhrhstou
eîs pê tîmêma tî diarhēmînês. proîteron ðe deîxhmen,
pôs àn òs eîn màliosta di òlîgoun kai tîn autów
thenâmâtîn eûmèthdýnton kai tâxhîan tîn ēpi-
bolhîan tîn prôs tâs pēlikóttíass autów pousiómeba,
dîpnow µη µónon ëktebhímena tà megebh tîw
eúdeiōn ëxîmewn anepistástow, ìllà kai diá tîs
ëk tîw grafhiôn mëthodikês autów sustásēwos
tîn èlêgchon ëx eûxerouì metaxhîrûmêba. kathôlou
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sions; for dimensions must be determinate, and determinate dimensions are along perpendicular straight lines, and it is not possible to find more than three straight lines at right angles one to another, two of them determining a plane and the third measuring depth; therefore, if any other were added after the third dimension, it would be completely unmeasurable and undetermined.

(b) Table of Sines

(i.) Introduction

Ptolemy, Syntaxis i. 10, ed. Heiberg i. pars i. 31. 7–32. 9

10. On the lengths of the chords in a circle

With a view to obtaining a table ready for immediate use, we shall next set out the lengths of these [chords in a circle], dividing the perimeter into 360 segments and by the side of the arcs placing the chords subtending them for every increase of half a degree, that is, stating how many parts they are of the diameter, which it is convenient for the numerical calculations to divide into 120 segments. But first we shall show how to establish a systematic and rapid method of calculating the lengths of the chords by means of the uniform use of the smallest possible number of propositions, so that we may not only have the sizes of the chords set out correctly, but may obtain a convenient proof of the method of calculating them based on geometrical considera-
GREEK MATHEMATICS

By the phrase "by means of lines" (διὰ τῶν γραμμῶν), in the text "On the Phaenomena of Eudoxus and Aratus," it is implied that certain facts about the risings of stars were used in rigorous calculations, not merely graphical methods. In other words, he was familiar with the main formulae of spherical trigonometry.

But the phrase "by means of lines" (διὰ τῶν γραμμῶν) means more than a graphical method; the phrase indicates a rigorous proof, as will be seen when the argument proceeds, and the use of διὰ τῶν γραμμῶν infra, p. 434. It may be inferred, therefore, that when Hipparchus proved "by means of lines" (On the Phaenomena of Eudoxus and Aratus, ed. Manitius 148-150) certain facts about the risings of stars, he used rigorous, and not merely graphical calculations; in other words, he was familiar with the main formulae of spherical trigonometry.

(iii.) sin 18° and sin 36°

Ibid. 32. 10-35. 16

"Εστω δὴ πρῶτον ἡμικύκλιον τὸ ABΓ ἐπὶ διαμέτρου τῆς ΔΔΓ περὶ κέντρου τὸ Δ, καὶ ἀπὸ τού Δ τῆς ΔΓ πρὸς ὀρθὰς γωνίας ἡχθω ἡ ΔΒ, καὶ τετμήσθω δίχα η ΔΓ κατὰ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΕΒ, καὶ κείσθω αὐτῇ ἵση ἡ EZ, καὶ ἐπεζεύχθω ἡ ΖΒ. λέγω, ὅτι ἡ μὲν ΖΔ δικαγώνου ἐστὶν πλευρά, ἡ δὲ ΒΖ πενταγώνου.

*a By διὰ τῆς ἐκ τῶν γραμμῶν μεθοδικῆς συντάσεως Ptolemy meant more than a graphical method; the phrase indicates a rigorous proof by means of geometrical considerations, as will be seen when the argument proceeds; cf. the use of διὰ τῶν γραμμῶν infra, p. 434. It may be inferred, therefore, that when Hipparchus proved "by means of lines" (διὰ τῶν γραμμῶν, On the Phaenomena of Eudoxus and Aratus, ed. Manitius 148-150) certain facts about the risings of stars, he used rigorous, and not merely graphical calculations; in other words, he was familiar with the main formulae of spherical trigonometry.

*b i.e., ΖΔ is equal to the side of a regular decagon, and ΒΖ to the side of a regular pentagon, inscribed in the circle ABΓ.
In general we shall use the sexagesimal system for the numerical calculations owing to the inconvenience of having fractional parts, especially in multiplications and divisions, and we shall aim at a continually closer approximation, in such a manner that the difference from the correct figure shall be inappreciable and imperceptible.

(ii.) \( \sin 18^\circ \) and \( \sin 36^\circ \)

*Ibid.* 32. 10–35. 16

First, let \( AB\Gamma \) be a semicircle on the diameter \( A\Delta\Gamma \) and with centre \( \Delta \), and from \( \Delta \) let \( \Delta B \) be drawn perpendicular to \( A\Gamma \), and let \( \Delta\Gamma \) be bisected at \( E \), and let \( EB \) be joined, and let \( EZ \) be placed equal to it, and let \( ZB \) be joined. I say that \( Z\Delta \) is the side of a decagon, and \( BZ \) of a pentagon.\(^6\)
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"Επει γὰρ εὐθεῖα γραμμὴ ἡ ΔΓ τέτμηται δίχα κατὰ τὸ Ε, καὶ πρόσκευται τις αὐτῇ εὐθεία ἡ ΔΖ, τὸ ὑπὸ τῶν ΓΖ καὶ ΖΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΔ τετραγώνων ἱσον ἐστὶν τῷ ἀπὸ τῆς ΕΖ τετραγώνω, τοινέστων τῷ ἀπὸ τῆς ΒΕ, ἐπεὶ ίση ἐστὶν ἡ ΕΒ τῇ ΖΕ. ἀλλὰ τῷ ἀπὸ τῆς ΕΒ τετραγώνω ἱσα ἐστὶ τὰ ἀπὸ τῶν ΕΔ καὶ ΔΒ τετράγωνα: τὸ ἀρὰ ὑπὸ τῶν ΓΖ καὶ ΖΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΔΕ τετραγώνου ἱσον ἐστὶν τοῖς ἀπὸ τῶν ΕΔ, ΔΒ τετραγώνοις. καὶ κοινῷ ἀφαιρεθάντος τοῦ ἀπὸ τῆς ΕΔ τετραγώνου λοιπὸν τὸ ὑπὸ τῶν ΓΖ καὶ ΖΔ ἱσον ἐστὶν τῷ ἀπὸ τῆς ΔΒ, τοινέστων τῷ ἀπὸ τῆς ΔΓ. ἡ ΖΓ ἀρὰ ἀκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Δ. ἐπεὶ οὖν ἡ τοῦ ἐξαγώνου καὶ ἡ τοῦ δεκαγώνου πλευρά τῶν ἐις τὸν αὐτὸν κύκλον ἐγγραφομένων ἐπὶ τῆς αὐτῆς εὐθείας ἀκρον καὶ μέσον λόγον τέμνονται, ἡ δὲ ΓΔ ἐκ τοῦ κέντρου ὀσα τὴν τοῦ ἐξαγώνου περιέχει πλευράν, ἡ ΔΖ ἀρὰ ἐστὶν ἵση τῇ τοῦ δεκαγώνου πλευρᾷ. ὅμοιως δὲ, ἐπεὶ ἡ τοῦ πενταγώνου πλευρᾶ δύναται τὴν τε τοῦ ἐξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων, τοῦ δὲ ΒΔΖ ὀρθογώνιον τὸ ἀπὸ τῆς ΒΖ τετράγωνων ἱσον ἐστὶν τῷ τῇ τῆς ΒΔ, ἦτις ἐστὶν ἐξαγώνου πλευρά, καὶ τῷ ἀπὸ τῆς ΔΖ, ἦτις ἐστὶν δεκαγώνου πλευρά, ἡ ΒΖ ἀρὰ ἴση ἐστὶν τῇ τοῦ πενταγώνου πλευρᾷ.

Ἐπεὶ οὖν, ὃς ἐφην, ὑποτιθέμεθα τὴν τοῦ κύκλου διάμετρον τμήματων ρξ, γίνεται διὰ τὰ προκείμενα ἡ μὲν ΔΕ ἡμίσεια οὕσα τῆς ἐκ τοῦ κέντρου

a Following the usual practice, I shall denote segments (τμήματα) of the diameter by ρξ, sixtieth parts of a τμήμα by ρξ 416
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For since the straight line $\Delta \Gamma$ is bisected at $E$, and the straight line $\Delta Z$ is added to it,

$$\Gamma Z \cdot Z\Delta + \Delta E^2 = E\Gamma^2$$  \[Eucl. ii. 6\]

$$= B\Gamma^2;$$

since $EB = ZE$.

But $E\Delta^2 + \Delta B^2 = EB^2$;  \[Eucl. i. 47\]

therefore $\Gamma Z \cdot Z\Delta + \Delta E^2 = E\Delta^2 + \Delta B^2$.

When the common term $E\Delta^2$ is taken away, the remainder $\Gamma Z \cdot Z\Delta = \Delta B^2$

i.e., $= \Delta \Gamma^2$;

therefore $Z\Gamma$ is divided in extreme and mean ratio at $\Delta$ [Eucl. vi., Def. 3]. Therefore, since the side of the hexagon and the side of the decagon inscribed in the same circle when placed in one straight line are cut in extreme and mean ratio [Eucl. xiii. 9], and $\Gamma \Delta$, being a radius, is equal to the side of the hexagon [Eucl. iv. 15, coroll.], therefore $\Delta Z$ is equal to the side of the decagon. Similarly, since the square on the side of the pentagon is equal to the rectangle contained by the side of the hexagon and the side of the decagon inscribed in the same circle [Eucl. xiii. 10], and in the right-angled triangle $B\Delta Z$ the square on $BZ$ is equal [Eucl. i. 47] to the sum of the squares on $B\Delta$, which is a side of the hexagon, and $\Delta Z$, which is a side of the decagon, therefore $BZ$ is equal to the side of the pentagon.

Then since, as I said, we made the diameter $a$ consist of $120^\circ$, by what has been stated $\Delta E$, being half the numeral with a single accent, and second-sixtieths by the numeral with two accents. As the circular associations of the system tend to be forgotten, and it is used as a general system of enumeration, the same notation will be used for the squares of parts.

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τμημάτων δ' καὶ τὸ ἀπ' αὐτῆς ᾿Η, ἡ δὲ ΒΔ ἐκ τοῦ κέντρου οὖσα τμημάτων ᾿ξ καὶ τὸ ἀπὸ αὐτῆς ᾿γχ, τὸ δὲ ἀπὸ τῆς ΕΒ, τούτεστιν τὸ ἀπὸ τῆς ΕΖ, τῶν ἐπὶ τὸ αὐτὸ ᾿δφ. μήκει ἀρα ἔσται ἡ ΕΖ τμημάτων ᾿ξ ᾿ς ᾿δ ᾿ν ἐγγιστα, καὶ λοιπῇ ἡ ΔΖ τῶν αὐτῶν ᾿ξ ᾿δ ᾿ν ἔ. ἡ ἀρα τοῦ δεκαγώνου πλευρά, ὑποτείνουσα δὲ περιφέρειαν τουούτων ᾿ξ, οἰων ἔστιν ὁ κύκλος ᾿δ, τουούτων ἐσται ᾿ξ ᾿δ ᾿ν, οἰων ἡ διάμετρος ᾿ρκ. πάλιν ἐπεὶ ἡ μὲν ΔΖ τμημάτων ἐστὶ ᾿ξ ᾿δ ᾿ν, τὸ δὲ ἀπὸ αὐτῆς ᾿ατοε ᾿δ ᾿ν ἔ, ἐστὶ δὲ καὶ τὸ ἀπὸ τῆς ΔΒ τῶν αὐτῶν ᾿γχ, ἀ συντεθέντα ποὺει τὸ ἀπὸ τῆς ΒΖ τετράγωνον ᾿δγκοε ᾿δ ᾿ν, μήκει ἀρα ἔσται ἡ ΒΖ τμημάτων ὁ ᾿λβ ᾿γ ἐγγιστα. καὶ ἡ τοῦ πεντα- γώνου ἄρα πλευρά, ὑποτείνουσα δὲ μοῖρας ᾿οβ, οἰων ἔστιν ὁ κύκλος ᾿δ, τουούτων ἐστὶν ὁ ᾿λβ ᾿γ, οἰων ἡ διάμετρος ᾿ρκ.

Φανερὸν δὲ αὐτόθεν, ὅτι καὶ ἡ τοῦ ἔξαγώνου πλευρά, ὑποτείνουσα δὲ μοῖρας ᾿ξ, καὶ ἵππη οὖσα τῇ ἐκ τοῦ κέντρου, τμημάτων ἐστὶν ᾿ξ. ὅμως δὲ, ἐπεὶ ἡ μὲν του τετραγώνου πλευρά, ὑποτείνουσα δὲ μοῖρας ᾿ξ, δυνάμει διπλασία ἐστὶν τῆς ἐκ τοῦ κέντρου, ἡ δὲ τοῦ τριγώνου πλευρά, ὑποτείνουσα δὲ μοῖρας ᾿ρκ, δυνάμει τῆς αὐτῆς ἐστὶν τριπλασίων, τὸ δὲ ἀπὸ τῆς ἐκ τοῦ κέντρου τμημάτων ἐστὶν ᾿γχ, συναχήσεται τὸ μὲν ἀπὸ τῆς του τετραγώνου πλευράς ᾿ξ, τὸ δὲ ἀπὸ τῆς του τριγώνου ᾿Μ ᾿ω. ὡστε καὶ μήκει ἡ μὲν τὰς ᾿ξ μοῖρας ὑποτείνουσα εὐθεία τουούτων ἐσται ὃδ ἄν ἓ ἐγγιστα, οἰων ἡ

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of the radius, consists of \(30^\circ\) and its square of \(900^2\), and \(B\Delta\), being the radius, consists of \(60^\circ\) and its square of \(3600^2\), while \(EB^2\), that is \(EZ^2\), consists of \(4500^2\); therefore \(EZ\) is approximately \(67^\circ 4' 55''\), and the remainder \(\Delta Z\) is \(37^\circ 4' 55''\). Therefore the side of the decagon, subtending an arc of \(36^\circ\) (the whole circle consisting of \(360^\circ\)), is \(37^\circ 4' 55''\) (the diameter being \(120^\circ\)). Again, since \(\Delta Z\) is \(37^\circ 4' 55''\), its square is \(1375^2 4' 15''\), and the square on \(\Delta B\) is \(3600^2\), which added together make the square on \(BZ\) \(4975^2 4' 15''\), so that \(BZ\) is approximately \(70^\circ 32' 3''\). And therefore the side of the pentagon, subtending \(72^\circ\) (the circle consisting of \(360^\circ\)), is \(70^\circ 32' 3''\) (the diameter being \(120^\circ\)).

Hence it is clear that the side of the hexagon, subtending \(60^\circ\) and being equal to the radius, is \(60^\circ\). Similarly, since the square on the side of the square,\(^b\) subtending \(90^\circ\), is double of the square on the radius, and the square on the side of the triangle, subtending \(120^\circ\), is three times the square on the radius, while the square on the radius is \(3600^2\), the square on the side of the square is \(7200^2\) and the square on the side of the triangle is \(10800^2\). Therefore the chord subtending \(90^\circ\) is approximately \(84^\circ 51' 10''\) (the diameter

\(^a\) Theon's proof that \(\sqrt{4500}\) is approximately \(67^\circ 4' 55''\) has already been given (vol. i. pp. 56-61).
\(^b\) This is, of course, the square itself; the Greek phrase is not so difficult. We could translate, "the second power of the side of the square," but the notion of powers was outside the ken of the Greek mathematician.
Let \( AB \) be a chord of a circle subtending an angle \( a \) at the centre \( O \), and let \( AKA' \) be drawn perpendicular to \( OB \) so as to meet \( OB \) in \( K \) and the circle again in \( A' \). Then

\[
\sin a (= \sin AB) = \frac{AK}{AO} = \frac{1}{2} \frac{AA'}{AO}.
\]

And \( AA' \) is the chord subtended by double of the arc \( AB \), while Ptolemy expresses the lengths of chords as so many 120th parts of the diameter; therefore \( \sin a \) is half the chord subtended by an angle \( 2a \) at the centre, which is conveniently abbreviated by Heath to \( \frac{1}{2} \text{(crd. } 2a) \), or, as we may alternatively express the relationship, \( \sin AB \) is “half the chord subtended by
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consisting of 120°), and the chord subtending 120° is 103° 55' 23".a

(iii.) \( \sin^2 \theta + \cos^2 \theta = 1 \)

Ibid. 35. 17-36. 12

The lengths of these chords have thus been obtained immediately and by themselves,b and it will be thence clear that, among the given straight lines, the lengths are immediately given of the chords subtending the remaining arcs in the semicircle, by reason of the fact that the sum of the squares on these chords is equal to the square on the diameter; for example, since the chord subtending 36° was shown to be 37° 4' 55" and its square 1375° 4' 15", while the square on the diameter is 14400°, therefore the square on the chord subtending the remaining 144° in the semicircle is double of the arc AB," which is the Ptolemaic form; as Ptolemy means by this expression precisely what we mean by \( \sin AB \), I shall interpolate the trigonometrical notation in the translation wherever it occurs. It follows that \( \cos a = \sin(90 - a) = \frac{1}{2} \text{crd.} (180° - 2a) \), or, as Ptolemy says, "half the chord subtended by the remaining angle in the semicircle." Tan \( a \) and the other trigonometrical ratios were not used by the Greeks.

In the passage to which this note is appended Ptolemy proves that

side of decagon (= crd. 36° = 2 sin 18°) = 37° 4' 55'',
side of pentagon (= crd. 72° = 2 sin 36°) = 70° 32' 3'',
side of hexagon (= crd. 60° = 2 sin 30°) = 60°,
side of square (= crd. 90° = 2 sin 45°) = 84° 51' 10'',
side of equilateral triangle (= crd. 120° = 2 sin 60°) = 103° 55' 23''.

b i.e., not deduced from other known chords.
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Μ.γκδ νε με, αυτή δὲ μήκει τῶν αυτῶν πρὶν ξ λδ ἐγγιστα, καὶ ἐπὶ τῶν ἄλλων ὀμοίως.

"Ον δὲ τρόπον ἀπὸ τούτων καὶ αἱ λοιπαὶ τῶν κατὰ μέρος δοθῆσονται, δείξομεν ἑφεξῆς προεκθέ-μενοι λημμάτιοι εὐχρηστον πάνυ πρὸς τὴν παροῦσαν πραγματείαν.

(iv.) "Ptolemy's Theorem"

Ibid. 36. 13-37. 18

"Εστω γὰρ κύκλος ἐγγεγραμμένον ἔξων τετρά-πλευρον τυχόν τὸ ΑΒΓΔ, καὶ ἐπεξεύθωσαν αἱ ΑΓ καὶ ΒΔ. δεικτέον, ὅτι τὸ ὑπὸ τῶν ΑΓ καὶ ΒΔ περιεχόμενον ὀρθογώνιον ἵσον ἐστὶ συναμφοτερος τῷ τε ὑπὸ τῶν ΑΒ, ΔΓ καὶ τῷ ὑπὸ τῶν ΑΔ, ΒΓ.

Κείσθω γὰρ τῇ ὑπὸ τῶν ΔΒΓ γωνία ἵση ἢ ὑπὸ ΑΒΕ. ἐὰν οὖν κοινὸν προσθῶμεν τὴν ὑπὸ ΕΒΔ,

* i.e., crd. $144^\circ = 2 \sin 72^\circ = 114^\circ 7' 37''$. If the given chord subtends an angle $2\theta$ at the centre, the chord subtended by the remaining arc in the semicircle subtends an angle $(180 - 2\theta)$, and the theorem asserts that

$$(\text{crd. } 2\theta)^2 + (\text{crd. } 180 - 2\theta)^2 = (\text{diameter})^2,$$

or

$$\sin^2 \theta + \cos^2 \theta = 1.$$
13024° 55' 45" and the chord itself is approximately 114° 7' 37", and similarly for the other chords.¹

We shall explain in due course the manner in which the remaining chords obtained by subdivision can be calculated from these, setting out by way of preface this little lemma which is exceedingly useful for the business in hand.

(iv.) "Ptolemy's Theorem"

Ibid. 36. 13–37. 18

Let \( AB\Delta \) be any quadrilateral inscribed in a circle, and let \( A\Gamma \) and \( B\Delta \) be joined. It is required to prove that the rectangle contained by \( A\Gamma \) and \( B\Delta \) is equal to the sum of the rectangles contained by \( AB, \Delta \Gamma \) and \( A\Delta, B\Gamma \).

For let the angle \( ABE \) be placed equal to the angle \( \Delta B\Gamma \). Then if we add the angle \( EBA\Delta \) to both, the
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έσται καὶ ἡ ὑπὸ ΑΒΔ γωνία ἴση τῇ ὑπὸ ΕΒΓ. ἔστιν δὲ καὶ ἡ ὑπὸ ΒΔΑ τῇ ὑπὸ ΒΓΕ ἴση· τὸ γὰρ αὐτὸ τμῆμα ὑποτείνουσιν· ἵσογώνιον ἀρα ἔστιν τὸ ΑΒΔ τρίγωνον τῷ ΒΓΕ τριγώνῳ. ὥστε καὶ ἀνάλογον ἔστιν, ὅτι ἡ ΒΓ πρὸς τὴν ΓΕ, οὔτως ἡ ΒΔ πρὸς τὴν ΔΑ· τὸ ἀρα ὑπὸ ΒΓ, ΑΔ ἴσον ἔστιν τῷ ὑπὸ ΒΔ, ΓΕ. πάλιν ἐπεὶ ἵση ἔστιν ἡ ὑπὸ ΑΒΕ γωνία τῇ ὑπὸ ΔΒΓ γωνία, ἔστιν δὲ καὶ ἡ ὑπὸ ΒΑΕ ἴση τῇ ὑπὸ ΒΔΓ, ἵσογώνιον ἀρα ἔστιν τὸ ΑΒΕ τρίγωνον τῷ ΒΓΔ τριγώνῳ· ἀνάλογον ἀρα ἔστιν, ὅτι ἡ ΒΑ πρὸς ΑΕ, ἡ ΒΔ πρὸς ΔΓ· τὸ ἀρα ὑπὸ ΒΑ, ΔΓ ἴσον ἔστιν τῷ ὑπὸ ΒΔ, ΑΕ. ἐδείχθη δὲ καὶ τὸ ὑπὸ ΒΓ, ΑΔ ἴσον τῷ ὑπὸ ΒΔ, ΓΕ· καὶ ὀλον ἀρα τὸ ὑπὸ ΑΓ, ΒΔ ἴσον ἔστιν συναμφότεροι τῷ τε ὑπὸ ΑΒ, ΔΓ καὶ τῷ ὑπέρ ΑΔ, ΒΓ· ὅπερ ἐδει δείξαι.

(v.) \( \sin(\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi \)

Ibid. 37. 19–39. 3

Τούτου προεκτεθέντος ἔστω ἡμικύκλιον τὸ ΑΒΓΔ ἐπὶ διαμέτρου τῆς ΑΔ, καὶ ἀπὸ τοῦ Α δύο δυνάσθησαν αἱ ΑΒ, ΑΓ, καὶ ἔστω ἑκάτερα αὐτῶν δοθεῖσα τῷ μεγέθει, οἷον ἡ διάμετρος δοθεῖσα ρκ, καὶ ἐπεζεύχθω ἡ ΒΓ. λέγω, ὅτι καὶ αὐτὴ δέδοται.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΔ, ΓΔ· δεδομέναι ἀρα εἰσὶν δηλονότι καὶ αὐταί διὰ τὸ λείπειν ἑκεῖνων εἰς τὸ ἡμικύκλιον. ἐπεὶ οὖν ἐν κύκλῳ τετράπλευρον ἔστιν τὸ ΑΒΓΔ, τὸ ἀρα ὑπὸ ΑΒ, ΓΔ μετὰ τοῦ 424
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angle $ABA = \text{the angle } EBG$. But the angle $BDA = \text{the angle } BGE$ [Eucl. iii. 21], for they subtend the same segment; therefore the triangle $ABD$ is equiangular with the triangle $BGE$.

\[ \therefore \quad BG : GE = BD : DA ; \quad \text{[Eucl. vi. 4]} \]

\[ \therefore \quad BG \cdot AD = BD \cdot GE. \quad \text{[Eucl. vi. 6]} \]

Again, since the angle $ABE$ is equal to the angle $BGE$, while the angle $BAE$ is equal to the angle $BGE$ [Eucl. iii. 21], therefore the triangle $ABE$ is equiangular with the triangle $BGD$;

\[ \therefore \quad BA : AE = BD : DG ; \quad \text{[Eucl. vi. 4]} \]

\[ \therefore \quad BA \cdot DG = BD \cdot AE. \quad \text{[Eucl. vi. 6]} \]

But it was shown that

\[ BG \cdot AD = BD \cdot GE ; \]

and \[ \therefore \quad AG \cdot BD = AB \cdot DG + AD \cdot BG ; \quad \text{[Eucl. ii. 1]} \]

which was to be proved.

\[(v.) \sin (\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi \]

Ibid. 37. 19–39. 3

This having first been proved, let $ABGDA$ be a semicircle having $AD$ for its diameter, and from $A$ let the two [chords] $AB$, $AG$ be drawn, and let each of them be given in length, in terms of the $120^\circ$ in the diameter, and let $BG$ be joined. I say that this also is given.

For let $BD$, $DG$ be joined; then clearly these also are given because they are the chords subtending the remainder of the semicircle. Then since $ABGDA$ is a quadrilateral in a circle,
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υπὸ τῶν ΑΔ, ΒΓ ἵσουν ἐστὶν τῷ υπὸ ΑΓ, ΒΔ. καὶ ἐστὶν τὸ τε υπὸ τῶν ΑΓ, ΒΔ δοθέν καὶ τὸ υπὸ ΑΒ, ΓΔ. καὶ λοιπὸν ἀρα τὸ υπὸ ΑΔ, ΒΓ δοθέν ἐστὶν. καὶ ἐστὶν ἡ ΑΔ διάμετρος· δοθεὶσα ἀρα ἐστὶν καὶ ἡ ΒΓ εὐθεία.

Καὶ φανερὸν ἡμῖν γέγονεν, ὅτι, εὰν δοθῶσιν δύο περιφέρειαι καὶ αἱ υπὸ αὐτὰς εὐθείαι, δοθεὶσα ἐσται καὶ ἡ τὴν ὑπεροχὴν τῶν δύο περιφερείων ὑποτείνουσα εὐθεία. δῆλον δέ, ὅτι διὰ τούτου τοῦ θεωρήματος ἀλλας τε ὅλιγας εὐθείας ἐγγράψομεν ἀπὸ τῶν ἐν ταῖς καθ’ αὐτὰς δεδομένων ὑπεροχών καὶ δὴ καὶ τὴν υπὸ τὰς δῶδεκα μοίρας, ἐπειδὴ ἐπεξεργασθήσεται τὴν τε υπὸ τὰς ξ καὶ τὴν υπὸ τὰς οβ.
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\[ AB \cdot \Gamma\Delta + \Lambda\Delta \cdot \beta\Gamma = \alpha\Gamma \cdot \beta\Delta. \]

["Ptolemy's theorem"]

And \( \alpha\Gamma \cdot \beta\Delta \) is given, and also \( AB \cdot \Gamma\Delta \); therefore the remaining term \( \Lambda\Delta \cdot \beta\Gamma \) is also given. And \( \Lambda\Delta \) is the diameter; therefore the straight line \( \beta\Gamma \) is given.\(^*\)

And it has become clear to us that, if two arcs are given and the chords subtending them, the chord subtending the difference of the arcs will also be given. It is obvious that, by this theorem we can inscribe\(^b\) many other chords subtending the difference between given chords, and in particular we may obtain the chord subtending 12°, since we have that subtending 60° and that subtending 72°.

\(^*\) If \( \alpha\Gamma \) subtends an angle 2\( \theta \) and \( AB \) an angle 2\( \phi \) at the centre, the theorem asserts that

\[
\text{crd.} \left(\frac{\theta - \phi}{2}\right) \cdot \text{crd.} \left(\frac{180°}{2}\right) = \text{crd.} \left(\frac{\theta}{2}\right) \cdot \text{crd.} \left(\frac{180° - \phi}{2}\right) - \text{crd.} \left(\frac{\phi}{2}\right) \cdot \text{crd.} \left(\frac{180° - \theta}{2}\right)
\]

i.e.,

\[
\sin (\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi.
\]

\(^b\) Or "calculate," as we might almost translate \( \gamma\gamma\rho\alpha\gamma\omicron\omicron\nu \); \( cf. \ supra, \ p. \ 414 \ n. \ \alpha \) on \( \epsilon\kappa \ \tau\omicron\nu \ \gamma\rho\alpha\mu\mu\omicron\nu \).
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(vi.) \( \sin^2 \frac{1}{2} \theta = \frac{1}{2}(1 - \cos \theta) \)

Ibid. 39. 4–41. 3

Πάλιν προκείσθω δοθείσης τινὸς εὐθείας ἐν
κύκλῳ τῇ ὑπὸ τὸ ἤμισυ τῆς ὑποτεινομένης περι-
φερείας εὐθείαν εὑρεῖν. καὶ ἔστω ἡμικύκλιον τὸ
ΑΒΓ ἐπὶ διαμέτρου τῆς ΑΓ καὶ δοθείσα εὐθεία ἡ
ΓΒ, καὶ ἡ ΓΒ περιφέρεια δίχα τετμῆσθω κατὰ
tὸ Δ, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΑΔ, ΒΔ, ΔΓ,
cαὶ ἀπὸ τοῦ Δ ἐπὶ τὴν ΑΓ κάθετος ἡχθῳ ἡ ΔΖ.
λέγω, ὅτι ἡ ΖΓ ἡμίσειά ἐστι τῆς τῶν ΑΒ καὶ
ΑΓ ὑπεροχῆς.

Κεῖσθω γὰρ τῇ ΑΒ ἴση ἡ ΑΕ, καὶ ἐπεζεύχθω
ἡ ΔΕ. ἔπει ἴση ἐστὶν ἡ ΑΒ τῇ ΑΕ, κοινὴ δὲ ἡ
ΑΔ, δύο δὴ αἱ ΑΒ, ΑΔ δύο ταῖς ΑΕ, ΑΔ ἴσαι
εἰσὶν ἐκατέρα ἐκατέρα. καὶ γωνία ἡ ὑπὸ ΒΑΔ
γωνία τῇ ὑπὸ ΕΑΔ ἴση ἐστὶν. καὶ βάσις ἀρα ἡ
ΒΔ βάσει τῇ ΔΕ ἴση ἐστὶν. ἀλλὰ ἡ ΒΔ τῇ ΔΓ
ἴσῃ ἐστὶν καὶ ἡ ΔΓ ἀρα τῇ ΔΕ ἴση ἐστὶν. ἔπει
οὐν ὑσοσκέλοις ὅντος τριγώνου τοῦ ΔΕΓ ἀπὸ
tῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἢκται ἡ ΔΖ,
ἴση ἐστὶν ἡ ΕΖ τῇ ΖΓ. ἀλλʼ ἡ ΕΓ ὀλὴ ἡ ὑπερ-
οχῆ ἐστὶν τῶν ΑΒ καὶ ΑΓ εὐθείων· ἡ ἀρα ΖΓ ἡμί-
σειά ἐστὶν τῆς τῶν αὐτῶν ὑπεροχῆς. ὡστε, ἔπει
τῆς ὑπὸ τὴν ΒΓ περιφέρειαν εὐθείας ὑποκειμένης
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Again, given any chord in a circle, let it be required to find the chord subtending half the arc subtended by the given chord. Let $AB\Gamma$ be a semicircle upon the diameter $A\Gamma$ and let the chord $\Gamma B$ be given, and let the arc $\Gamma B$ be bisected at $\Delta$, and let $AB, A\Delta, B\Delta, \Delta\Gamma$ be joined, and from $\Delta$ let $\Delta Z$ be drawn perpendicular to $A\Gamma$. I say that $Z\Gamma$ is half of the difference between $AB$ and $A\Gamma$.

For let $AE$ be placed equal to $AB$, and let $\Delta E$ be joined. Since $AB = AE$ and $A\Delta$ is common, [in the triangles $AB\Delta, AE\Delta$] the two [sides] $AB, A\Delta$ are equal to $AE, A\Delta$ each to each; and the angle $BA\Delta$ is equal to the angle $EA\Delta$ [Eucl. iii. 27]; and therefore the base $B\Delta$ is equal to the base $\Delta E$ [Eucl. i. 4]. But $B\Delta = \Delta\Gamma$; and therefore $\Delta\Gamma = \Delta E$. Then since the triangle $\Delta E\Gamma$ is isosceles and $\Delta Z$ has been drawn from the vertex perpendicular to the base, $EZ = Z\Gamma$ [Eucl. i. 26]. But the whole $E\Gamma$ is the difference between the chords $AB$ and $A\Gamma$; therefore $Z\Gamma$ is half of the difference. Thus, since the chord subtending the arc $B\Gamma$ is given, the chord $AB$ subtending the remainder...
αὐτόθεν δέδοται καὶ ἡ λείπονσα εἰς τὸ ἡμικύκλιον ἡ ΑΒ, δοθῆσεται καὶ ἡ ΖΓ ἡμίσεια οὖσα τῆς τῶν ΑΓ καὶ ΑΒ ὑπεροχής. ἀλλ' ἔπει ἐν ὀρθογώνιοι τῷ ΑΓΔ καθέτου ἀχθείσης τῆς ΔΖ ἱσογώνιον γίνεται τὸ ΛΔΓ ὀρθογώνιον τῷ ΔΓΖ, καὶ ἠστιν, ὡς ἡ ΑΓ πρὸς ΓΔ, ἡ ΓΔ πρὸς ΓΖ, τὸ ἂρα ὑπὸ τῶν ΑΓ, ΓΖ περιεχόμενον ὀρθογώνιον ἱσον ἠστιν τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ. δοθέν δὲ τὸ ὑπὸ τῶν ΑΓ, ΓΖ. δοθέν ἄρα ἠστιν καὶ τὸ ἀπὸ τῆς ΓΔ τετρά-γωνῳ. ὡστε καὶ μῆκει ἡ ΓΔ εὐθεία δοθῆσεται τῆς ἡμίσειας ὑποτείνουσα τῆς ΒΓ περιφερείας.

Καὶ διὰ τούτον δὴ πάλιν τοῦ θεωρήματος ἀλλαὶ τε ληφθῆσονται πλεῖστα κατὰ τὰς ἡμίσειάς τῶν προεκτεθειμένων, καὶ δὴ καὶ ἀπὸ τῆς τὰς ἱβ μοίρας ὑποτείνουσας εὐθείας ἢ τε ὑπὸ τὰς ἱβ καὶ ἡ ὑπὸ τὰς γ καὶ ἡ ὑπὸ τὴν μίαν ἡμισαν καὶ ἡ ὑπὸ τὸ ἡμισαν τεταρτον τῆς μίας μοίρας. εὐρίσκομεν δε ἐκ τῶν ἐπιλογισμῶν τὴν μὲν ὑπὸ τὴν μίαν ἡμισαν μοίραν τοιούτων α Ἴδ τε ἐγγυτά, οἷον ἠστιν ἡ διάμετρος ρκ, τῆν δὲ ὑπὸ τὸ Ζ' δ' τῶν αὐτῶν 0 μζ ἡ.

(vii.) \[ \cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \]

_Ibid._ 41. 4-43. 5

Πάλιν ἠστιν κύκλος ὁ ΑΒΓΔ περὶ διάμετρον μὲν τὴν ΑΔ, κέντρον δὲ τὸ Ζ, καὶ ἀπὸ τοῦ Α ἀπειληφθώσας δύο περιφέρειαι δοθεῖσαι κατὰ τὸ ἐξής αἱ ΑΒ, ΒΓ, καὶ ἐπεζεύξωσαν αἱ ΑΒ, ΒΓ ὑπ' αὐτᾶς εὐθείαι καὶ αὐταὶ δεδομέναι. λέγω ὅτι, ἐὰν ἐπεζεύξωσαν τὴν ΑΓ, δοθῆσεται καὶ αὐτὴν.

* If ΒΓ subtends an angle 2θ at the centre the proposition asserts that

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of the semicircle is immediately given, and $Z\Gamma$ will also be given, being half of the difference between $A\Gamma$ and $AB$. But since the perpendicular $\Delta Z$ has been drawn in the right-angled triangle $A\Gamma\Delta$, the right-angled triangle $A\Delta\Gamma$ is equiangular with $\Delta\Gamma Z$ [Eucl. vi. 8], and

$$A\Gamma : \Gamma\Delta = \Gamma\Delta : \Gamma Z,$$

and therefore $A\Gamma \cdot \Gamma Z = \Gamma\Delta^2$.

But $A\Gamma \cdot \Gamma Z$ is given; therefore $\Gamma\Delta^2$ is also given. Therefore the chord $\Gamma\Delta$, subtending half of the arc $B\Gamma$, is also given.

And again by this theorem many other chords can be obtained as the halves of known chords, and in particular from the chord subtending $12^\circ$ can be obtained the chord subtending $6^\circ$ and that subtending $3^\circ$ and that subtending $1\frac{1}{2}$ and that subtending $1\frac{1}{2} + 1\frac{1}{4}(-\frac{3}{4})$. We shall find, when we come to make the calculation, that the chord subtending $1\frac{1}{2}$ is approximately $1^\circ 34' 15''$ (the diameter being $120^\circ$) and that subtending $\frac{3}{4}$ is $0^\circ 47' 8''$.

(vii.) $\cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$

_Ibid. 41. 4-43. 5_

Again, let $AB\Gamma\Delta$ be a circle about the diameter $A\Delta$ and with centre $Z$, and from $A$ let there be cut off in succession two given arcs $AB$, $B\Gamma$, and let there be joined $AB$, $B\Gamma$, which, being the chords subtending them, are also given. I say that, if we join $A\Gamma$, it also will be given.

$$(\text{crd. } \theta)^2 = \frac{1}{2}(\text{crd. } 180) \cdot ((\text{crd. } 180^\circ) - \text{crd. } 180^\circ - 2\theta)$$

i.e., $\sin^2 \frac{1}{2}\theta = \frac{1}{2}(1 - \cos \theta)$.

The symbol in the Greek for $O$ should be noted; $v$. vol. i. p. 47 n. a.
Διήχθω γὰρ διὰ τοῦ Β διάμετρος τοῦ κύκλου ἡ ΒΖΕ, καὶ ἐπεξεύχθωσαν αἱ ΒΔ, ΔΓ, ΓΕ, ΔΕ·
δὴ λοιπὸν ὅτι διὰ μὲν τὴν ΒΓ δοθήσεται καὶ ἡ ΓΕ, διὰ δὲ τὴν ΑΒ δοθήσεται ἡ τε ΒΔ καὶ
ἡ ΔΕ. καὶ διὰ τὰ αὐτὰ τοὺς ἐμπροσθεν, ἐπεὶ ἐν
κύκλῳ τετράπλευρόν ἐστιν τὸ ΒΓΔΕ, καὶ διηγοῦμέναι
eἰσιν αἱ ΒΔ, ΓΕ, τὸ ὑπὸ τῶν διηγούμενων περι-
εχόμενον ὀρθογώνιον ἵππον ἐστὶν συναμφοτέρως
τοῖς ὑπὸ τῶν ἀπεναντίων· ὡστε, ἐπεὶ δεδομένου
τοῦ ὑπὸ τῶν ΒΔ, ΓΕ δέδοται καὶ τὸ ὑπὸ τῶν ΒΓ,
ΔΕ, δέδοται ἀρα καὶ τὸ ὑπὸ ΒΕ, ΓΔ. δέδοται
de καὶ ἡ ΒΕ διάμετρος, καὶ λοιπὴ ἡ ΓΔ ἐσται
dεδομένη, καὶ διὰ τοῦτο καὶ ἡ λειτουρσα εἰς τὸ
ἡμικύκλιον ἡ ΓΑ· ὡστε, ἐὰν δοθῶσιν δύο περι-
φέρειαι καὶ αἱ ὑπ’ αὐτὰς εὐθεῖαι, δοθήσεται καὶ
ἡ συναμφοτέρας τὰς περιφερεῖας κατὰ σύνθεσιν
ὑποτείνουσα εὐθεία διὰ τούτου τοῦ θεωρήματος.

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* If AB subtends an angle $2\theta$ and BG an angle $2\phi$ at the centre, the theorem asserts that

$$(\text{crd. } 180^\circ) \cdot (\text{crd. } 180^\circ - 2\theta - 2\phi) = (\text{crd. } 180^\circ - 2\theta) \cdot (\text{crd. } 180^\circ - 2\phi),$$

i.e.,

$$\cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi.$$
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For through B let BZE, the diameter of the circle, be drawn, and let BΔ, ΔΓ, ΓE, ΔE be joined; it is then immediately obvious that, by reason of BΓ being given, ΓE is also given, and by reason of AB being given, both BΔ and ΔE are given. And by the same reasoning as before, since BΓΔE is a quadrilateral in a circle, and BΔ, ΓE are the diagonals, the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides. And so, since BΔ. ΓE is given, while BΓ. ΔE is also given, therefore BE. ΔΩ is given. But the diameter BE is given, and [therefore] the remaining term ΓΔ will be given, and therefore the chord ΓΔ subtending the remainder of the semicircle a; accordingly, if two arcs be given, and the chords subtending them, by this theorem the chord subtending the sum of the arcs will also be given.
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Φανερὸν δὲ, ὅτι συντιθέντες ἄεὶ μετὰ τῶν προ-
εκτεθειμένων πασῶν τὴν ὑπὸ ἀ λέντι ποτε στί πάντως ἐγγράφομεν ὅσοι δις γινόμεναι τρίτον μέρος ἔξουσιν, καὶ μόνοι ἐτε περιλειφθήσονται αἱ μεταξὺ τῶν ἁνὰ ἀ λέντι μοιρὰν διαστημάτων δύο καθ' ἐκαστὸν ἐσόμεναι, ἐπειδήπερ καθ' ἡμιμοιρίων ποιούμεθα τὴν ἐγγραφήν. ὡστε, ἐὰν τὴν ὑπὸ τὸ ἡμιμοῖρον εὐθείαν εὕρωμεν, αὐτὴ κατὰ τὴν σύνθεσιν καὶ τὴν ὑπεροχήν τὴν πρὸς τὰς τὰ διαστήματα περιεχούσας καὶ δεδομένας εὐθείας καὶ τὰς λοιπὰς τὰς μεταξὺ πάσας ἥμιν συνανα-
πληρώσει. ἐπεὶ δὲ δοθείσης τυνὸς εὐθείας ὡς τῆς ὑπὸ τὴν ἀ λέντι μοίραν ἢ τὸ τρίτον τῆς αὐτῆς περι-
φερείας ὑποτείνουσα διὰ τῶν γραμμῶν οὐ δίδοται πως· εἰ δὲ γε δυνατὸν ἦν, εἰχομεν ἃν αὐτόθεν καὶ τὴν ὑπὸ τὸ ἡμιμοῖρον· πρὸτερον μεθοδεύσομεν τὴν ὑπὸ τὴν ἀ λέντι μοίραν ἀπὸ τε τῆς ὑπὸ τὴν ἀ λέντι μοίραν καὶ τῆς ὑπὸ ἀ λέντι δ' ὑποτεθήκηνοι λημμάτιον, ὃ, κἂν μὴ πρὸς τὸ καθόλου δύνηται τὰς πηλι-
kότητας ὀρίζειν, ἐπὶ γε τῶν οὕτως ἐλαχίστων τὸ πρὸς τὰς ὑψημένας ἀπαράλλακτον δύνατ' ἀν συντηρεῖν.

(viii.) Method of Interpolation

Ibid. 43. 6–46. 20

Δέγω γὰρ, ὅτι ἐὰν ἐν κύκλῳ διαχθῶσιν ἄνισοι δύο εὐθείαι, ἢ μείζων πρὸς τὴν ἐλάσσονα ἐλάσσονα λόγον ἔχει ἦτερ ἢ ἐπὶ τῆς μείζονος εὐθείας περι-
φέρεια πρὸς τὴν ἐπὶ τῆς ἐλάσσονας.

Ἐπο τὸ κύκλος ὁ ΑΒΓΔ, καὶ διήθησαν ἐν αὐτῷ δύο εὐθείαι ἄνισοι ἐλάσσον μὲν ἡ ΑΒ,
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It is clear that, by continually putting next to all known chords a chord subtending $1\frac{1}{2}^\circ$ and calculating the chords joining them, we may compute in a simple manner all chords subtending multiples of $1\frac{1}{2}^\circ$, and there will still be left only those within the $1\frac{1}{2}^\circ$ intervals—two in each case, since we are making the diagram in half degrees. Therefore, if we find the chord subtending $\frac{1}{2}^\circ$, this will enable us to complete, by the method of addition and subtraction with respect to the chords bounding the intervals, both the given chords and all the remaining, intervening chords. But when any chord subtending, say, $1\frac{1}{2}^\circ$, is given, the chord subtending the third part of the same arc is not given by the [above] calculations—if it were, we should obtain immediately the chord subtending $\frac{1}{2}^\circ$; therefore we shall first give a method for finding the chord subtending $1^\circ$ from the chord subtending $1\frac{1}{2}^\circ$ and that subtending $\frac{3}{4}^\circ$, assuming a little lemma which, even though it cannot be used for calculating lengths in general, in the case of such small chords will enable us to make an approximation indistinguishable from the correct figure.

(viii.) Method of Interpolation

Ibid. 43. 6–46. 20

For I say that, if two unequal chords be drawn in a circle, the greater will bear to the less a less ratio than that which the arc on the greater chord bears to the arc on the lesser.

For let $\triangle ABC$ be a circle, and in it let there be drawn two unequal chords, of which $AB$ is the lesser
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μείζων δὲ ἢ ΒΓ. λέγω, ὅτι ἢ ΓΒ εὐθείᾳ πρὸς τὴν ΒΑ εὐθείαν ἐλάσσονα λόγον ἔχει ἂπερ ἢ ΒΓ περιφέρεια πρὸς τὴν ΒΑ περιφέρειαν.

Τετμήσθω γὰρ ἢ ὑπὸ ΑΒΓ γωνία δίχα ὑπὸ τῆς ΒΔ, καὶ ἐπεζεύχωσαν ἢ τε ἉΕΓ καὶ ἢ ΑΔ καὶ ἢ ΓΔ. καὶ ἔπει ἢ ὑπὸ ΑΒΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΒΕΔ εὐθείας, ἵση μὲν ἐστὶν ἢ ΓΔ εὐθεία τῆς ΑΔ, μείζων δὲ ἢ ΓΕ τῆς ΕΑ. ἦχθω δὴ ἀπὸ τοῦ Δ κάθετος ἐπὶ τὴν ΑΕΓ ἢ ΔΖ. ἔπει τοῖς μείζων ἐστὶν ἢ μὲν ΑΔ τῆς ΕΔ, ἢ δὲ ΕΔ τῆς ΔΖ, δὴ ἃρα κέντρῳ μὲν τῷ Δ, διαστήματι δὲ τῷ ΔΕ γραφόμενος κύκλος τὴν μὲν ΑΔ τεμεῖ, ὑπερπεσεῖται δὲ τὴν ΔΖ. γεγράφθω δὴ ὁ ΗΕΘ, καὶ ἐκβεβλήσθω ἢ ΔΖΘ. καὶ ἔπει ὁ μὲν ΔΕΘ τομεὺς μείζων ἐστὶν τοῦ ΔΕΖ τριγώνου, τὸ δὲ ΔΕΑ τρίγωνον μείζον τοῦ ΔΕΗ τομέως, τὸ ἃρα ΔΕΖ

* Lit. “let ΔΖΘ be produced.”
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and $BG$ the greater. I say that

$\Gamma B : BA < \text{arc } BG : \text{arc } BA$.

For let the angle $ABG$ be bisected by $B\Delta$, and let

$AE\Gamma$ and $A\Delta$ and $G\Delta$ be joined. Then since the angle $ABG$ is bisected by the chord $BE\Delta$, the chord $G\Delta=A\Delta$ [Eucl. iii. 26, 29], while $GE > EA$ [Eucl. vi. 3].

Now let $\Delta Z$ be drawn from $\Delta$ perpendicular to $AE\Gamma$. Then since $A\Delta > E\Delta$, and $E\Delta > \Delta Z$, the circle described with centre $\Delta$ and radius $E\Delta$ will cut $A\Delta$, and will fall beyond $\Delta Z$. Let [the arc] $HE\Theta$ be described, and let $\Delta Z$ be produced to $\Theta$. Then since

sector $\Delta E\Theta >$ triangle $\Delta EZ$, 
and triangle $\Delta EA >$ sector $\Delta EH$, 

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τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον ἐλάσσονα λόγον ἔχει ἕπερ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ. ἀλλ' ὡς μὲν τὸ ΔΕΖ τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον, οὔτως ἡ ΕΖ εὐθεία πρὸς τὴν ΕΑ, ὡς δὲ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ τομέα, οὔτως ἡ ἕπο ΖΔΕ γωνία πρὸς τὴν ἕπο ΕΔΑ. ἡ ἀρα ΖΕ εὐθεία πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἕπερ ἡ ἕπο ΖΔΕ γωνία πρὸς τὴν ἕπο ΕΔΑ. καὶ συνθέντει ἡ ΖΑ εὐθεία πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἕπερ ἡ ἕπο ΖΔΑ γωνία πρὸς τὴν ἕπο ΑΔΕ. καὶ τὰν ἡγομένων τὰ διπλάσια, ἡ ΓΑ εὐθεία πρὸς τὴν ΑΕ ἐλάσσονα λόγον ἔχει ἕπερ ἡ ἕπο ΓΔΑ γωνία πρὸς τὴν ἕπο ΕΔΑ. καὶ διελόντι ἡ ΓΕ εὐθεία πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἕπερ ἡ ἕπο ΓΔΕ γωνία πρὸς τὴν ἕπο ΕΔΑ. ἀλλ' ὡς μὲν ἡ ΓΕ εὐθεία πρὸς τὴν ΕΑ, οὔτως ἡ ΓΒ εὐθεία πρὸς τὴν ΒΑ, ὡς δὲ ἡ ἕπο ΓΔΒ γωνία πρὸς τὴν ἕπο ΒΔΑ, οὔτως ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ. ἡ ΓΒ ἀρα εὐθεία πρὸς τὴν ΒΑ ἐλάσσονα λόγον ἔχει ἕπερ ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ περι-

φέρειαν.

Τούτου δὴ οὖν ὑποκειμένου ἔστω κύκλος ὁ 

ἈΒΓ, καὶ διήκθωσαν ἐν αὐτῷ δύο εὐθείαι ἡ ὁ 

ἈΒ καὶ ἡ ΑΓ, ὑποκείσθω δὲ πρῶτον ἡ μὲν ἈΒ 

ὑποτείνουσα μιᾶς μοίρας ζ' δ', ἡ δὲ ΑΓ μοίραν 

ἀ. ἐπεὶ ἡ ΑΓ εὐθεία πρὸς τὴν ΒΑ εὐθεῖαν 

ἐλάσσονα λόγον ἔχει ἕπερ ἡ ΑΓ περιφέρεια πρὸς 

τὴν ΑΒ, ἡ δὲ ΑΓ περιφέρεια ἐπίτροπος ἐστὶν 

τὴς 

ἈΒ, ἡ ΓΑ ἀρα εὐθεία τῆς ΒΑ ἐλάσσων ἐστὶν ἡ 

ἐπίτροπος. ἀλλὰ ἡ ἈΒ εὐθεία ἐδείχθη τοιούτων 

ὁ μὲν ἡ, οὖν ἐστὶν ἡ διάμετρος ἰδ. ἡ ἀρα ΓΑ
TRIGONOMETRY

\[ \cdot \cdot \cdot \text{triangle } \Delta EZ : \text{triangle } \Delta EA < \text{sector } \Delta E\Theta : \]
\[ \text{sector } \Delta EH. \]

But \[ \text{triangle } \Delta EZ : \text{triangle } \Delta EA = EZ : EA, \]
\[ [\text{Eucl. vi. 1} \]
and

\[ \text{sector } \Delta E\Theta : \text{sector } \Delta EH = \text{angle } Z\Delta E : \text{angle } E\Delta A. \]
\[ \cdot \cdot \cdot ZE : EA < \text{angle } Z\Delta E : \text{angle } E\Delta A. \]
\[ \cdot \cdot \cdot \text{componendo, } ZA : EA < \text{angle } Z\Delta A : \text{angle } A\Delta E; \]
and, by doubling the antecedents,

\[ \Gamma A : AE < \text{angle } \Gamma \Delta A : \text{angle } E\Delta A; \]
and \textit{dirimendo,}

\[ \Gamma E : EA < \text{angle } \Gamma \Delta E : \text{angle } E\Delta A. \]

But \[ \Gamma E : EA = \Gamma B : BA, \]
\[ [\text{Eucl. vi. 3} \]
and

\[ \text{angle } \Gamma \Delta B : \text{angle } B\Delta A = \text{arc } \Gamma B : \text{arc } BA; \]
\[ [\text{Eucl. vi. 33} \]
\[ \cdot \cdot \cdot \Gamma B : BA < \text{arc } \Gamma B : \text{arc } BA. \]

On this basis, then, let \( \Delta B \Gamma \) be a circle, and in it let there be drawn the two chords \( AB \) and \( A\Gamma \), and let it first be supposed that \( AB \) subtends an angle of \( \frac{2}{3} \) and \( A\Gamma \) an angle of \( 1^\circ \). Then since

\[ A\Gamma : BA < \text{arc } A\Gamma : \text{arc } AB, \]
while \[ \text{arc } A\Gamma = \frac{4}{3} \cdot \text{arc } AB, \]
\[ \cdot \cdot \cdot \Gamma A : BA < \frac{4}{3}. \]

But the chord \( AB \) was shown to be \( 0^\circ 47' 8'' \) (the diameter being \( 120^\circ \)); therefore the chord \( \Gamma A \)

\[ a \text{ If the chords } \Gamma B, BA \text{ subtend angles } 2\theta, 2\phi \text{ at the centre, this is equivalent to the formula,} \]

\[ \frac{\sin \theta}{\sin \phi} < \frac{\theta}{\phi}, \]

where \( \theta < \phi < \frac{1}{2} \pi \).
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eυθεία ἐλάσσων ἐστίν τῶν αὐτῶν ἃ β ἐν ταύτα γὰρ ἐπίτριτά ἐστιν ἔγγιστα τῶν οὗ ἲ.

Πάλιν ἐπὶ τῆς αὐτῆς καταγραφῆς ἢ μὲν ἈΒ εὐθεία ὑποκείσθω ὑποτείνουσα μοῖραν ἃ, ἢ δὲ ΑΓ μοίραν ἁ Λ. κατὰ τὰ αὐτὰ δὴ, ἕπει ἢ ΑΓ περιφέρεια τῆς ΑΒ ἐστὶν ἡμιολία, ἢ ΓΑ ἀρα εὐθεία τῆς ΒΑ ἐλάσσων ἐστὶν ἡ ἡμιόλιος. ἀλλὰ τὴν ΑΓ ἀπεδείξαμεν τοιούτων οὖσαν ἃ λ ἢ, οἷον ἐστὶν ἡ διάμετρος ἰκ. ἢ ἀρα ΑΒ εὐθεία μείζων ἐστὶν τῶν αὐτῶν ἃ β ἐν τούτων γὰρ ἡμι-

ολία ἐστὶν τὰ προκείμενα ἃ λ ἢ. ὡστε, ἕπει τῶν αὐτῶν ἐδείχθη καὶ μείζων καὶ ἐλάσσων ἡ τὴν μίαν μοίραν ὑποτείνουσα εὐθεία, καὶ ταύτην δηλον- ότι ἔξομεν τοιούτων ἃ β ἐγγίστα, οἶον ἐστὶν ἡ διάμετρος ἰκ, καὶ διὰ τὰ προδεδειγμένα καὶ τὴν ὑπὸ τὸ ἡμιμοίριον, ἢτις εὑρίσκεται τῶν αὐτῶν

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\[ <1^\circ 2' 50'' \text{; for this is approximately four-thirds of } 0^\circ 47' 8''. \]

Again, with the same diagram, let the chord \( AB \)

be supposed to subtend an angle of \( 1^\circ \), and \( \Delta \Gamma \) an angle of \( 1\frac{1}{2}^\circ \). By the same reasoning,

since \( \arcsin \Delta \Gamma = \frac{3}{2} \cdot \arcsin AB \),

\( \therefore \) \( \Delta \Gamma : BA < \frac{3}{2} \).

But we have proved \( \Delta \Gamma \) to be \( 1^\circ 34' 15'' \) (the diameter being \( 120'' \)); therefore the chord \( AB > 1^\circ 2' 50'' \); for \( 1^\circ 34' 15'' \) is one-and-a-half times this number. Therefore, since the chord subtending an angle of \( 1^\circ \) has been shown to be both greater and less than [approximately] the same [length], manifestly we shall find it to have approximately this identical value \( 1^\circ 2' 50'' \) (the diameter being \( 120'' \)), and by what has been proved before we shall obtain the chord subtending \( \frac{1}{2}^\circ \), which is found to be approximately 441
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Ο Δα κε ἐγνιστα. καὶ συνανασπηρωθήσεται τὰ λοιπά, ὡς ἔφαμεν, διαστήματα ἐκ μὲν τῆς πρὸς τὴν μίαν ἡμισυ μοῖραν λόγου ἐνεκεν ὡς ἐπὶ τοῦ πρῶτου διαστήματος συνθέσεως τοῦ ἡμιμοίρου δεικνυμένης τῆς ὑπὸ τὰς β μοίρας, ἐκ δὲ τῆς ὑπεροχῆς τῆς πρὸς τὰς γ μοίρας καὶ τῆς ὑπὸ τὰς β ζ' διδομένης ὁμαίτως δὲ καὶ ἐπὶ τῶν λοιπῶν.

(ix.) The Table

Ibid. 46. 21–63. 46

Ἡ μὲν οὖν πραγματεία τῶν ἐν τῷ κύκλῳ εὐθειῶν οὕτως ἂν οἶμαι ράστα μεταχειρισθείη. Ἰνα δέ, ὡς ἐφην, ἐφ' ἐκάστης τῶν χρείαν εὖ ἐτοίμου τὰς πηλικότητας ἐξωμεν τῶν εὐθειῶν ἐκκεμένας, κανόνια ὑποτάξουμεν ἀνὰ στίγχους μὲ διὰ τὸ σύμμετρον, ὅν τὰ μὲν πρῶτα μέρη περιέξει τὰς πηλικότητας τῶν περιφερεῖων καθ' ἡμιμοίριον παρηξημένας, τὰ δὲ δεύτερα τὰς τῶν παρακειμένων ταῖς περιφερείαις εὐθειῶν πηλικότητας ὡς τῆς διαμέτρου τῶν ἰκ τμημάτων ὑποκειμένης, τὰ δὲ τρίτα τὸ λ' μέρος τῆς καθ' ἐκαστον ἡμιμοίριον τῶν εὐθειῶν παραυξήσεως, ἢν ἔχοντες καὶ τὴν τοῦ ἐνὸς εὖκοστοῦ μέσην ἐπιβολὴν ἀδιαφοροῦσαν πρὸς αἰσθησίων τῆς ἀκριβούς καὶ τῶν μεταξί τοῦ ἡμίσους μερῶν εὖ ἐτοίμων ἐπίβαλλουσας πηλικότητας ἐπιλογίζεσθαι δυνάμεθα. εὐκατανόητον δ', ὅτι διὰ τῶν αὐτῶν καὶ προκειμένων θεωρημάτων, κἂν ἐν δισταγμῷ γενώμεθα γραφικῆς ἀμαρτίας περὶ τινα τῶν ἐν τῷ κανονίῳ παρακειμένων εὐθειῶν, ῥαδίαν ποιησόμεθα τὴν τε εξέτασιν καὶ τὴν 442
TRIGONOMETRY

0° 31' 25''. The remaining intervals may be completed, as we said, by means of the chord subtending 1° 1/2—in the case of the first interval, for example, by adding 1° we obtain the chord subtending 2°, and from the difference between this and 3° we obtain the chord subtending 2 1/2°, and so on for the remainder.

(ix.) The Table
Ibid. 46. 21–63. 46

The theory of the chords in the circle may thus, I think, be very easily grasped. In order that, as I said, we may have the lengths of all the chords in common use immediately available, we shall draw up tables arranged in forty-five symmetrical rows. The first section will contain the magnitudes of the arcs increasing by half degrees, the second will contain the lengths of the chords subtending the arcs measured in parts of which the diameter contains 120, and the third will give the thirtieth part of the increase in the chords for each half degree, in order that for every sixtieth part of a degree we may have a mean approximation differing imperceptibly from the true figure and so be able readily to calculate the lengths corresponding to the fractions between the half degrees. It should be well noted that, by these same theorems now before us, if we should suspect an error in the computation of any of the chords in the table, we can easily make a test and

* As there are 360 half degrees in the table, the statement appears to mean that the table occupied eight pages each of 45 rows; so Manitius, Des Claudius Ptolemäus Handbuch der Astronomie, 1er Bd., p. 35 n. a.

b Such an error might be accumulated by using the approximations for 1° and 1/2°; but, in fact, the sines in the table are generally correct to five places of decimals.
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ἐπανόρθωσιν ἢτοι ἀπὸ τῆς ὑπὸ τὴν διπλασίονα τῆς ἐπιζητουμένης ἢ τῆς πρὸς ἀλλας τινὰς τῶν δεδομένων ὑπεροχῆς ἢ τῆς τὴν λείπουσαν εἰς τὸ ἡμικύκλιον περιφέρειαν ὑποτευνόσης εὐθείας. καὶ ἐστὶν ἡ τοῦ κανονίου καταγραφὴ τοιαύτη.

ια'. Κανόνιον τῶν ἐν κύκλῳ εὐθείῶν

<table>
<thead>
<tr>
<th>περιφερεῖῶν</th>
<th>εὐθείῶν</th>
<th>ἔξηκοστῶν</th>
</tr>
</thead>
<tbody>
<tr>
<td>λ'</td>
<td>ο</td>
<td>λα</td>
</tr>
<tr>
<td>a</td>
<td>α</td>
<td>β</td>
</tr>
<tr>
<td>α λ'</td>
<td>a</td>
<td>λδ</td>
</tr>
<tr>
<td>β</td>
<td>β</td>
<td>ε</td>
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<tr>
<td>β λ'</td>
<td>β</td>
<td>λζ</td>
</tr>
<tr>
<td>γ</td>
<td>γ</td>
<td>η</td>
</tr>
<tr>
<td>γ λ'</td>
<td>γ</td>
<td>λθ</td>
</tr>
<tr>
<td>δ</td>
<td>δ</td>
<td>ια</td>
</tr>
<tr>
<td>δ λ'</td>
<td>δ</td>
<td>μβ</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ροζ λ'</th>
<th>ροζ</th>
<th>ροζ λ'</th>
<th>ροζ η</th>
<th>ροζ η λ'</th>
<th>ροζ θ</th>
<th>ροζ λ'</th>
<th>ροθ</th>
<th>ροθ λ'</th>
<th>ρπ</th>
</tr>
</thead>
<tbody>
<tr>
<td>μθ</td>
<td>νε</td>
<td>ιη</td>
<td>νθ</td>
<td>νη</td>
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<td>μθ</td>
<td>μθ</td>
</tr>
</tbody>
</table>


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apply a correction, either from the chord subtending double of the arc which is under investigation, or from the difference with respect to any others of the given magnitudes, or from the chord subtending the remainder of the semicircular arc. And this is the diagram of the table:

11. **Table of the Chords in a Circle**

<table>
<thead>
<tr>
<th>Arcs</th>
<th>Chords</th>
<th>Sixtieths</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{3}^\circ)</td>
<td>0° 31' 25&quot;</td>
<td>0° 1' 2&quot; 50&quot;</td>
</tr>
<tr>
<td>1°</td>
<td>1 2 50</td>
<td>0 1 2 50</td>
</tr>
<tr>
<td>1(\frac{1}{2})°</td>
<td>1 34 15</td>
<td>0 1 2 50</td>
</tr>
<tr>
<td>2°</td>
<td>2 5 40</td>
<td>0 1 2 50</td>
</tr>
<tr>
<td>2(\frac{1}{2})°</td>
<td>2 37 4</td>
<td>0 1 2 48</td>
</tr>
<tr>
<td>3°</td>
<td>3 8 28</td>
<td>0 1 2 48</td>
</tr>
<tr>
<td>3(\frac{1}{2})°</td>
<td>3 39 52</td>
<td>0 1 2 48</td>
</tr>
<tr>
<td>4°</td>
<td>4 11 16</td>
<td>0 1 2 47</td>
</tr>
<tr>
<td>4(\frac{1}{2})°</td>
<td>4 42 40</td>
<td>0 1 2 47</td>
</tr>
<tr>
<td>6°</td>
<td>60 0 0 0</td>
<td>0 0 54 21</td>
</tr>
<tr>
<td>176°</td>
<td>119 55 38</td>
<td>0 0 2 3</td>
</tr>
<tr>
<td>176(\frac{1}{2})°</td>
<td>119 56 39</td>
<td>0 0 1 47</td>
</tr>
<tr>
<td>177°</td>
<td>119 57 32</td>
<td>0 0 1 30</td>
</tr>
<tr>
<td>177(\frac{1}{2})°</td>
<td>119 58 18</td>
<td>0 0 1 17</td>
</tr>
<tr>
<td>178°</td>
<td>119 58 55</td>
<td>0 0 0 57</td>
</tr>
<tr>
<td>178(\frac{1}{2})°</td>
<td>119 59 24</td>
<td>0 0 0 41</td>
</tr>
<tr>
<td>179°</td>
<td>119 59 44</td>
<td>0 0 0 25</td>
</tr>
<tr>
<td>179(\frac{1}{2})°</td>
<td>119 59 56</td>
<td>0 0 0 9</td>
</tr>
<tr>
<td>180°</td>
<td>120 0 0</td>
<td>0 0 0 0</td>
</tr>
</tbody>
</table>
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(c) Menelaus’s Theorem

(i.) Lemmas

Ibid. 68. 14–74. 8

γ’. Προλαμβανόμενα εἰς τὰς σφαιρικὰς
deίξεις

'Ακολούθου δ' οὖντος ἀποδείξαι καὶ τὰς κατὰ
μέρος γινομένας πηλικότητας τῶν ἀπολαμβανο-
μένων περιφερειῶν μεταξὺ τοῦ τε ἱσημερινοῦ καὶ
tοῦ διὰ μέσων τῶν Ζωδίων κύκλου τῶν γραφο-
μένων μεγίστων κύκλων διὰ τῶν τοῦ ἱσημερινοῦ
πόλων προεκθησόμεθα λημμάτια βραχέα καὶ ἐὐ-
χρηστα, δι' ὧν τὰς πλείστας σχεδὸν δείξεις τῶν
σφαιρικῶς θεωρουμένων, ὡς ἐνι μάλιστα, ἀπλοῦ-
στερον καὶ μεθοδικότερον ποιησόμεθα.

Εἰς δύο δὴ εὐθείας τὰς ΑΒ καὶ ΑΓ διαχθεῖσαι
dύο εὐθείαι η τε ΒΕ καὶ η ΓΔ τεμνετωσαν ἀλλήλας
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(c) **Menelaus’s Theorem**

(i.) **Lemmas**

*Ibid.* 68. 14-74. 8

13. Preliminary matter for the spherical proofs

The next subject for investigation being to show the lengths of the arcs, intercepted between the celestial equator and the zodiac circle, of great circles drawn through the poles of the equator, we shall set out some brief and serviceable little lemmas, by means of which we shall be able to prove more simply and more systematically most of the questions investigated spherically.

Let two straight lines BE and ΓΔ be drawn so as to meet the straight lines AB and AΓ and to cut one...
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κατά τὸ Ζ σημεῖον. λέγω, ὅτι ὁ τῆς ΓΑ πρὸς ΑΕ λόγος συνῆπται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΖΒ πρὸς ΒΕ.

"Ἡχθο γὰρ διὰ τοῦ Ε τῇ ΓΔ παράλληλος ἡ ΕΗ. ἔπει παράλληλοί εἰσιν αἱ ΓΔ καὶ ΕΗ, ὁ τῆς ΓΑ πρὸς ΕΑ λόγος ὁ αὐτὸς ἐστὶν τῷ τῆς ΓΔ πρὸς ΕΗ. ἔσωθεν δὲ ἡ ΖΔ· ὁ ἄρα τῆς ΓΔ πρὸς ΕΗ λόγος συγκείμενος ἐστιν ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΔΖ πρὸς ΗΕ· ὡστε καὶ ὁ τῆς ΓΑ πρὸς ΑΕ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΔΖ πρὸς ΗΕ. ἐστὶν δὲ καὶ ὁ τῆς ΔΖ πρὸς ΗΕ λόγος ὁ αὐτὸς τῷ τῆς ΖΒ πρὸς ΒΕ διὰ τὸ παραλλήλους πάλιν εἰναι τὰς ΕΗ καὶ ΖΔ· ὁ ἄρα τῆς ΓΑ πρὸς ΑΕ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΖΒ πρὸς ΒΕ· ὁπερ προέκειτο δεῖξαι.

Κατὰ τὰ αὐτὰ δὲ δειχθῆσται, ὅτι καὶ κατὰ διαίρεσιν ὁ τῆς ΓΕ πρὸς ΕΑ λόγος συνῆπται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ, διὰ τοῦ Α τῇ ΕΒ παραλλήλου ἀξθείσης καὶ

* Lit. "the ratio of ΓΑ to ΑΕ is compounded of the ratio of ΓΔ to ΔΖ and ΖΒ to ΒΕ."
another at the point $Z$. I say that
\[ \Gamma A : AE = (\Gamma \Delta : \Delta Z)(ZB : BE). \]

For through $E$ let $EH$ be drawn parallel to $\Gamma \Delta$. Since $\Gamma \Delta$ and $EH$ are parallel,
\[ \Gamma A : EA = \Gamma \Delta : EH. \] [Eucl. vi. 4]
But $Z\Delta$ is an external [straight line];
\[ \therefore \quad \Gamma \Delta : EH = (\Gamma \Delta : \Delta Z)(\Delta Z : HE); \]
\[ \therefore \quad \Gamma A : AE = (\Gamma \Delta : \Delta Z)(\Delta Z : HE). \]

But \[ \Delta Z : HE = ZB : BE, \] [Eucl. vi. 4]
by reason of the fact that $EH$ and $Z\Delta$ are parallels;
\[ \therefore \quad \Gamma A : AE = (\Gamma \Delta : \Delta Z)(ZB : BE); \quad (1) \]
which was set to be proved.

With the same premises, it will be shown by transformation of ratios that
\[ \Gamma E : EA = (\Gamma Z : \Delta Z)(\Delta B : BA), \]

\[ \text{a parallel to } EB \text{ being drawn through } A \text{ and } \Gamma \Delta \]
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προσεκβληθείσης ἐπ’ αὐτὴν τῆς ΓΔΗ. ἐπεὶ γὰρ πάλιν παράλληλός ἐστιν ᾧ ΑΗ τῇ EZ, ἔστω, ὡς ᾧ ΓΕ πρὸς ΕΑ, ᾧ ΓΖ πρὸς ΖΗ. ἀλλὰ τῆς ΖΔ ἐξωθεὶς λαμβανομένης ὁ τῆς ΓΖ πρὸς ΖΗ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΖΔ καὶ τοῦ τῆς ΔΖ πρὸς ΖΗ. ἔστω δὲ ὁ τῆς ΔΖ πρὸς ΖΗ λόγος ὁ αὐτὸς τῷ τῆς ΔΒ πρὸς ΒΑ διὰ τὸ εἰς παραλλήλους τὰς ΑΗ καὶ ΖΒ διήχθαι τὰς ΒΑ καὶ ΖΗ. ὁ ἀρα τῆς ΓΖ πρὸς ΖΗ λόγος συνήπται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ. ἀλλὰ τῷ τῆς ΓΖ πρὸς ΖΗ λόγῳ ὁ αὐτὸς ἔστω ὁ τῆς ΓΕ πρὸς ΕΑ· καὶ ὁ τῆς ΓΕ ἀρα πρὸς ΕΑ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ· ὥπερ ἔδει δεῖξαι.

Πάλιν ἔστω κύκλος ὁ ΑΒΓ, οὐ κέντρον τὸ Δ, καὶ εἰλήφθω ἐπὶ τὰς περιφερείας αὐτοῦ τυχόντα 450
being produced to it. For, again, since AH is parallel to EZ,
\[ \Gamma E : EA = \Gamma Z : ZH. \]  
[Eucl. vi. 2]
But, an external straight line ZΔ having been taken,
\[ \Gamma Z : ZH = (\Gamma Z : Z\Delta)(\Delta Z : ZH); \]
and
\[ \Delta Z : ZH = \Delta B : BA, \]
by reason of BA and ZH being drawn to meet the parallels AH and ZB;
\[ \therefore \quad \Gamma Z : ZH = (\Gamma Z : \Delta Z)(\Delta B : BA). \]
But \[ \Gamma Z : ZH = \Gamma E : EA; \]  
[supra and \]
and \[ \therefore \quad \Gamma E : EA = (\Gamma Z : \Delta Z)(\Delta B : BA); \]  
(2)
which was to be proved.

Again, let AΓ Δ be a circle with centre Δ, and let

there be taken on its circumference any three points.

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tría σημεῖα τὰ Α, Β, Γ, ὥστε ἐκατέραν τῶν ΑΒ, ΒΓ περιφερεῖῶν ἐλάσσονα εἶναι ἡμικυκλίου· καὶ ἐπὶ τῶν ἐξῆς δὲ λαμβανομένων περιφερειῶν τὸ ὁμιοῦν ὑπακονεύσθω· καὶ ἐπεξεύχθωσαν αἱ ΑΓ καὶ ΔΕΒ. λέγω, ὅτι ἐστὶν, ὡς ἡ ὑπὸ τὴν διπλῆν τῆς ΑΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ, οὕτως ἡ ΑΕ εὐθεῖα πρὸς τὴν ΕΓ εὐθείαν.

"Ἡχθωσαν γὰρ κάθετοι ἀπὸ τῶν Α καὶ Γ σημεῖων ἐπὶ τὴν ΔΒ ἡ τε ΑΖ καὶ ἡ ΓΗ. ἐπεὶ παράλληλος ἐστὶν ἡ ΑΖ τῇ ΓΗ, καὶ διήκται εἰς αὐτὰς εὐθεία ἡ ΑΕΓ, ἐστὶν, ὡς ἡ ΑΖ πρὸς τὴν ΓΗ, οὕτως ἡ ΑΕ πρὸς ΕΓ. ἄλλ' ὁ αὐτὸς ἐστὶν λόγος ὁ τῆς ΑΖ πρὸς ΓΗ καὶ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ. ἡμίσεια γὰρ ἐκατέρα ἐκατέρας· καὶ ὁ τῆς ΑΕ ἀρα πρὸς ΕΓ λόγος ὁ αὐτὸς ἐστὶν τῷ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ· ὅπερ ἐδει δεῖξαι.

Παρακολούθει δ' αὐτόθεν, ὅτι, κἂν δοθῶσιν ἡ τε ΑGamma διπλῆν περιφέρεια καὶ ὁ λόγος ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ, δοθήσεται καὶ ἐκατέρα τῶν ΑΒ καὶ ΒΓ περιφερειῶν. ἐκτεθέισις γὰρ τῆς αὐτῆς καταγραφῆς ἐπεξεύχθω ἡ ΑΔ, καὶ ἡχθω ἀπὸ τοῦ Δ κάθετος ἐπὶ τὴν ΑΕΓ ἡ ΔΖ. ὅτι μὲν οὖν τῆς ΑΓ περι-

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A, B, Γ, in such a manner that each of the arcs AB, BG is less than a semicircle; and upon the arcs taken in succession let there be a similar relationship; and let ΔΓ be joined and ΔEB. I say that

the chord subtended by double of the arc AB:
the chord subtended by double of the arc BG

[i.e., sin AB : sin BG] = AE : EG.

For let perpendiculareas AZ and ΓH be drawn from the points A and Γ to ΔB. Since AZ is parallel to ΓH, and the straight line AEG has been drawn to meet them,

AZ : ΓH = AE : EG.  [Eucl. vi. 4
But AZ : ΓH = the chord subtended by double of the arc AB:
the chord subtended by double of the arc BG,
for each term is half of the corresponding term;
and therefore

AE : EG = the chord subtended by double of the arc AB:
the chord subtended by double of the arc BG.  .  .  .  (3)

[= sin AB : sin BG],

which was to be proved.

It follows immediately that, if the whole arc ΔΓ be given, and the ratio of the chord subtended by double of the arc AB to the chord subtended by double of the arc BG [i.e. sin AB : sin BG], each of the arcs AB and BG will also be given. For let the same diagram be set out, and let ΔΔ be joined, and from Δ let ΔZ be drawn perpendicular to AEG. If the arc

v. supra, p. 420 n. a.


ΓΕΡΙΚ ΜΑΘΗΜΑΤΙΚΑ

θείας δοθείσης ἢ τε ὑπὸ ΑΔΖ γωνία τῆς ἡμι-
σειαν αὗτῆς ὑποτείνουσα δεδομένη ἔσται καὶ
ὁλον τὸ ΑΔΖ τρίγωνον, δῆλον. ἐπεὶ δὲ τῆς ΑΓ

εὐθείας ὅλης δεδομένης ὑπόκειται καὶ ὁ τῆς ΑΕ
πρὸς ΕΓ λόγος ὅ αὐτὸς ὃν τῷ τῆς ὑπὸ τῆν διπλῆν
τῆς ΑΒ πρὸς τῆν ὑπὸ τῆν διπλῆν τῆς ΒΓ, ἢ τε
ΑΕ ἔσται δοθείσα καὶ λοιπὴ ἢ ΖΕ. καὶ διὰ τοῦτο
καὶ τῆς ΔΖ δεδομένης δοθήσεται καὶ ἢ τε ὑπὸ
ΕΔΖ γωνία τοῦ ΕΔΖ ὀρθογωνίου καὶ ὅλη ἢ ὑπὸ
ΑΔΒ· ὡστε καὶ ἢ τε ΑΒ περιφέρεια δοθήσεται
καὶ λοιπὴ ἢ ΒΓ· ὅπερ ἔδει δεῖξαι.

Πάλιν ἔστω κύκλος ὁ ΑΒΓ περὶ κέντρου τὸ Δ,
καὶ ἐπὶ τῆς περιφερείας αὐτοῦ ἐιλήφθω τρία
σημεῖα τὰ Α, Β, Γ, ὡστε ἐκατέραν τῶν ΑΒ, ΑΓ
περιφερείων ἐλάσσονα εἶναι ἡμικυκλίου· καὶ ἐπὶ
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$\triangle A\Gamma$ is given, it is then clear that the angle $A\Delta Z$, subtending half the same arc, will also be given and therefore the whole triangle $A\Delta Z$; and since the whole chord $A\Gamma$ is given, and by hypothesis

$AE : E\Gamma =$ the chord subtended by double of the arc $AB$:

the chord subtended by double of the arc $B\Gamma$,

[i.e. $= \sin AB : \sin B\Gamma$],

therefore $AE$ will be given [Eucl. Dat. 7], and the remainder $ZE$. And for this reason, $\Delta Z$ also being given, the angle $E\Delta Z$ will be given in the right-angled triangle $E\Delta Z$, and [therefore] the whole angle $A\Delta B$; therefore the arc $AB$ will be given and also the remainder $B\Gamma$; which was to be proved.

Again, let $AB\Gamma$ be a circle about centre $\Delta$, and let

three points $A$, $B$, $\Gamma$ be taken on its circumference so that each of the arcs $AB$, $A\Gamma$ is less than a semicircle;
τῶν ἑξῆς δὲ λαμβανομένων περιφερειῶν τὸ ὁμοιὸν ὑπακούεσθω· καὶ ἐπιζευχθεῖσα ἡ τε ΔΑ καὶ ἡ ΓΒ ἐκβεβλήθησαν καὶ συμπυπτέτωσαν κατὰ τὸ Ε σημεῖον. λέγω, ὅτι ἐστίν, ὡς ἡ ὑπὸ τὴν διπλὴν τῆς ΓΑ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς AB, οὕτως ἡ ΓΕ εὐθείᾳ πρὸς τὴν BE.

'Ομοίως γὰρ τῷ πρωτέρῳ λημματὶ, ἕαν ἀπὸ τῶν B καὶ Γ ἀγάγωμεν καθέτους ἐπὶ τὴν ΔΑ τῆς τε ΒΖ καὶ τῆς ΓΗ, ἔσται διὰ τὸ παραλλήλους αὐτὰς εἶναι, ὡς ἡ ΓΗ πρὸς τὴν ΒΖ, οὕτως ἡ ΓΕ πρὸς τὴν EB· ὡστε καὶ, ὡς ἡ ὑπὸ τὴν διπλὴν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς AB, οὕτως ἡ ΓΕ πρὸς τὴν EB· ὅπερ ἐδει δειξαί.

Καὶ ἐνταῦθα δὲ αὐτὸθεν παρακολουθεῖ, διότι, κἂν ἡ ΓΒ περιφέρεια μόνη δοθῇ, καὶ ὁ λόγος ὁ τῆς ὑπὸ τὴν διπλὴν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς AB δοθῇ, καὶ ἡ AB περιφέρεια δοθήσεται. πάλιν γὰρ ἐπὶ τῆς ὁμοίως καταγραφῆς ἐπιζευχθεῖσας τῆς ΔΒ καὶ καθέτου ἀχθείσης ἡπὶ

\[
\begin{array}{c}
\text{Ε}
\end{array}
\begin{array}{c}
\text{Β}
\end{array}
\begin{array}{c}
\text{Ζ}
\end{array}
\begin{array}{c}
\text{Γ}
\end{array}
\begin{array}{c}
\text{Α}
\end{array}
\begin{array}{c}
\text{Δ}
\end{array}
\begin{array}{c}
\text{Τριγωνομετρία}
\end{array}
\begin{array}{c}
\text{τῆς ΒΓ τῆς ΔΖ ἡ μὲν ὑπὸ ΒΔΖ γωνία τῆς ἡμι-
\text{σειαν ὑποτείνουσα τῆς ΒΓ περιφερείας ἔσται}
\end{array}
\begin{array}{c}
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\end{array}
and upon the arcs taken in succession let there be a similar relationship; and let $\Delta A$ be joined and let $\Gamma B$ be produced so as to meet it at the point $E$. \ I say that

the chord subtended by double of the arc $\Gamma A$:

the chord subtended by double of the arc $AB$

$[i.e., \sin \Gamma A : \sin AB] = \Gamma E : BE$.

For, as in the previous lemma, if from $B$ and $\Gamma$ we draw $BZ$ and $\Gamma H$ perpendicular to $\Delta A$, then, by reason of the fact that they are parallel,

$$\Gamma H : BZ = \Gamma E : EB.$$  \ [Eucl. vi. 4

$\therefore$ the chord subtended by double of the arc $\Gamma A$:

the chord subtended by double of the arc $AB$

$[i.e., \sin \Gamma A : \sin AB] = \Gamma E : EB$; $\therefore \therefore \therefore (4)$

which was to be proved.

And thence it immediately follows why, if the arc $\Gamma B$ alone be given, and the ratio of the chord subtended by double of the arc $\Gamma A$ to the chord subtended by double of the arc $AB [i.e., \sin \Gamma A : \sin AB]$, the arc $AB$ will also be given. \ For again, in a similar diagram let $\Delta B$ be joined and let $\Delta Z$ be drawn perpendicular to $B\Gamma$; then the angle $B\Delta Z$ subtended by half the arc $B\Gamma$ will be given; and therefore the
(ii.) *The Theorem*

*Ibid.* 74. 9-76. 9

Τούτων προλήφθεντων γεγράφθησαν ἐπὶ σφαιρικῆς ἑπιφανείας μεγίστων κύκλων περιφέρειαί, ὡστε εἰς δύο τὰς ΑΒ καὶ ΑΓ δύο γραφείσας τὰς ΒΕ καὶ ΓΔ τέμνειν ἀλλήλας κατὰ τὸ Ζ σημείον. ἔστω δὲ ἐκάστη αὐτῶν ἔλασσον ἡμικυκλίου τὸ δὲ αὐτὸ καὶ ἐπὶ πασῶν τῶν καταγραφῶν ὑποκονέσθων.

Λέγω δὴ, ὅτι ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΕ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΕΑ λόγος συνήπται ἐκ τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΖ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΖΔ καὶ τοῦ τῆς ὑπὸ τὴν διπλῆν τῆς ΔΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΑ.

Εἰλήφθω γὰρ τὸ κέντρον τῆς σφαίρας καὶ ἔστω τὸ Ῥ, καὶ ἤχθωσαν ἀπὸ τοῦ Ῥ ἐπὶ τὰς Β, Ζ, Ε τομᾶς τῶν κύκλων ἢ τε ΗΒ καὶ ἢ ΗΖ καὶ ἢ ΗΕ, καὶ ἐπιζευγθεῖσα ἢ ΑΔ ἐκβεβλήσθω καὶ συμπτέτω τῇ ΗΒ ἐκβληθείσῃ καὶ αὐτῇ κατὰ τὸ Θ σημεῖον, ὁμοίως δὲ ἐπιζευγθεῖσαι αἰ ΔΓ καὶ ΑΓ τεμνέτωσαν τὰς ΗΖ καὶ ΗΕ κατὰ τὸ Κ καὶ Λ 458
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whole of the right-angled triangle \( B\Delta Z \). But since the ratio \( \Gamma E : EB \) is given and also the chord \( \Gamma B \), therefore \( EB \) will also be given and, further, the whole [straight line] \( EBZ \); therefore, since \( \Delta Z \) is given, the angle \( E\Delta Z \) in the same right-angled triangle will be given, and the remainder \( E\Delta B \). Therefore the arc \( AB \) will be given.

(ii.) The Theorem

*Ibid. 74. 9–76. 9*

These things having first been grasped, let there be described on the surface of a sphere arcs of great circles such that the two arcs \( BE \) and \( \Gamma \Delta \) will meet the two arcs \( AB \) and \( \Lambda \Gamma \) and will cut one another at the point \( Z \); let each of them be less than a semi-circle; and let this hold for all the diagrams.

Now I say that the ratio of the chord subtended by double of the arc \( \Gamma E \) to the chord subtended by double of the arc \( EA \) is compounded of \((a)\) the ratio of the chord subtended by double of the arc \( \Gamma Z \) to the chord subtended by double of the arc \( Z\Delta \), and \((b)\) the ratio of the chord subtended by double of the arc \( \Delta B \) to the chord subtended by double of the arc \( BA \),

\[
\left[ \text{i.e., } \frac{\sin \Gamma E}{\sin EA} = \frac{\sin \Gamma Z}{\sin Z\Delta} \cdot \frac{\sin \Delta B}{\sin BA} \right].
\]

For let the centre of the sphere be taken, and let it be \( H \), and from \( H \) let \( HB \) and \( HZ \) and \( HE \) be drawn to \( B, Z, E \), the points of intersection of the circles, and let \( \Lambda \Delta \) be joined and produced, and let it meet \( HB \) produced at the point \( \Theta \), and similarly let \( \Delta \Gamma \) and \( \Lambda \Gamma \) be joined and cut \( HZ \) and \( HE \) at \( K \) and the point 459
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σημείον· ἐπὶ μιᾶς δὴ γίνεται εὐθείας τὰ Θ, Κ, Λ σημεῖα διὰ τὸ ἐν δυσὶν ἀμα εἶναι ἐπιπέδους τῷ τε τοῦ ΑΓΔ τριγώνου καὶ τῷ τοῦ ΒΖΕ κύκλου, ἢτις

ἐπιζευχθεῖσα ποιεῖ εἰς δύο εὐθείας τὰς ΘΑ καὶ ΓΑ διηγμένας τὰς ΘΔ καὶ ΓΔ τεμνούσας ἀλλήλας κατὰ τὸ Κ σημείον· ὁ ἄρα τῆς ΓΑ πρὸς ΔΑ λόγος συνήπται ἐκ τε τοῦ τῆς ΓΚ πρὸς ΚΔ καὶ τοῦ τῆς ΔΘ πρὸς ΘΑ. ἀλλ' ὡς μὲν ἡ ΓΑ πρὸς ΛΑ, οὕτως ἢ ὑπὸ τὴν διπλῆν τῆς ΓΕ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΕΑ περιφερείας, ὡς δὲ ἡ ΓΚ πρὸς ΚΔ, οὕτως ἢ ὑπὸ τὴν διπλῆν τῆς ΓΖ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΖΔ, ὡς δὲ ἡ ΘΔ

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$\Lambda$; then the points $\Theta, K, \Lambda$ will lie on one straight line because they lie simultaneously in two planes, that of the triangle $\Delta \Gamma \Delta$ and that of the circle $\text{BZE}$, and therefore we have straight lines $\Theta \Lambda$ and $\Gamma \Delta$ meeting the two straight lines $\Theta A$ and $\Gamma A$ and cutting one another at the point $K$; therefore

$$\Gamma \Lambda : \Lambda \Lambda = (\Gamma K : K \Delta)(\Delta \Theta : \Theta \Lambda).$$

[by (2)]

But $\Gamma \Lambda : \Lambda \Lambda =$ the chord subtended by double of the arc $\Gamma E$:

the chord subtended by double of the arc $E A$

[i.e., $\sin \Gamma E : \sin EA$],

while $\Gamma K : K \Delta =$ the chord subtended by double of the arc $\Gamma Z$:

the chord subtended by double of the arc $Z \Delta$ [by (3)]

[i.e., $\sin \Gamma Z : \sin Z \Delta$].

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From the Arabic version, it is known that “Menelaus’s Theorem” was the first proposition in Book iii. of his Sphaerica, and several interesting deductions follow.
and \( \Theta \Delta : \Theta A = \) the chord subtended by double of the arc \( \Delta B \):

the chord subtended by double of the arc \( BA \) \[\text{by (4)}\] 

\[\text{i.e., } \sin \Delta B : \sin BA,\]

and therefore the ratio of the chord subtended by double of the arc \( GE \) to the chord subtended by double of the arc \( EA \) is compounded of \((a)\) the ratio of the chord subtended by double of the arc \( GZ \) to the chord subtended by double of the arc \( Z\Delta \), and \((b)\) the ratio of the chord subtended by double of the arc \( \Delta B \) to the chord subtended by double of the arc \( BA \),

\[
\left[ \text{i.e., } \frac{\sin \Gamma E}{\sin \Gamma A} = \frac{\sin \Gamma Z}{\sin \Delta} \cdot \frac{\sin \Delta B}{\sin BA} \right].
\]

Now with the same premises, and as in the case of the straight lines in the plane diagram \[\text{by (1)}\], it is shown that the ratio of the chord subtended by double of the arc \( GA \) to the chord subtended by double of the arc \( EA \) is compounded of \((a)\) the ratio of the chord subtended by double of the arc \( GA \) to the chord subtended by double of the arc \( \Delta Z \), and \((b)\) the ratio of the chord subtended by double of the arc \( ZB \) to the chord subtended by double of the chord \( BE \),

\[
\left[ \text{i.e., } \frac{\sin \Gamma A}{\sin \Gamma \Delta} = \frac{\sin \Gamma \Delta}{\sin \Delta Z} \cdot \frac{\sin ZB}{\sin BE} \right];
\]

which was set to be proved.*

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XXII. MENSURATION: HERON OF ALEXANDRIA
XXII. MENSURATION: HERON OF ALEXANDRIA

(a) DEFINITIONS

Heron, Deff., ed. Heiberg (Heron iv.) 14. 1-24

Кαὶ τὰ μὲν πρὸ τῆς γεωμετρικῆς στοιχειώσεως τεχνολογούμενα ὑπογράφων σοι καὶ ὑποτυποῦμενος, ὡς ἔχει μάλιστα συντόμως, Διονύσιε λαμπρότατε, τὴν τε ἀρχὴν καὶ τὴν ὅλην σύνταξιν ποιήσομαι κατὰ τὴν τοῦ Εὐκλείδου τοῦ Στοιχειωτοῦ τῆς ἐν γεωμετρίᾳ θεωρίας διδασκαλίαν: οἴμαι γὰρ οὕτως οὐ μόνον τὰς ἐκείνου πραγματείας

* The problem of Heron's date is one of the most disputed questions in the history of Greek mathematics. The only details certainly known are that he lived after Apollonius, whom he quotes, and before Pappus, who cites him, say between 150 B.C. and A.D. 250. Many scraps of evidence have been thrown into the dispute, including the passage here first cited; for it is argued that the title λαμπρότατος corresponds to the Latin clarissimus, which was not in common use in the third century A.D. Both Heiberg (Heron, vol. v. p. ix) and Heath (H.G.M. ii. 306) place him, however, in the third century A.D., only a little earlier than Pappus.

The chief works of Heron are now definitively published in five volumes of the Teubner series. Perhaps the best known are his Pneumatica and the Automata, in which he shows how to use the force of compressed air, water or steam; they are of great interest in the history of physics, and have led some to describe Heron as "the father of the turbine," but
XXII. MENSURATION : HERON OF ALEXANDRIA

(a) Definitions

Heron, Definitions, ed. Heiberg (Heron iv.) 14. 1-24

In setting out for you as briefly as possible, O most excellent Dionysius, a sketch of the technical terms premised in the elements of geometry, I shall take as my starting point, and shall base my whole arrangement upon, the teaching of Euclid, the writer of the elements of theoretical geometry; for in this way I think I shall give you a good general understanding, as they have no mathematical interest they cannot be noticed here. Heron also wrote a Belopoeica on the construction of engines of war, and a Mechanics, which has survived in Arabic and in a few fragments of the Greek.

In geometry, Heron's elaborate collection of Definitions has survived, but his Commentary on Euclid's Elements is known only from extracts preserved by Proclus and an-Nairizi, the Arabic commentator. In mensuration there are extant the Metrica, Geometrica, Stereometrica, Geodaesia, Mensurae and Liber Gei'-ponicus. The Metrica, discovered in a Constantinople ms. in 1896 by R. Schöne and edited by his son H. Schöne, seems to have preserved its original form more closely than the others, and will be relied on here in preference to them. Heron's Dioptra, describing an instrument of the nature of a theodolite and its application to surveying, is also extant and will be cited here.

For a full list of Heron's many works, v. Heath, H.G.M. ii. 308-310.
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εὐσωνόττους ἐσεσθαι σοι, ἀλλὰ καὶ πλείστας ἀλλάς τῶν εἰς γεωμετρίαν ἀνὴρ κοντων. ἄρξομαι τούν ἀπὸ σημείου.

α’. Σημείον ἔστιν, οὐδέ μέρος οὐθὲν ἡ πέρας ἀδιάστατον ἡ πέρας γραμμῆς, πέφυκε δὲ διανοία μόνη ληπτον εἶναι ὡσανεί ἀμερές τε καὶ ἀμέγεθες τυγχάνον. τοιοῦτον οὖν αὐτὸ φασιν εἶναι οἷον ἐν χρόνῳ τὸ ἔνεστος καὶ οἷον μονάδα θέσιν ἔχουσαν. ἔστι μὲν οὖν τῇ οὐσίᾳ ταὐτῷ τῇ μονάδι· ἀδιαίρετα γὰρ ἀμφω καὶ ἀσώματα καὶ ἀμέριστα· τῇ δὲ ἐπι- φανείᾳ καὶ τῇ σχέσει διαφέρει· ἢ μὲν γὰρ μονάς ἄρχη ἄριθμοῦ, τὸ δὲ σημεῖον τῆς γεωμετρουμένης οὐσίας ἄρχη, ἄρχη δὲ κατὰ ἐκθέσιν, οὐχ ὡς μέρος ὅν τῆς γραμμῆς, ὥς τοῦ ἄριθμοῦ μέρος ἢ μονάς, προεπιοῦμενον δὲ αὐτῆς· κινήθεντος γὰρ ἢ μᾶλλον νοηθέντος ἐν ρύσει νοεῖται γραμμῆ, καὶ οὕτω σημείον ἄρχη ἐστι γραμμῆς, ἐπιφάνεια δὲ στερεοῦ σώματος.

Ibid. 60. 22-62. 9

ζ’. Σπείρα γίνεται, ὅταν κύκλος ἐπὶ κύκλου τὸ κέντρον ἐχων ὀρθός ὅν πρὸς τὸ τοῦ κύκλου ἐπίπεδον περιενέχειες εἰς τὸ αὐτὸ πάλιν ἀποκατα- σταθῇ· τὸ δὲ αὐτὸ τοῦτο καὶ κρίκος καλεῖται. διεξῆς μὲν οὖν ἐστὶ σπείρα ἡ ἐχονσα διάλειμμα, συνεχῆς δὲ ἡ καθ’ ἐν σημείον συμπιπτουσα, ἐπαλ- λάττουσα δὲ, καθ’ ἣν ὁ περιφερόμενος κύκλος

1 ἔστι Friedlein, ὅτι codd..

* The first definition is that of Euclid i. Def. 1, the third in effect that of Plato, who defined a point as ἄρχη γραμμῆς (Aristot. Metaph. 992 a 20); the second is reminiscent of Nicomachus, Arith. Introd. ii. 7. 1, v. vol. i. pp. 86-89.

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not only of Euclid's works, but of many others pertaining to geometry. I shall begin, then, with the point.

1. *A point* is that which has no parts, or an extremity without extension, or the extremity of a line, and, being both without parts and without magnitude, it can be grasped by the understanding only. It is said to have the same character as the moment in time or the unit having position. It is the same as the unit in its fundamental nature, for they are both indivisible and incorporeal and without parts, but in relation to surface and position they differ; for the unit is the beginning of number, while the point is the beginning of geometrical being—but a beginning by way of setting out only, not as a part of a line, in the way that the unit is a part of number—and is prior to geometrical being in conception; for when a point moves, or rather is conceived in motion, a line is conceived, and in this way a point is the beginning of a line and a surface is the beginning of a solid body.

*Ibid.* 60. 22-62. 9

97. *A spire* is generated when a circle revolves and returns to its original position in such a manner that its centre traces a circle, the original circle remaining at right angles to the plane of this circle; this same curve is also called a *ring*. A spire is open when there is a gap, *continuous* when it touches at one point, and *self-crossing* when the revolving circle cuts itself.

(b) Measurement of Areas and Volumes

(i.) Area of a Triangle Given the Sides
Heron, Metr. i. 8, ed. H. Schöne (Heron iii.) 18. 12-24. 21

"Εστι δὲ καθολικὴ μέθοδος ὡστε τριών πλευρῶν δοθεισῶν ὁιονποτότων τριγώνου τὸ ἐμβαδὸν εὑρεῖν χωρίς καθέτου. οἶον ἐστωσαν αἴ τοῦ τριγώνου πλευραὶ μονάδων ξ, η, θ. σύνθες τὰ ξ καὶ τὰ η καὶ τὰ θ. γίγνεται κδ. τούτων λαβὲ τὸ ἣμισυν. γίγνεται ἵβ. ἀφελε τὰς μονάδας. λοιπαὶ ἐ. πάλιν ἀφελε ἀπὸ τῶν ἵβ τὰς ἵλ. λοιπαὶ δ. καὶ ἔτι τὰς θ. λοιπαὶ ἵ. ποίησον τὰ ἵβ ἐπὶ τὰ ἵ. γίγνονται ξ. ταῦτα ἐπὶ τὸν δ. γίγνονται σιμ. ταῦτα ἐπὶ τὸν ἵ. γίγνεται ψκ. τούτων λαβὲ πλευρὰν καὶ ἔσται τὸ ἐμβαδὸν τοῦ τριγώνου. ἔπει οὖν αἴ ψκ ῥητὴν τὴν πλευρὰν ὡς ἔχουσι. ληψόμεθα μετὰ διαφόρου ἐλαχίστου τὴν πλευρὰν οὐτως. ἔπει ὁ συνεγγύζων τῷ ψκ τετράγωνὸς ἐστιν ὁ ψκθ καὶ πλευρὰν ἔχει τὸν κζ, μέρισον τὰς ψκ εἰς τὸν κζ. γίγνεται κζ καὶ τρίτα δύο. πρόσθες τὰς κζ. γίγνεται νγ. τρίτα δύο. τούτων τὸ ἣμισυν. γίγνεται κζ ἀγ'. ἔσται ἀρα τὸν ψκ ἡ πλευρὰ ἐγγίστα τὰ κζ ἀγ'. τὰ γὰρ κζ ἀγ' ἐφ' ἐαυτὰ γίγνεται ψκ λζ'. ὡστε ὁ διάφορον μονάδως ἐστὶ μόριον λζ'. ἐὰν δὲ βουλώμεθα.
Certain special curves are generated by sections of these spires. But the square rings are prismatic sections of cylinders; various other kinds of prismatic sections are formed from spheres and mixed surfaces.

(b) Measurement of Areas and Volumes

(i.) Area of a Triangle Given the Sides

Heron, *Metrica* i. 8, ed. H. Schöne (Heron iii.) 18. 12–24. 21

There is a general method for finding, without drawing a perpendicular, the area of any triangle whose three sides are given. For example, let the sides of the triangle be 7, 8 and 9. Add together 7, 8 and 9; the result is 24. Take half of this, which gives 12. Take away 7; the remainder is 5. Again, from 12 take away 8; the remainder is 4. And again 9; the remainder is 3. Multiply 12 by 5; the result is 60. Multiply this by 4; the result is 240. Multiply this by 3; the result is 720. Take the square root of this and it will be the area of the triangle. Since 720 has not a rational square root, we shall make a close approximation to the root in this manner. Since the square nearest to 720 is 729, having a root 27, divide 27 into 720; the result is $26\frac{2}{3}$; add 27; the result is $53\frac{2}{3}$. Take half of this; the result is $26\frac{1}{3} + \frac{1}{3}(=26\frac{5}{9})$. Therefore the square root of 720 will be very nearly $26\frac{5}{9}$. For $26\frac{5}{9}$ multiplied by itself gives $720\frac{1}{9}$; so that the difference is $\frac{1}{36}$. If we wish to make the difference less than $\frac{1}{36}$,

* The passage should be read in conjunction with those from Proclus cited supra, pp. 360-365; note the slight difference in terminology—self-crossing for interlaced.
If a non-square number $A$ is equal to $a^2 \pm b$, Heron's method gives as a first approximation to $\sqrt{A}$,

$$a_1 = \frac{1}{2} \left( a + \frac{A}{a} \right),$$

and as a second approximation,

$$a_2 = \frac{1}{2} \left( a_1 + \frac{A}{a_1} \right).$$

An equivalent formula is used by Rhabdas (v. vol. i. p. 30 n. b) and by a fourteenth century Calabrian monk Barlaam, who wrote in Greek and who indicated that the process could be continued indefinitely. Several modern writers have used the formula to account for Archimedes' approximations to $\sqrt{3}$ (v. vol. i. p. 322 n. a).

Heron had previously shown how to do this.
instead of 729 we shall take the number now found, 720\(\frac{1}{3}\), and by the same method we shall find an approximation differing by much less than \(\frac{1}{3}\).a

The geometrical proof of this is as follows: In a triangle whose sides are given to find the area. Now it is possible to find the area of the triangle by drawing one perpendicular and calculating its magnitude,b but let it be required to calculate the area without the perpendicular.

Let \(\Delta\) be the given triangle, and let each of

\(AB, \beta\Gamma, \Gamma\alpha\) be given; to find the area. Let the
δόν. ἐγγεγράφθη εἰς τὸ τρίγωνον κύκλος ὁ ΔΕΖ, οὗ κέντρον ἐστὶ τὸ Η, καὶ ἐπεζεύχθωσαν αἱ ΑΗ, ΒΗ, ΓΗ, ΔΗ, ΕΗ, ΖΗ. τὸ μὲν ἄρα ὑπὸ ΒΓ, ΕΗ διπλάσιον ἐστὶ τοῦ ΒΗΓ τριγώνου, τὸ δὲ ὑπὸ ΓΑ, ΖΗ τοῦ ΑΓΗ τριγώνου, τῷ δὲ ὑπὸ ΑΒ, ΔΗ τοῦ ΑΒΗ τριγώνου. τὸ ἄρα ὑπὸ τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου καὶ τῆς ΕΗ, τούτου τῆς ἐκ τοῦ κέντρου τοῦ ΔΕΖ κύκλου, διπλάσιον ἐστὶ τοῦ ΑΒΓ τριγώνου. ἐκβεβλήθη τῇ ΓΒ, καὶ τῇ ΑΔ ἵσθα κείσθω τῇ ΒΘ. ἢ ἄρα ΓΘ ἡμίσεια ἐστὶ τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου διὰ τὸ ἵσθα εἶναι τὴν μὲν ΑΔ τῇ ΑΖ, τὴν δὲ ΔΒ τῇ ΒΕ, τὴν δὲ ΖΓ τῇ ΓΕ. τὸ ἄρα ὑπὸ τῶν ΓΘ, ΕΗ ἵσον ἐστὶ τῷ ΑΒΓ τριγώνῳ. ἀλλὰ τὸ ὑπὸ τῶν ΓΘ, ΕΗ πλευρά ἐστὶ τοῦ ἀπὸ τῆς ΓΘ ἐπὶ τὸ ἀπὸ τῆς ΕΗ. ἐσται ἄρα τοῦ ΑΒΓ τριγώνου τὸ ἐμβαδὸν ἔφ᾽ ἐαυτὸ γενόμενον ἵσον τῷ ἀπὸ τῆς ΘΓ ἐπὶ τὸ ἀπὸ τῆς ΕΗ. ἡχθω τῇ μὲν ΓΗ πρὸς ὀρθὰς ἡ ΗΛ, τῇ δὲ ΓΒ ἡ ΒΛ, καὶ ἐπεζεύχθω τῇ ΓΛ. ἐπεὶ οὖν ὀρθή ἐστιν ἐκατέρα τῶν ὑπὸ ΓΗΛ, ΓΒΛ, ἐν κύκλῳ ἄρα ἐστὶ τῷ ΓΗΒΛ τετράπλευρον. αἱ ἄρα ὑπὸ ΓΗΒ, ΓΛΒ δυσὶν ὀρθαῖς εἰσιν ἵσαι. εἰσὶν δὲ καὶ αἱ ὑπὸ ΓΗΒ, ΑΗΔ δυσὶν ὀρθαῖς ἵσαι διὰ τὸ δίχα τετμηθαί τὰς πρὸς τῷ Η.γωνιάς ταῖς ΑΗ, ΒΗ, ΓΗ καὶ ἵσας εἰναι τὰς ὑπὸ τῶν ΓΗΒ, ΑΗΔ ταῖς ὑπὸ τῶν ΑΗΓ, ΔΗΒ καὶ τὰς πάσας τέτρασιν ὀρθαῖς ἵσας εἰναι. ἵση ἄρα ἐστὶ τῇ ὑπὸ ΑΗΔ τῇ ὑπὸ ΓΛΒ. ἐστὶ δὲ καὶ ὀρθή ἡ ὑπὸ ΑΔΗ ὀρθῆ τῇ ὑπὸ ΓΒΛ ἵση. ὀμοιον ἄρα ἐστὶ τῷ ΑΗΔ τριγώνον τῷ ΓΒΛ τριγώνῳ. ὡς ἄρα ἡ ΒΓ πρὸς 474

1 τὸ δὲ ... τριγώνου: these words, along with several
circle \( \triangle EZ \) be inscribed in the triangle with centre \( H \) [Eucl. iv. 4], and let \( AH, BH, GH, DH, EH, ZH \) be joined. Then

\[
BG \cdot EH = 2 \cdot \text{triangle } BHG, \quad [\text{Eucl. i. 41}]
\]

\[
GA \cdot ZH = 2 \cdot \text{triangle } AHG, \quad [\text{ibid.}]
\]

\[
AB \cdot DH = 2 \cdot \text{triangle } ABH. \quad [\text{ibid.}]
\]

Therefore the rectangle contained by the perimeter of the triangle \( \triangle ABG \) and \( EH \), that is the radius of the circle \( \triangle EZ \), is double of the triangle \( \triangle ABG \). Let \( GB \) be produced and let \( BO \) be placed equal to \( AD \); then \( GB \Theta \) is half of the perimeter of the triangle \( \triangle ABG \) because \( AD = AZ, DB = BE, ZG = GE \) [by Eucl. iii. 17]. Therefore

\[
\Gamma \Theta \cdot EH = \text{triangle } ABG. \quad [\text{ibid.}]
\]

But

\[
\Gamma \Theta \cdot EH = \sqrt{\Gamma \Theta^2 \cdot EH^2};
\]

therefore

\[
(\text{triangle } ABG)^2 = \Theta G^2 \cdot EH^2.
\]

Let \( HA \) be drawn perpendicular to \( GH \) and \( BA \) perpendicular to \( GB \), and let \( GA \) be joined. Then since each of the angles \( \Theta HA, GB \) is right, a circle can be described about the quadrilateral \( \Theta HBA \) [by Eucl. iii. 31]; therefore the angles \( \Theta HB, \Theta AB \) are together equal to two right angles [Eucl. iii. 22]. But the angles \( \Theta HB, AH \) are together equal to two right angles because the angles at \( H \) are bisected by \( AH, BH, GH \) and the angles \( \Theta HB, AH \) together with \( AH, DH \) are equal to four right angles; therefore the angle \( AH \) is equal to the angle \( HBA \). But the right angle \( AH \) is equal to the right angle \( HBA \); therefore the triangle \( AH \) is similar to the triangle \( HBA \).

Other obvious corrections not specified in this edition, were rightly added to the text by a fifteenth-century scribe.
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BL, ἡ ΑΔ πρὸς ΔΗ, τουτέστιν ἡ ΒΘ πρὸς ΕΗ, καὶ ἐναλλάξ, ὅς ἡ ΓΒ πρὸς ΒΘ, ἡ ΒΑ πρὸς ΕΗ, τουτέστιν ἡ ΒΚ πρὸς ΚΕ διὰ τὸ παράλληλον εἶναι τὴν ΒΑ τῇ ΕΗ, καὶ συνθέντι, ὅς ἡ ΓΘ πρὸς ΒΘ, οὕτως ἡ ΒΕ πρὸς ΕΚ· ὡστε καὶ ὡς τὸ ἀπὸ τῆς ΓΘ πρὸς τὸ ὑπὸ τῶν ΓΘ, ΘΒ, οὕτως τὸ ὑπὸ ΒΕΓ πρὸς τὸ ὑπὸ ΓΕΚ, τουτέστι πρὸς τὸ ἀπὸ ΕΗ· ἐν ὀρθωγωνίῳ γὰρ ἀπὸ τῆς ὀρθῆς ἐπὶ τὴν βάσιν κάθετος ἦκται ἡ ΕΗ· ὡστε τὸ ἀπὸ τῆς ΓΘ ἐπὶ τὸ ἀπὸ τῆς ΕΗ, οὗ πλευρά ἦν τὸ ἐμβαδὸν τοῦ ΑΒΓ τριγώνου, ἵσον ἦσται τῷ ὑπὸ ΓΘΒ ἐπὶ τὸ ὑπὸ ΓΕΒ. καὶ ἦστι δοθεῖσα ἐκάστη τῶν ΓΘ, ΘΒ, ΒΕ, ΓΕ· ἡ μὲν γὰρ ΓΘ ἡμίσειά ἦστι τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου, ἡ δὲ ΒΘ ἡ ὑπεροχή, ἡ ὑπέρεχε ἡ ἡμίσεια τῆς περιμέτρου τῆς ΓΒ, ἡ δὲ ΒΕ ἡ ὑπεροχή, ἡ ὑπέρεχε ἡ ἡμίσεια τῆς περιμέτρου τῆς ΑΓ, ἡ δὲ ΕΓ ἡ ὑπεροχή, ἡ ὑπέρεχε ἡ ἡμίσεια τῆς περιμέτρου τῆς ΑΒ, ἐπειδὴ ἐπὶ ἵση ἦστιν ἡ μὲν ΕΓ τῇ ΓΖ, ἡ δὲ ΒΘ τῇ ΑΖ, ἐπεὶ καὶ τῇ ΑΔ ἦστιν ἵση. δοθὲν ἄρα καὶ τὸ ἐμβαδὸν τοῦ ΑΒΓ τριγώνου.

(ii.) Volume of a Spire

Ibid. ii. 13, ed. H. Schöne (Heron iii.) 126. 10–130. 3

"Ἑστω γὰρ τις ἐν ἐπιπέδῳ εὐθεῖα ἡ ΑΒ καὶ δύο τυχόντα ἐπ' αὐτῆς σημεῖαν. εἰλήφθων δὲ ΒΓΔΕ τὸν κύκλος ὅρθος ὃν πρὸς τὸ ὑποκείμενον ἐπίπεδον, ἐν ζ ἦστιν ἡ ΑΒ εὐθεία, καὶ μένοντος τοῦ Α

1 κύκλος add. H. Schöne.
Therefore \( \frac{\Gamma \Theta}{\Theta B} = \frac{\Delta \Theta}{\Delta H} = \frac{\Delta H}{\Theta H} \)
and permutando, \( \frac{\Gamma B}{\Theta B} = \frac{\Delta A}{\Theta B} = \frac{\Theta B}{\Theta H} \),
because \( \Delta A \) is parallel to \( \Theta H \),
and componendo \( \frac{\Delta \Theta}{\Theta B} = \frac{\Theta B}{\Theta E} \);
therefore \( \frac{\Delta \Theta^2}{\Theta B} = \frac{\Theta B}{\Theta E} \cdot \frac{\Theta B}{\Theta E} = \frac{\Theta B}{\Theta E} \cdot \Theta E \cdot \Theta K \),
i.e. \( \frac{\Delta \Theta^2}{\Theta B} = \frac{\Theta B}{\Theta E} \cdot \Theta E \cdot \Theta K \),
for in a right-angled triangle \( \Theta H \) has been drawn from the right angle perpendicular to the base; therefore \( \frac{\Delta \Theta^2}{\Theta B} \cdot \Theta H^2 \), whose square root is the area of the triangle \( \Delta \Theta \), is equal to \( (\Theta \Theta \cdot \Theta B)(\Theta E \cdot \Theta B) \). And each of \( \Theta \Theta \), \( \Theta B \), \( \Theta E \), \( \Theta E \) is given; for \( \Theta \Theta \) is half of the perimeter of the triangle \( \Delta \Theta \), while \( \Theta B \) is the excess of half the perimeter over \( \Gamma \Theta \), \( \Theta E \) is the excess of half the perimeter over \( \Delta \Theta \), and \( \Theta E \) is the excess of half the perimeter over \( \Delta \Theta \), inasmuch as \( \Theta E = \Delta \Theta^2 \), \( \Theta B = \Delta \Theta = \AZ \). Therefore the area of the triangle \( \Delta \Theta \) is given. 

(ii.) Volume of a Spire

*Ibid.* ii. 13, ed. H. Schöne (Heron iii.) 126. 10–130. 3

Let \( \Delta \Theta \) be any straight line in a plane and \( \Delta \Theta \), \( \Delta \Theta \) any two points taken on it. Let the circle \( \Delta \Theta \Delta \Theta \) be taken perpendicular to the plane of the horizontal, in which lies the straight line \( \Delta \Theta \), and, while the point

- If the sides of the triangle are \( a, b, c \), and \( s = \frac{1}{2}(a + b + c) \), Heron's formula may be stated in the familiar terms,

\[
\text{area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}.
\]

Heron also proves the formula in his *Dioptra* 30, but it is now known from Arabian sources to have been discovered by Archimedes.
σημείου περιφερέσθω κατὰ τὸ ἐπίπεδον ἤ ΑΒ, ἀχρι οὗ εἰς τὸ αὐτὸ ἀποκατασταθῇ συμπεριφερομένου καὶ τοῦ ΒΓΔΕ κύκλου ὀρθοῦ διαμένοντος πρὸς τὸ ύποκείμενον ἐπίπεδον. ἀπογεννήσει ἄρα τινὰ ἐπιφάνειαν ἢ ΒΓΔΕ περιφέρεια, ἢν ἡ σπειρικήν καλούσιν· καὶ μὴ ἢ ἢ δὲ ὅλος ὁ κύκλος, ἀλλὰ τμῆμα αὐτοῦ, πάλιν ἀπογεννήσει τὸ τοῦ κύκλου τμῆμα σπειρικῆς ἐπιφάνειας τμῆμα, καθ' ἀπερ εἰς καὶ αἱ ταῖς κύσιν ύποκείμεναι σπειραί· τριῶν γὰρ οὐσῶν ἐπιφάνειῶν ἐν τῷ καλουμένῳ ἀναγραφεῖ, ὃν δὴ τινὲς καὶ ἐμβολέα καλοῦσιν, δῦο μὲν κοιλων τῶν ἄκρων, μιᾶς δὲ μέσης καὶ κυρτῆς, ἀμα περιφερόμεναι αἱ τρεῖς ἀπογεννῆσι τὸ εἶδος τῆς τοῖς κύσιν ύποκειμένης σπειρας.

Δέον οὖν ἔστω τὴν ἀπογεννηθεῖσαν σπειραν ύπὸ τοῦ ΒΓΔΕ κύκλου μετρῆσαι. δεδοσθω ἢ μὲν ΑΒ μονάδων ἢ, ἢ δὲ ΒΓ διάμετρος μονάδων ἢβ.

εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Ζ, καὶ ἀπὸ τῶν Α, Ζ τῶ ύποκειμένω ἐπιπέδῳ πρὸς ὀρθὰς ἄχθωσαν αἱ ΔΖΕ, ΧΑΘ. καὶ διὰ τῶν Δ, Ε τῇ ΑΒ παράλ-478
A remains stationary, let $AB$ revolve in the plane until it concludes its motion at the place where it started, the circle $B\Gamma\Delta E$ remaining throughout perpendicular to the plane of the horizontal. Then the circumference $B\Gamma\Delta E$ will generate a certain surface, which is called spiric; and if the whole circle do not revolve, but only a segment of it, the segment of the circle will again generate a segment of a spiric surface, such as are the spirae on which columns rest; for as there are three surfaces in the so-called anagrapheus, which some call also emboleus, two concave (the extremes) and one (the middle) convex, when the three are moved round simultaneously they generate the form of the spira on which columns rest.\(^a\)

Let it then be required to measure the spire generated by the circle $B\Gamma\Delta E$. Let $AB$ be given as 20, and the diameter $B\Gamma$ as 12. Let $Z$ be the centre of the circle, and through $A$, $Z$ let $HA\Theta$, $\Delta ZE$ be drawn perpendicular to the plane of the horizontal. And through $\Delta$, $E$ let $\Delta H$, $E\Theta$ be drawn parallel to

\(^a\) The ἀναγραφεύς or ἐμβολεύς is the pattern or templet for applying to an architectural feature, in this case an Attic-Ionic column-base. The Attic-Ionic base consists essentially of two convex mouldings, separated by a concave one. In practice, there are always narrow vertical ribbons between the convex mouldings and the concave one, but Heron ignores them. In the templet, there are naturally two concave surfaces separated by a convex, and the kind of figure Heron had in mind appears to be that here illustrated. I am indebted to Mr. D. S. Robertson, Regius Professor of Greek in the University of Cambridge, for help in elucidating this passage.

\(^b\) Lit. "from."
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ληλου ἡχωσαν αἱ ΔΗ, ΕΘ. δέδεικται δὲ Διονυσodorφων ἐν τῷ Περὶ τῆς σπείρας ἐπιγραφομένω, ὅτι ὃν λόγον ἔχει ὁ ΒΓΔΕ κύκλος πρὸς τὸ ἡμισιν τοῦ ΔΕΘο παραλληλογράμμου, τούτον ἔχει καὶ ἡ γεννηθεῖσα σπείρα ύπο τοῦ ΒΓΔΕ κύκλου πρὸς τὸν κύλινδρον, οὐ ἄξων μὲν ἐστὶν ὁ ΗΘ, ἡ δὲ ἐκ τοῦ κέντρου τῆς βάσεως ἡ ΕΘ. ἐπεὶ οὖν ἡ ΒΓ μονάδων πῆς ἐστίν, ἡ ἀρα ΖΓ ἐσται μονάδων ἤ. ἐστὶ δὲ καὶ ἡ ΑΓ μονάδων ἤ. ἐσται ἀρα ἡ ΑΖ μονάδων ἢ, τουτέστω τὴ ἙΘ, ἦτις ἐστὶν ἐκ τοῦ κέντρου τῆς βάσεως τοῦ εἰρημένου κυλίνδρου, δοθεὶς ἀρα ἐστὶν ὁ κύκλος· ἀλλὰ καὶ ὁ ἄξων δοθεὶς· ἐστὶ γὰρ μονάδων πῆς, ἐπεὶ καὶ ἡ ΔΕ. ὡστε δοθεὶς καὶ ὁ εἰρημένος κυλίνδρος· καὶ ἐστι τὸ ΔΘ παραλληλόγραμμον θ.' δοθε'ν. ὡστε καὶ τὸ ἡμισιν αὐτοῦ· ἀλλὰ καὶ ὁ ΒΓΔΕ κύκλος· δοθεῖσα γὰρ ἡ ΓΒ διάμετρος. λόγος ἀρα τοῦ ΒΓΔΕ κύκλου πρὸς τὸ ΔΘ παραλληλόγραμμον δοθείς· ὡστε καὶ τῆς σπείρας πρὸς τὸν κύλινδρον λόγος ἐστὶ δοθείς· καὶ ἐστὶ δοθείς ὁ κύλινδρος· δοθε'ν ἀρα καὶ τὸ στερεὸν τῆς σπείρας.

Συνεθήσεται δὴ ἀκολούθως τῇ ἀναλύσει οὕτως. ἀφελε ἀπὸ τῶν κ τὰ πῆς λοιπὰ ἦ, καὶ πρόσθες τὰ κ. γίγνεται κῆ. καὶ μέτρησον κύλινδρον, οὐ ἡ μὲν διάμετρος τῆς βάσεως ἐστὶ μονάδων κῆ, τὸ δὲ ύψος πῆς καὶ γίγνεται τὸ στερεὸν αὐτοῦ ,π. καὶ μέτρησον κύκλον, οὐ διάμετρος ἐστὶ μονάδων πῆς. γίγνεται τὸ ἐμβαδόν αὐτοῦ, καθώς ἐμάθομεν, ἀργ ζ' καὶ λαβὴ τῶν κῆ τὸ ἡμισιν γίγνεται ἢ. ἐπὶ τὸ ἡμισιν τῶν πῆς γίγνεται π. καὶ πολλα-

1 δοθε'ν add. H. Schöne.
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AB. Now it is proved by Dionysodorus in the book which he wrote On the Spire that the circle $\Gamma\Delta\Gamma$ bears to half of the parallelogram $\Delta\Theta\Theta$ the same ratio as the spire generated by the circle $\Gamma\Delta\Gamma$ bears to the cylinder having $\Theta\Theta$ for its axis and $\Theta\Theta$ for the radius of its base. Now, since $\Gamma\Gamma$ is 12, $Z\Gamma$ will be 6. But $A\Gamma$ is 8; therefore $AZ$ will be 14, and likewise $E\Theta$, which is the radius of the base of the aforesaid cylinder. Therefore the circle is given; but the axis is also given; for it is 12, since this is the length of $\Delta\Theta$. Therefore the aforesaid cylinder is also given; and the parallelogram $\Delta\Theta$ is given, so that its half is also given. But the circle $\Gamma\Delta\Gamma$ is also given; for the diameter $\Gamma\Gamma$ is given. Therefore the ratio of the circle $\Gamma\Delta\Gamma$ to the parallelogram is given; and so the ratio of the spire to the cylinder is given. And the cylinder is given; therefore the volume of the spire is also given.

Following the analysis, the synthesis may thus be done. Take 12 from 20; the remainder is 8. And add 20; the result is 28. Let the measure be taken of the cylinder having for the diameter of its base 28 and for height 12; the resulting volume is 7392. Now let the area be found of a circle having a diameter 12; the resulting area, as we learnt, is 1134.1. Take the half of 28; the result is 14. Multiply it by the half of 12; the result is 84. Now multiply

* For Dionysodorus v. supra, p. 162 n. a and p. 364 n. a.

If $\Delta\Theta = H\Theta = 2r$ and $E\Theta = a$, then the volume of the spire bears to the volume of the cylinder the ratio $2\pi a : 2r \cdot \pi a^2$ or $\pi r : a$, which, as Dionysodorus points out, is identical with the ratio of the circle to half the parallelogram, that is, $\pi r^2 : ra$ or $\pi r : a$. 481
πλαισίασας τὰ ἕπὶ τὰ πίν 
καὶ τὰ γενόμενα

παράβαλε παρά τὸν πίν. γίγνεται, θ' νυς ἕκτον \( \zeta \) τοσ-

ούτου ἐσται τὸ στερεὸν τῆς σπείρας.

(iii.) \textit{Division of a Circle}

\textit{Ibid. iii. 18, ed. H. Schöne (Heron iii.)} 172. 13–174. 2

Τὸν δοθέντα κύκλον διελεῖν εἰς τρία ἵσα δυσὶν
eὐθείας. τὸ μὲν οὖν πρόβλημα ὅτι οὐ βητῶν ἔστι,
δῆλον, τῆς εὐχρηστίας δὲ ἐνεκεν διελοῦμεν αὐτῶν

ὡς ἐγγιστά ὦτῳ. ἐστω ὦ δοθεῖς κύκλος, οὐ
cέντρων τὸ Α, καὶ ἐνημόσθω eἰς αὐτῶν τρίγωνον

ιούπλευρον, οὐ πλευρὰ ἡ ΒΓ, καὶ παράλληλος αὐτῇ

ηχθῶν ἡ ΔΑΕ καὶ ἐπεζεύχῳσαν αἱ ΒΔ, ΔΓ. λέγω,

ὅτι τὸ ΔΒΓ τμήμα τρίτων ἐγγιστὰ ἐστὶ μέρος τοῦ

ὁλου κύκλου. ἐπεζεύχῳσαν γὰρ αἱ ΒΑ, ΑΓ. ὦ

ἄρα ΑΒΓΖΒ τομεὺς τρίτων ἐστὶ μέρος τοῦ ὥλου

κύκλου. καὶ ἐστών ἴσων τὸ ΑΒΓ τρίγωνον τῷ

ΒΔΓ τριγώνως· τὸ ἄρα ΒΔΓΖ σχήμα τρίτων

μέρος ἐστὶ τοῦ ὥλου κύκλου, ὥ δὴ μείζον ἐστὶν

αὐτοῦ τὸ ΔΒΓ τμήμα ἀνεπαισθήτου ὡντος ὡς

πρὸς τὸν ὥλον κύκλου. ὁμοίως δὲ καὶ ἐτέραν

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7392 by $113\frac{1}{7}$ and divide the product by 84; the result is $9956\frac{4}{7}$. This will be the volume of the spire.

(iii.) *Division of a Circle*

*Ibid. iii. 18, ed. H. Schöne (Heron iii.) 172. 13–174. 2*

*To divide a given circle into three equal parts by two straight lines.* It is clear that this problem is not rational, and for practical convenience we shall make the division as closely as possible in this way. Let the given circle have A for its centre, and let there be inserted in it an equilateral triangle with side $B\Gamma$, and let $\Delta AE$ be drawn parallel to it, and let $B\Delta, \Delta \Gamma$

be joined. I say that the segment $\Delta B\Gamma$ is approximately a third part of the whole circle. For let $BA, AG$ be joined. Then the sector $AB\Gamma ZB$ is a third part of the whole circle. And the triangle $AB\Gamma$ is equal to the triangle $B\Gamma \Delta$ [Eucl. i. 37]; therefore the figure $B\Delta \Gamma Z$ is a third part of the whole circle, and the excess of the segment $\Delta B\Gamma$ over it is negligible in comparison with the whole circle. Similarly, if we
(iv.) *Measurement of an Irregular Area*

Heron, *Diopt.* 23, ed. H. Schöne (Heron iii.) 260. 18–264. 15

To dothèn χωρίον μετρήσαι διὰ διόπτρας. Ἐστώ τὸ δοθὲν χωρίον περιεχόμενον ὑπὸ γραμμῆς ἀτάκτου τῆς ΑΒΓΔΕΖΗΘ. Ἐπεὶ οὖν ἐμάθομεν διὰ τῆς κατάσκευασθείσης διόπτρας διάγειν πάση τῇ δοθείσῃ εὐθείᾳ. (ἔτεραν)² πρὸς ὀρθάς, ἐλαβόν τι σημεῖον ἐπὶ τῆς περιεχούσης τὸ χωρίον γραμμῆς 484
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inscribe another side of the equilateral triangle, we may take away another third part; and therefore the remainder will also be a third part of the whole circle.a

(iv.) Measurement of an Irregular Area

Heron, Diphra b 23, ed. H. Schöne (Heron iii.) 260. 18–264. 15

To measure a given area by means of the dioptra. Let the given area be bounded by the irregular line ABΓΔEZΘ. Since we learnt to draw, by setting the dioptra, a straight line perpendicular to any other straight line, I took any point B on the line en-

a Euclid, in his book On Divisions of Figures which has partly survived in Arabic, solved a similar problem—to draw in a given circle two parallel chords cutting off a certain fraction of the circle; Euclid actually takes the fraction as one-third. The general character of the third book of Heron'sMetrics is very similar to Euclid's treatise.

It is in the course of this book (iii. 20) that Heron extracts the cube root of 100 by a method already noted (vol. i. pp. 60–63).

b The dioptra was an instrument fulfilling the same purposes as the modern theodolite. An elaborate description of the instrument prefaces Heron's treatise on the subject, and it was obviously a fine piece of craftsmanship, much superior to the "parallactic" instrument with which Ptolemy had to work—another piece of evidence against an early date for Heron.
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tο Β, καὶ ἦγαγον εὐθεῖαν τυχόσαν διὰ τῆς διόπτρας τὴν ΒΗ, καὶ ταύτη πρὸς ὀρθᾶς τὴν ΒΓ, \(\text{kai tau'th} \)\(^1\) ἔτεραν πρὸς ὀρθὰς τὴν ΓΖ, καὶ ὀμοίως τῇ ΓΖ πρὸς ὀρθᾶς τὴν ΖΘ. καὶ ἔλαβον ἐπὶ τῶν ἀχθεισῶν εὐθείαν συνεχῆ σημεῖα, ἐπὶ μὲν τῆς ΒΗ τὰ Κ, Λ, Μ, Ν, Ε, Θ. ἔπι δὲ τῆς ΒΓ τὰ Π, Ρ. ἔπι δὲ τῆς ΓΖ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, Ω. ἔπι δὲ τῆς ΖΘ τὰ ζ, γ. καὶ ἀπὸ τῶν ληφθέντων σημείων ταῖς εὐθείαις, ἐφ' ἄν ἐστὶ τὰ σημεῖα, πρὸς ὀρθᾶς ἦγαγον τὰς Κλ, ΔΑ, ΜΑ, ΝΒ, Ε,Γ, ΟΔ, ΠΕ, ΡΣ, ΣΖ, ΤΗ, ΥΘ, ΦΔ, ΧΜ, ΨΜ, ΩΕ, ΣΜ, ΣΜ ὁποῖς ὁπλε [τὰς ἔπι]\(^2\) τὰ πέρατα τῶν ἀχθεισῶν πρὸς ὀρθᾶς [ἐπιζευγνυμένας]\(^3\) ἀπολαμβάνειν γραμμάς ἀπὸ τῆς περιεχούσης τὸ χωρίον γραμμῆς σύνεγγυς εὐθείας: καὶ τούτων γενηθέντων ἔσται δυνατὸν τὸ χωρίον μετρεῖν. τὸ μὲν γὰρ ΒΓΖΜ παραλληλόγραμμον ὀρθογώνιον ἐστιν· ἔπειτα τὰς πλευρὰς ἀλύσει ἡ σχοινίω βεβασανισμένως, τοιτεστίν μήτε ἐκτείνεσθαι μήτε συστέλλεσθαι δυναμένως, μετρήσαντες ἐξομεν τὸ ἐμβαδὸν τοῦ παραλληλογράμμου. τὰ δ' ἐκτὸς τούτων τρίγωνα ὀρθογώνια καὶ τριπέζια ὀμοίως μετρήσομεν, ἔχοντες τὰς πλευρὰς αὐτῶν· ἔσται γὰρ τρίγωνα μὲν ὀρθογώνια τὰ ΒΚλ, ΒΠΕ, ΓΡΣ, ΓΣΖ, ΖΩΕ, ΖΣΜ, ΘΗΜ· τὰ δὲ λοιπὰ τριπέζια ὀρθογώνια. τὰ μὲν οὖν τρίγωνα μετρεῖται τῶν περὶ τὴν ὀρθὴν γωνίαν πολλαπλασιαζομένων ἐπὶ ἄλληλα· καὶ τοῦ γενομένου τὸ ἡμιον. τὰ δὲ τριπέζια· συναμφοτέρων τῶν παραλληλῶν τὸ ἡμιον ἔπὶ τὴν ἐπὶ αὐτῶς κάθετον οὖσαν, οἶον 486
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closing the area, and by means of the dioptra drew any straight line BH, and drew BG perpendicular to it, and drew another straight line GZ perpendicular to this last, and similarly drew ZΘ perpendicular to GZ. And on the straight lines so drawn I took a series of points—on BH taking K, Λ, M, N, \( \Xi \), O, on BG taking Π, P, on GZ taking Σ, T, Υ, Φ, X, Ψ, Ω, and on ZΘ taking \( \zeta \), \( \zeta' \). And from the points so taken on the straight lines designated by the letters, I drew the perpendiculars \( K\gamma \), ΛΛ, M, A, N, Β, \( \Xi, \Gamma \), O, Α, Π, E, P, \( \zeta' \), Σ, Z, T, H, ΥΘ, ΦΔ, XM, ΨΜ, ΩΕ, \( \zeta\)M, \( \zeta'\)M in such a manner that the extremities of the perpendiculars cut off from the line enclosing the area approximately straight lines. When this is done it will be possible to measure the area. For the parallelogram BGZM is right-angled; so that if we measure the sides by a chain or measuring-rod, which has been carefully tested so that it can neither expand nor contract, we shall obtain the area of the parallelogram. We may similarly measure the right-angled triangles and trapezia outside this by taking their sides; for BK\( \gamma \), BΠE, \( \Gamma P\zeta' \), \( \Gamma Z\zeta \), ZΩE, \( Z\zeta\)M, ΘΗM are right-angled triangles, and the remaining figures are right-angled trapezia. The triangles are measured by multiplying together the sides about the right angle and taking half the product. As for the trapezia—take half of the sum of the two parallel sides and multiply it by the perpendicular upon

\[ \text{1} \ kαι \ ταύτη \ \text{add. H. Schöne.} \]
\[ \text{2} \ \tauάς \ επί \ \text{om. H. Schöne.} \]
\[ \text{3} \ \epsilonπιζευγνυμένας \ \text{om. H. Schöne.} \]
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tων Κ��, ΑΔ το ήμισυ ἐπὶ την ΚΛ· καὶ των λοιπῶν δὲ ὁμοίως. ἐσται ἄρα μεμετρημένον ὅλον τὸ χωρίον διὰ τε τοῦ μέσου παραλληλογράμμου καὶ τῶν ἐκτὸς αὐτῶν τριγώνων καὶ τραπέζων. ἐάν δὲ τύχῃ ποτε μεταξὺ αὐτῶν τῶν ἀκθεισῶν πρὸς ὅρθας ταῖς τοῦ παραλληλογράμμου πλευραῖς καμπύλη γραμμή μῆς συνεγγίζουσα εὐθεία (ὅλον μεταξὺ τῶν Ξ,Γ, ΟΔ γραμμή ἡ ,Γ,Δ), ἄλλα περιφερεῖ, μετρήσομεν οὔτως· ἀγαγόντες τῇ

Ο,Δ πρὸς ὅρθας τῆν ,ΔΜ, καὶ ἐπί αὐτῆς λαβόντες σημεῖα συνεχῆ τὰ Μ, Μ, καὶ ἀπ’ αὐτῶν πρὸς ὅρθας ἀγαγόντες τῇ Μ,Δ τὰς ΜΜ, ΜΜ, ὥστε τὰς μεταξὺ τῶν ἀκθεισῶν σύνεγγυς εὐθείας εἶναι, πάλιν μετρήσομεν τὸ τε ΜΕΟ,Δ παραλληλόγραμμον καὶ τὸ ΜΜ,Δ τρίγωνον, καὶ τὸ ΓΜΜΜ τραπέζιον, καὶ ἐτὶ τὸ ἐτερον τραπέζιον, καὶ ἔξομεν τὸ περιεχόμενον χωρίον ὑπὸ τε τῆς ΓΜΜ,Δ γραμμῆς καὶ τῶν ,ΓΕ, <ΞΟ,> Ω,Δ εὐθειῶν μεμετρημένον.

(c) Mechanics

Heron, Diopt. 37, ed. H. Schöne (Heron iii.) 306. 22-312. 22

Τῇ δοθεῖσθαι δυνάμει τὸ δοθὲν βάρος κινῆσαι

1 τῇ add. H. Schöne. 2 ΞΟ add. H. Schöne.

* Heron's Mechanics in three books has survived in Arabic, but has obviously undergone changes in form. It begins with the problem of arranging toothed wheels so as to move 488
them, as, for example, half of $K\vartriangleright$, $\Delta \Lambda$ by $K\Lambda$; and similarly for the remainder. Then the whole area will have been measured by means of the parallelogram in the middle and the triangles and trapezia outside it. If perchance the curved line between the perpendiculars drawn to the sides of the parallelogram should not approximate to a straight line (as, for example, the curve $\Gamma, \Delta$ between $\Xi, \Gamma, \mathrm{O}, \Delta$), but to an arc, we may measure it thus: Draw $\Delta M$ perpendicular to $\mathrm{O}, \Delta$, and on it take a series of points $\theta, \eta, \xi$, and from them draw $MM, MM$ perpendicular to $\mathrm{M}, \Delta$, so that the portions between the straight lines so drawn approximate to straight lines, and again we can measure the parallelogram $M\Xi O, \Delta$ and the triangle $MM, \Delta$, and the trapezium $\Gamma MM, \mathrm{M}$, and also the other trapezium, and so we shall obtain the area bounded by the line $\Gamma \mathrm{M}, \Delta$ and the straight lines $\Gamma \Xi, \Xi O, \mathrm{O}, \Delta$.

(c) Mechanics

Heron, *Dioptis* 37, ed. H. Schöne (Heron iii.) 306. 22–312. 22

*With a given force to move a given weight by the* a given weight by a given force. This account is the same as that given in the passage here reproduced from the *Dioptis*, and it is obviously the same as the account found by Pappus (viii. 19, ed. Hultsch 1060. 1–1068. 23) in a work of Heron's (now lost) entitled Βαροινεκός ("weight-lifter")—though Pappus himself took the ratio of force to weight as 4 : 160 and the ratio of successive diameters as 2 : 1. It is suggested by Heath (*H.G.M.* ii. 346–347) that the chapter from the
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διὰ τυμπάνων ὁδοντωτῶν παραθέσεως. κατεσκευάσθω πήγμα καθάπερ γλωσσόκομον ἐἰς τοὺς μακροὺς καὶ παραλλήλους τοῖχους διακείσθωσαν ἀξόνες παράλληλοι ἐαυτοῖς, ἐν διαστημάσι κείμενοι

&oacute;ς τά συμφυή αὐτοῖς ὁδοντωτά τύμπανα παρακείσθαι καὶ συμπεπλέχθαι ἀλλήλοις, καθα μέλλομεν δῆλοιν. ἔστω τὸ εἰρημένον γλωσσόκομον τὸ ΑΒΓΔ, ἐν ὃ ἀξῶν ἔστω διακείμενος, ὡς εἰρηται, καὶ δυνάμενος εὐλύτως στρέφεσθαι, ὅ ΕΖ. τούτω δὲ συμφύες ἔστω τύμπανον ὁδοντωμένον τὸ ἩΘ ἔχον τὴν διάμετρον, εἰ τούχοι, πενταπλασίονα 〈τῆς〉 τοῦ ΕΖ ἀξόνος διαμέτρου. καὶ ἵνα ἐπὶ παραδεῖγματος τὴν κατασκευὴν ποιησῶμεθα, ἔστω τὸ μὲν ἀγόμενον βάρος ταλάντων χιλίων, ἡ δὲ κινοῦσα δύναμις ἔστω ταλάντων ἐ, τοῦτοῦτω δὲ χωνῶν ἄνθρωπος ἡ παιδάριον, ὅστε δύνασθαι καὶ ἐαυτὸν ἄνευ μηχανῆς ἐλκειν τάλαντα ἐ. οὐκοῦν ἐὰν τὰ ἐκ τοῦ φορτίου ἐκδεδεμένα ὅπλα διὰ τῖνος ὑπής

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juxtaposition of toothed wheels. Let a framework be prepared like a chest; and in the long, parallel walls let there lie axles parallel one to another, resting at such intervals that the toothed wheels fitting on to them will be adjacent and will engage one with the other, as we shall explain. Let ΑΒΓΔ be the aforesaid chest, and let EZ be an axle lying in it, as stated above, and able to revolve freely. Fitting on to this axle let there be a toothed wheel ΗΘ whose diameter, say, is five times the diameter of the axle EZ. In order that the construction may serve as an illustration, let the weight to be raised be 1000 talents, and let the moving force be 5 talents, that is, let the man or slave who moves it be able by himself, without mechanical aid, to lift 5 talents. Then if the rope holding the load passes through some aperture in

Βαρούλκος was substituted for the original opening of the Mechanics, which had become lost.

Other problems dealt with in the Mechanics are the paradox of motion known as Aristotle’s wheel, the parallelogram of velocities, motion on an inclined plane, centres of gravity, the five mechanical powers, and the construction of engines. Edited with a German translation by L. Nix and W. Schmidt, it is published as vol. ii. in the Teubner Heron.

a Perhaps "rollers."

1 τῆς add. Vincentius.
οὖσας) ἐν τῷ ΑΒ τοῖχῳ ἐπειληθῇ περὶ τὸν ΕΖ ἀξονα (. . . . ) κατειλουμένα τὰ ἐκ τοῦ φορτίου ὅπλα κινήσει τὸ βάρος. ίνα δὲ κινηθῇ τὸ ΗΘ τύμπανον, ἔδει δυνάμει ὑπάρχειν πλέον ταλάντων διακοσίων, διὰ τὸ τὴν διάμετρον τοῦ τύμπανον τῆς διαμέτρου τοῦ ἀξονός, ὥς ὑπεθέμεθα, πενταπλῆς ἐν ταῖς τῶν ἐ δυνάμεων ἀποδείξεισιν. ἀλλ' (. . . . ) ἔχομεν τί τὴν δύναμιν ταλάντων διακοσίων, ἀλλὰ πέντε. γεγονέτω ὅτι ἔτερος ἰξον (παράλληλος) διακεί- μενος τῷ ΕΖ, ὁ ΚΛ, ἔχων συμφυές τύμπανον ωδοντωμένον τὸ ΜΝ. οὐδοντὼδες δὲ καὶ τὸ ΗΘ τύμπανον, ὥστε ἐναρμόζειν ταῖς οὐδοντώσει τοῦ ΜΝ τυμπάνου. τῷ δὲ αὐτῷ ἰξον τῷ ΚΛ συμφυές τύμπανον τὸ ΞΘ, ἔχων ὁμοιώς τὴν διάμετρον πενταπλασίονα τῆς τοῦ ΜΝ τυμπάνου διαμέτρου. διὰ δὴ τούτῳ δεήσει τὸν βουλόμενον κίνησι διὰ τοῦ ΞΘ τυμπάνου τὸ βάρος ἔχειν δύναμιν ταλάντων μ., ἐπειδήπερ τῶν ὁ ταλάντων τὸ πέμπτον ἐστὶ ταλάντα μ. πάλιν οὖν παρακείσθω (τῷ ΞΘ τυμπάνῳ ωδοντωμένῳ) τύμπανον οὐδοτωθέν ἔτερ- ρον (τῷ ΠΡ, καὶ ἔστω τῷ) τύμπανο ωδοντωμένῳ τῷ ΠΡ συμφυές ἔτερον τύμπανον τὸ ΣΤἔχον ὁμοιῶς πενταπλῆ ἡ διάμετρον τῆς ΠΡ τυμπάνου διαμέτρου. καὶ ἐδὲ ἀ(νάλογος ἐσται δύναμις) τοῦ ΣΤ τυμπάνου ἢ ἔχουσα τὸ βάρος ταλάντων ἦ.

1 ὁπῆς ὁπῆς add. Hultsch et H. Schöne.
2 After ἰξον there is a lacuna of five letters.
3 τὰ ἐκ τοῦ φορτίου ὅπλα κινήσει τὸ βάρος H. Schöne, τὰ ἐκ τοῦ φορτίου ἐπλακών en τοῖς τῷ βάρος cod.
4 ἐδει δυνάμει—“septem litteris madore absumptis, supplevi dubitantem,” H. Schöne.
5 εἶναι add. H. Schöne.

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the wall AB and is coiled round the axle EZ, the rope holding the load will move the weight as it winds up. In order that the wheel HΘ may be moved, a force of more than 200 talents is necessary, owing to the diameter of the wheel being, as postulated, five times the diameter of the axle; for this was shown in the proofs of the five mechanical powers. We have [not, however . . .] a force of 200 talents, but only of 5. Therefore let there be another axle KA, lying parallel to EZ, and having the toothed wheel MN fitting on to it. Now let the teeth of the wheel HΘ be such as to engage with the teeth of the wheel MN. On the same axle KA let there be fitted the wheel ΞO, whose diameter is likewise five times the diameter of the wheel MN. Now, in consequence, anyone wishing to move the weight by means of the wheel ΞO will need a force of 40 talents, since the fifth part of 200 talents is 40 talents. Again, then, let another toothed wheel ΠΠ lie alongside the toothed wheel ΞO, and let there be fitted to the toothed wheel ΠΠ another toothed wheel ΣΤ whose diameter is likewise five times the diameter of the wheel ΠΠ; then the force needing to be applied to the wheel ΣΤ will be 8 talents; but the force actually available

The wheel and axle, the lever, the pulley, the wedge and the screw, which are dealt with in Book ii. of Heron’s Mechanics.

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6 After ἀλλ’ is a special sign and a lacuna of 22 letters.
7 παράλληλος add. H. Schöne.
8 τῷ ΞΟ τυμπάνῳ ὀδοντωμένῳ add. H. Schöne.
9 τῷ ΠΠ, καὶ ἔστω τῷ add. H. Schöne.
10 τύμπανον τῷ ΣΤ, so I read in place of the συμφώνει in Schöne’s text.
11 ἀνάλογος ἔσται δύναμι—so H. Schöne completes the lacuna.
GRÈEK MATHEMATICS

άλλη ἡ ὑπάρχουσα ἡμῖν δύναμις δέδοται ταλάντων ἐκ. ὀμοίως ἔτερον παρακείσθω τύμπανον ὁδοντωμένον τὸ ΨΤ τῷ ὁδοντωθέντι· τούτῳ τοῦ ΨΤ τυμπάνον τῷ ἄξονι συμφυές ἐστω τύμπανον τὸ ΧΨ ὁδοντωμένον, ὅλῳ ἡ διάμετρος πρὸς τὴν τοῦ ΨΤ τυμπάνου διάμετρον λόγον ἔχετω, ὅν τὰ ὀκτὼ τάλαντα πρὸς τὰ τῆς δοθείσης δυνάμεως τάλαντα ἐκ.

Καὶ τούτων παρασκευασθέντων, εὰν ἐπινοήσωμεν τὸ ΑΒΓΔ ἡγλωσσόκομον ᾗ μετέωρων κείμενον, καὶ ἐκ μὲν τοῦ ΕΖ ἄξονος τὸ βάρος ἐξαίφωμεν, ἐκ δὲ τοῦ ΧΨ τυμπάνου τὴν ἐλκουσαι δύναμιν, ὀυδοπότερον αὐτῶν κατενεχθῆσθαι, εὐλύτως στρεφόμενων τῶν ἄξωνων, καὶ τῆς τῶν τυμπάνων παραθέσεως καλῶς ἀρμοζοῦσης, ἀλλ' ὅσπερ ξυγοῦ τινος ἱσορροπήσει ἡ δύναμις τῶ βάρει. εὰν δὲ ἐνὶ αὐτῶν προσθώμεν ολίγον ἐτερον βάρος, καταρρέσει καὶ ἐνεχθῆσθαι ἐφ' ὁ προσεθῆ βάρος, ώστε εὰν ἐν τῶν ἐ ταλάντων δυνάμει 〈. . . . . .〉2 εἰ τύχωι μναίαιον προσεθῆ βάρος, κατακρατήσει καὶ ἐπιστάπεται τὸ βάρος. ἀντὶ δὲ τῆς προσθέσεως τούτων παρακείσθω κοιχίας ἐξων τὴν ἐλκα ἀρμοστὴν τῶν ὀδοῦ τοῦ τυμπάνου, στρεφόμενος εὐλύτως περὶ τόρμους ἐνότας ἐν τρήμασι στρογγυλοῖς, ὃν ὁ μὲν ἔτερος ὑπερεχέτω εἰς τὸ ἐκτὸς μέρος τοῦ γλωσσοκόμου κατὰ τὸν ΓΔ ἡγλωσσὸν τῶν παρακείμενον, τῷ κοιχίας· ἡ ἀρα ὑπεροχὴ τετραγωνισθεία λαβέτω χειρολάβην τὴν ξυγοῦ, δι' ἡς ἐπιλαμβανόμενος τις καὶ ἐπιστρέφων ἐπιστρέφει τὸν κοιχίαν καὶ τὸ ΧΨ τύμπανον, ώστε καὶ τὸ ΨΤ συμφυές αὐτῷ. διὰ δὲ τούτῳ καὶ τὸ παρακείμενον τὸ ΓΔ ἐπιστραφῆςται, καὶ τὸ συμφυές αὐτῷ τὸ ΠΡ, καὶ τὸ τούτω παρακείμενον τὸ ΞΘ, 494
to us is 5 talents. Let there be placed another toothed wheel $\Psi$ engaging with the toothed wheel $\Sigma T$; and fitting on to the axle of the wheel $\Psi$ let there be a toothed wheel $X\Psi$, whose diameter bears to the diameter of the wheel $\Psi$ the same ratio as 8 talents bears to the given force 5 talents.

When this construction is done, if we imagine the chest $A\Gamma\Delta$ as lying above the ground, with the weight hanging from the axle $EZ$ and the force raising it applied to the wheel $X\Psi$, neither of them will descend, provided the axles revolve freely and the juxtaposition of the wheels is accurate, but as in a beam the force will balance the weight. But if to one of them we add another small weight, the one to which the weight was added will tend to sink down and will descend, so that if, say, a mina is added to one of the 5 talents in the force it will overcome and draw the weight. But instead of this addition to the force, let there be a screw having a spiral which engages the teeth of the wheel, and let it revolve freely about pins in round holes, of which one projects beyond the chest through the wall $\Gamma\Delta$ adjacent to the screw; and then let the projecting piece be made square and be given a handle $\zeta$. Anyone who takes this handle and turns, will turn the screw and the wheel $X\Psi$, and therefore the wheel $\Psi$ joined to it. Similarly the adjacent wheel $\Sigma T$ will revolve, and $\Pi\Pi$ joined to it, and then the adjacent wheel $\Xi O$, and then $MN$ fitting

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1 γλωσσόκομον add. H. Schöne.
2 After δυνάμει is a lacuna of seven letters.
3 In Schöne's text δὲ is printed after τοῦτῳ.
4 τοιχον τον παρακέλμενον add. H. Schöne.
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(d) Optics: Equality of Angles of Incidence and Reflection


'Απεδείξε γάρ ὁ μηχανικὸς Ἡρων ἐν τοῖς αὐτοῦ Κατοπτρικοῖς, ὅτι αἱ πρὸς ἑσας γωνίας κλώμεναι εὐθεῖαι ἐλάχισται εἰςι πασῶν τῶν ἀπὸ τῆς αὐτῆς καὶ ὀμοιομερῶς γραμμῆς πρὸς τὰ αὐτὰ κλωμένων [πρὸς ἀνίσους γωνίας]. Τοῦτο δὲ ἀποδείξας φησίν ὅτι εἰ μὴ μέλλοι ἡ φύσις μάτην περιάγειν τὴν ἠμετέραν ὄμην, πρὸς ἑσας αὐτὴν ἀνακλάσει γωνίας.

Olympiod. In Meteor. iii. 2 (Aristot. 371 b 18), ed. Stüve 212. 5-213. 21

'Επειδὴ γάρ τοῦτο ὀμολογημένον ἔστι παρὰ πᾶσιν, ὅτι οὐδὲν μάτην ἐργάζεται ἡ φύσις οὐδὲ ματαιοπονεῖ, ἕαν μὴ ὅσωσμεν πρὸς ἑσας γωνίας γίνεθαι τὴν ἀνάκλασιν, πρὸς ἀνίσους ματαιοπονεῖ

1 τῆν add. H. Schöne.
2 πασῶν G. Schmidt, τῶν μέσων codd.
3 πρὸς ἀνίσους γωνίας om. R. Schöne.
on to this last, and then the adjacent wheel ΗΘ, and so finally the axle EZ fitting on to it; and the rope, winding round the axle, will move the weight. That it will move the weight is obvious because there has been added to the one force that moving the handle which describes a circle greater than that of the screw; for it has been proved that greater circles prevail over lesser when they revolve about the same centre.

(d) Optics: Equality of Angles of Incidence and Reflection

Damianus, *On the Hypotheses in Optics 14*,
ed. R. Schöne 20. 12-18

For the mechanician Heron showed in his Catoptrica that of all [mutually] inclined straight lines drawn from the same homogenous straight line [surface] to the same [points], those are the least which are so inclined as to make equal angles. In his proof he says that if Nature did not wish to lead our sight in vain, she would incline it so as to make equal angles.

Olympiodorus, *Commentary on Aristotle’s Meteora iii. 2* (371 b 18), ed. Stüve 212. 5-213. 21

For this would be agreed by all, that Nature does nothing in vain nor labours in vain; but if we do not grant that the angles of incidence and reflection are equal, Nature would be labouring in vain by following

* Damianus, or Heliodorus, of Larissa (date unknown) is the author of a small work on optics, which seems to be an abridgement of a large work based on Euclid’s treatise. The full title given in some mss.—Δαμιανοῦ φιλοσόφου τοῦ Ἡλιοδώρου Δαρίσατον Περὶ ὀπτικῶν ὑποθέσεων βιβλία β leaves uncertain which was his real name.
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η φύσις, καὶ ἀντὶ τοῦ διὰ βραχείας περιόδου φθάσαι τὸ ὀρώμενον τὴν ὀψιν, διὰ μακρᾶς περιόδου τοῦτο φανῆσεται καταλαμβάνουσα.1 εὑρέθησονται γὰρ αἱ τοὺς ἁνίσοντας γωνίας περιέχουσαι εὐθείας, αἵτινες ἀπὸ τῆς ὀψεως [περιέχουσαι]2 φέρονται3 πρὸς τὸ κάτωπτρον κάκειθεν πρὸς τὸ ὀρώμενον, μείζονες οὐσα τῶν τὰς ἴσας γωνίας περιεχουσῶν εὐθείῶν. καὶ ὥστε τοῦτο ἀληθὲς, δῆλον ἐντεῦθεν.

Ὑποκείσθω γὰρ τὸ κάτωπτρον εὐθεία τῆς Ἡ ἈΒ, καὶ ἔστω τὸ μὲν ὀρῶν Γ, τὸ δὲ ὀρώμενον τὸ Δ, τὸ δὲ Ε σημεῖον τοῦ κατόπτρου, ἐν ᾗ προσπίπτουσα ἡ ὀψις ἀνακλάται πρὸς τὸ ὀρώμενον, ἔστω,

καὶ ἐπεζεύχθω ἡ ΓΕ, ΕΔ. λέγω ὅτι ἡ ὑπὸ ΑΕΓ γωνία ἴση ἐστὶ τῇ ὑπὸ ΔΕΒ.

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unequal angles, and instead of the eye apprehending the visible object by the shortest route it would do so by a longer. For straight lines so drawn from the eye to the mirror and thence to the visible object as to make unequal angles will be found to be greater than straight lines so drawn as to make equal angles. That this is true, is here made clear.

For let the straight line AB be supposed to be the mirror, and let Γ be the observer, Δ the visible object, and let E be a point on the mirror, falling on which the sight is bent towards the visible object, and let ΓE, EΔ be joined. I say that the angle AEG is equal to the angle ΔEB.  

* Different figures are given in different mss., with corresponding small variants in the text. With G. Schmidt, I have reproduced the figure in the Aldine edition.

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1 καταλαμβάνουσα om. Ideler.
2 περιέχουσαι om. R. Schöne, περιέχουσαi Ideler, Stüve.
3 φέρονται R. Schöne, φερομένας codd.
Εἰ γὰρ μὴ ἐστιν ἵση, ἐστω ἐτερον σημειον τοῦ κατόπτρου, ἐν ὀ προσπιτουσα ἢ ὁμις πρὸς ἀνίσους γωνίας ἀνακλᾶται, τὸ Ζ, καὶ ἐπεξεύρηκα γνωρίας μεῖζων ἐστὶ τῆς ὑπὸ ΔΖΕ γωνίας. λέγω ὅτι αἱ ΓΖ, ΖΔ εὐθεῖαι, αἰτινες τὰς ἀνίσους γωνίας περιέχουσιν ὑποκειμένης τῆς ΑΒ εὐθείας, μεῖζονες εἰς τῶν ΓΕ, ΕΔ εὐθείων, αἰτινες τὰς ἵσας γωνίας περιέχουσιν μετὰ τῆς ΑΒ. ἤξθω γὰρ κάθετος ἀπὸ τοῦ Δ ἐπὶ τῆν ΑΒ κατὰ τὸ Η σημεῖον καὶ ἐκβεβληθὼς ἐπὶ εὐθείας ὡς ἐπὶ τὸ Θ. φανερὸν δὴ ὅτι αἱ πρὸς τῷ Η γωνίαι ἵσαι εἰσὶν ὁρθαὶ γὰρ εἰσὶ. καὶ ἐστω ἡ ΔΗ τῇ ΗΘ ἵση, καὶ ἐπεξεύρηκα γνωρίας ΘΖ καὶ ἡ ΘΕ. αὐτὴ μὲν ἡ κατασκευὴ. ἐπεὶ οὖν ἵση ἐστὶν ἡ ΔΗ τῇ ΗΘ, ἀλλὰ καὶ ἡ ὑπὸ ΔΗΕ γωνία τῆς ὑπὸ ΘΗΕ γωνία ἵση ἐστὶ, κοινὴ δὲ πλευρά τῶν δύο τριγώνων ἡ ΗΕ, [καὶ βάσις ἡ ΘΕ βάσει τῇ ΕΔ ἵση ἐστὶ, καὶ] τῇ ΗΘΕ τρίγωνον τῷ ΔΗΕ τριγώνῳ ἵσον ἐστὶ, καὶ ἂι λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις εἰσίν ἵσαι, ὡς ὅ ἂς αἱ ἵσαι πλευραὶ ὑποτείνουσιν. ἵση ἀρα ἡ ΘΕ τῇ ΕΔ. πάλιν ἐπειδὴ τῇ ΗΘ ἵση ἐστὶν ἡ ΗΔ καὶ γωνία ἡ ὑπὸ ΔΗΖ γωνία τῆς ὑπὸ ΘΗΖ ἵση ἐστὶ, κοινὴ δὲ τῇ ΗΖ τῶν δύο τριγώνων τῶν ΔΗΖ καὶ ΘΗΖ, [καὶ βάσις ἀρα ἡ ΘΖ βάσει τῇ ΔΖ ἵσῃ ἐστὶ, καὶ] τῇ ΖΗΔ τρίγωνον τῷ ΘΗΖ τριγώνῳ ἵσον ἐστὶν. ἵση ἀρα ἐστὶν ἡ ΘΖ τῇ ΖΔ. καὶ ἐπεὶ ἵση ἐστὶν ἡ ΘΕ τῇ ΕΔ, κοινὴ προσκείνθω ἡ ΕΖ. δύο ἀρα αἱ ΓΕ, ΕΔ δυοι ταῖς ΓΕ, ΕΘ ἵσαι εἰσίν. ὅλη ἀρα ἡ ΓΘ δυοι ταῖς ΓΕ, ΕΔ ἵσῃ ἐστὶ. καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ
MENSURATION: HERON OF ALEXANDRIA

For if it be not equal, let there be another point Z, on the mirror, falling on which the sight makes unequal angles, and let \( \Gamma Z, Z \Delta \) be joined. It is clear that the angle \( \Gamma ZA \) is greater than the angle \( \Delta ZE \). I say that the sum of the straight lines \( \Gamma Z, Z \Delta \) which make unequal angles with the base line \( AB \), is greater than the sum of the straight lines \( \Gamma E, E \Delta \), which make equal angles with \( AB \). For let a perpendicular be drawn from \( \Delta \) to \( AB \) at the point \( H \) and let it be produced in a straight line to \( \Theta \).

Then it is obvious that the angles at \( H \) are equal; for they are right angles. And let \( \Delta H = H \Theta \), and let \( \Theta Z \) and \( \Theta E \) be joined. This is the construction. Then since \( \Delta H = H \Theta \), and the angle \( \Delta HE \) is equal to the angle \( \Theta HE \), while \( HE \) is a common side of the two triangles, the triangle \( H \Theta E \) is equal to the triangle \( \Delta HE \), and the remaining angles, subtended by the equal sides are severally equal one to the other [Eucl. i. 4]. Therefore \( \Theta E = E \Delta \). Again, since \( H \Delta = H \Theta \) and angle \( \Delta HZ = \) angle \( \Theta HZ \), while \( HZ \) is common to the two triangles \( \Delta HZ \) and \( \Theta HZ \), the triangle \( ZH \Delta \) is equal to the triangle \( \Theta HZ \) [ibid.]. Therefore \( \Theta Z = Z \Delta \). And since \( \Theta E = E \Delta \), let \( E \Gamma \) be added to both. Then the sum of the two straight lines \( \Gamma E, E \Delta \) is equal to the sum of the two straight lines \( \Gamma E, E \Theta \). Therefore the whole \( \Gamma \Theta \) is equal to the sum of the two straight lines \( \Gamma E, E \Delta \). And since in any triangle the sum of two sides is always greater than

1 \( \kappa\alpha \ldots \kappa\alpha \). These words are out of place here and superfluous.
2 \( \alpha \) add. Schmidt. But possibly \( \kappa\alpha \ldots \upsilon\pi\sigma\tau\epsilon\iota\nu\omega\nu\alpha\nu \), being superfluous, should be omitted.
3 \( \kappa\alpha \ldots \kappa\alpha \). These words are out of place here and superfluous.
GREEK MATHEMATICS

The proof here given appears to have been taken by Olympiodorus from Heron’s Catoptrica, and it is substantially identical with the proof in De Speculis 4. This work was formerly attributed to Ptolemy, but the discovery of Ptolemy’s Optics in Arabic has encouraged the belief, now...
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the remaining side, in whatever way these may be taken [Eucl. i. 20], therefore in the triangle $\Theta Z \Gamma$ the sum of the two sides $\Theta Z$, $Z \Gamma$ is greater than the one side $\Gamma \Theta$. But

$$\Gamma \Theta = \Gamma E + E \Delta;$$

$$\therefore \quad \Theta Z + Z \Gamma > \Gamma E + E \Delta.$$  

But

$$\Theta Z = Z \Delta;$$

$$\therefore \quad Z \Gamma + Z \Delta > \Gamma E + E \Delta.$$  

And $\Gamma Z$, $Z \Delta$ make unequal angles; therefore the sum of straight lines making unequal angles is greater than the sum of straight lines making equal angles; which was to be proved.  

(e) Quadratic Equations

Heron, Geometrica 21. 9-10, ed. Heiberg
(Heron iv.) 380. 15-31

Given the sum of the diameter, perimeter and area of a circle, to find each of them separately. It is done thus: Let the given sum be 212. Multiply this by 154; the result is 32648. To this add 841, making 33489, whose square root is 183. From this take away 29, leaving 154, whose eleventh part is 14; this will be the diameter of the circle. If you wish to find the circumference, take 29 from 183, leaving 154; double this, making 308, and take the seventh part, which is 44; this will be the perimeter. To usually held, that it is a translation of Heron's Catoptrica. The translation, made by William of Moerbeke in 1269, can be shown by internal evidence to have been made from the Greek original and not from an Arabic translation. It is published in the Teubner edition of Heron's works, vol. ii. part i.
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περίμετρος. τὸ δὲ ἐμβαδὸν εὑρεῖν. ποίει οὕτως. τὰ ἰδ τῆς διαμέτρου ἐπὶ τὰ μὲ ἥ τῆς περιμέτρου. γίνονται χις. τούτων λαβὲ μέρος τέταρτον· γίνονται ρῦδ. τοσοῦτον τὸ ἐμβαδὸν τοῦ κύκλου. ὁμοὶ τῶν τριῶν ἀριθμῶν μονάδες σιβ.

(f) Indeterminate Analysis

Heron, Geom. 24. 1, ed. Heiberg
(Heron iv.) 414. 28–415. 10

Εὑρεῖν δύο χωρία τετράγωνα, ὅπως τὸ τοῦ πρώτον ἐμβαδὸν τοῦ τοῦ δευτέρου ἐμβαδοῦ ἔσται τριπλάσιον. ποιῶ οὕτως· τὰ γ' κύβισον· γίνονται

*If* $d$ is the diameter of the circle, then the given relation is that

$$d + \frac{22}{7}d + \frac{11}{14}d^2 = 212,$$

i.e.

$$\frac{11}{14}d^2 + \frac{29}{7}d = 212.$$

To solve this quadratic equation, we should divide by $\frac{11}{14}$ so as to make the first term a square; Heron makes the first term a square by multiplying by the lowest requisite factor, in this case 154, obtaining the equation

$$11^2d^2 + 2 \cdot 29 \cdot 11d = 154 \cdot 212.$$

By adding 841 he completes the square on the left-hand side

$$(11d + 29)^2 = 154 \cdot 212 + 841$$

$$= 32648 + 841$$

$$= 33489.$$

∴

$$11d + 29 = 183.$$

∴

$$11d = 154,$$

and

$$d = 14.$$

The same equation is again solved in Geom. 24. 46 and a similar one in Geom. 24. 47. Another quadratic equation is solved in Geom. 24. 3 and the result of yet another is given in Metr. iii. 4.

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find the area. It is done thus: Multiply the diameter, 14, by the perimeter, 44, making 616; take the fourth part of this, which is 154; this will be the area of the circle. The sum of the three numbers is 212. 

(f) Indeterminate Analysis

Heron, Geometrica 24. 1, ed. Heiberg  
(Heron iv.) 414. 28–415. 10

To find two rectangles such that the area of the first is three times the area of the second. I proceed thus:

The Constantinople ms. in which Heron's Metrica was found in 1896 contains also a number of interesting problems in indeterminate analysis; and two were already extant in Heron's Geēponicus. The problems, thirteen in all, are now published by Heiberg in Heron iv. 414. 28–426. 29.

It appears also to be a condition that the perimeter of the second should be three times the perimeter of the first. If we substitute any factor $n$ for 3 the general problem becomes: To solve the equations

\[ u + v = n(x + y) \quad \vdots \quad (1) \]
\[ xy = n^2 uv \quad \vdots \quad (2) \]

The solution given is equivalent to

\[ x = 2n^3 - 1, \quad y = 2n^3 \]
\[ u = n(4n^3 - 2), \quad v = n. \]

Zeuthen (Bibliotheca mathematica, viii. (1907–1908), pp. 118–134) solves the problem thus: Let us start with the hypothesis that $v = n$. It follows from (1) that $u$ is a multiple of $n$, say $nz$. We have then

\[ x + y = 1 + z, \]
while by (2)

\[ xy = n^3z, \]
whence

\[ xy = n^3(x + y) - n^3 \]
or

\[ (x - n^3)(y - n^3) = n^3(n^3 - 1). \]

An obvious solution of this equation is

\[ x - n^3 = n^3 - 1, \quad y - n^3 = n^3, \]
which gives $z = 4n^3 - 2$, whence $u = n(4n^3 - 2)$. The other values follow.
ГРЕКСКИЕ МАТЕМАТИКИ

κλ. ταύτα δίς γίνονται νῦ. νῦν ἀρον μονάδα ἀ.
λοιπὸν γίνονται νῦ. ἐστώ οὖν ἡ μὲν μία πλευρά
ποδῶν νῦ, ἡ δὲ ἑτέρα πλευρά ποδῶν νῦ. καὶ τοῦ ἀλλοῦ χωρίου οὖτως: θές ὁμοί τὰ νῦ καὶ τὰ νῦ
γίνονται πόδες ρκ. ταύτα ποιεῖ ἐπὶ τὰ νῦ... λοιπὸν γίνονται πόδες τῆ. ἐστώ οὖν ἡ τοῦ προ-
τέρου πλευρά ποδῶν τῆ, ἡ δὲ ἑτέρα πλευρά ποδῶν
γ. τά δὲ ἐμβαδά τοῦ ἐνὸς γίνεται ποδῶν ἕνδ καὶ
τοῦ ἀλλοῦ ποδῶν βωξβ.

Ibid. 24. 10, ed. Heiberg (Heron iv.) 422. 15–424. 5

Τριγώνου ὀρθογωνίου τὸ ἐμβαδὸν μετὰ τῆς περι-
μέτρου ποδῶν σ'π. ἀποδιαστείλα τὰς πλευρὰς καὶ
ἐξεισὶ τὸ ἐμβαδὸν. ποὺ ὀὐτως: αἰ τῆτε τοὺς
ἀπαρτίζοντας ἀριθμοὺς: ἀπαρτίζει δὲ τὸν σ'π. ὁ δ' ἀς
τὸν ρμ, δ' τὸν ο, δ' ἐ' τὸν νη, ὁ ρ' τὸν μ, ὁ ρ' τὸν
λε, ὁ ι' τὸν κη, ὁ υ' τὸν κ. ἐσκεφάλημ, ὅτι ὁ ρ
καὶ λε ποιήσουσι τὸ δοθὲν ἐπίταγμα. τῶν σ'π. τὸ
τη' γίνονται πόδες λε. διὰ παντὸς λάμβανε δυνάδα
τῶν τη. λοιπὸν μένουσιν σ' πόδες. τὰ οὖν λε καὶ
τὰ σ' ὁμοί γίνονται πόδες μα. ταύτα ποιεῖ ὁφ'
ἐαυτά' γίνονται πόδες ,αχτά. τὰ λε ἐπὶ τὰ σ'
γίνονται πόδες σι. ταύτα ποίει αἰ ἐπὶ τὰ τη. γίνον-
ται πόδες ,αχτ. ταύτα ἄρον ἀπὸ τῶν ,αχτα:
λοιπὸν μένει ἄ. ὁ ἐν πλευρὰ τετραγωνική γίνεται ἄ.
ἀρτι θές τὰ μα καὶ ἄρον μονάδα ἄ. λοιπὸν μ. ὁν
ζ' γίνεται κ. τοῦτο ἐστιν ἡ κάθετος, ποδῶν κ.
καὶ θές πάλιν τὰ μα καὶ πρόσθες ἄ. γίνονται πόδες
μβ. ὁν ζ' γίνεται πόδες κα. ἐστώ ἡ βάσις ποδῶν
κα. καὶ θές τὰ λε καὶ ἄρον τὰ σ'. λοιπὸν μένουσι
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Take the cube of 3, making 27; double this, making 54. Now take away 1, leaving 53. Then let one side be 53 feet and the other 54 feet. As for the other rectangle, [I proceed] thus: Add together 53 and 54, making 107 feet; multiply this by 3, [making 321; take away 3], leaving 318. Then let one side be 318 feet and the other 3 feet. The area of the one will be 954 feet and of the other 2862 feet.¹

*Ibid. 24. 10, ed. Heiberg (Heron iv.) 422. 15–424. 5

In a right-angled triangle the sum of the area and the perimeter is 280 feet; to separate the sides and find the area. I proceed thus: Always look for the factors; now 280 can be factorized into 2 . 140, 4 . 70, 5 . 56, 7 . 40, 8 . 35, 10 . 28, 14 . 20. By inspection, we find 8 and 35 fulfil the requirements. For take one-eighth of 280, getting 35 feet. Take 2 from 8, leaving 6 feet. Then 35 and 6 together make 41 feet. Multiply this by itself, making 1681 feet. Now multiply 35 by 6, getting 210 feet. Multiply this by 8, getting 1680 feet. Take this away from the 1681, leaving 1, whose square root is 1. Now take the 41 and subtract 1, leaving 40, of which the half is 20; this is the perpendicular, 20 feet. And again take 41 and add 1, getting 42 feet, of which the half is 21; and let this be the base, 21 feet. And take 35 and subtract 6, leaving 29 feet. Now multiply

* The term "feet," πόδες, is used by Heron indiscriminately of lineal feet, square feet and the sum of numbers of lineal and square feet.
πόδες καθ. ἀρτι θες τὴν κάθετον ἐπὶ τὴν βάσιν· οὖν ζ' γίνεται πόδες τὸν. καὶ αἱ τρεῖς πλευραὶ περιμετρούμεναι ἔχουσι πόδας δ' ὁμοί σύνθες μετὰ τοῦ ἐμβαδοῦ· γίνονται πόδες τὸν.

* Heath (H.G.M. ii. 446-447) shows how this solution can be generalized. Let $a$, $b$ be the sides of the triangle containing the right angle, $c$ the hypotenuse, $S$ the area of the triangle, $r$ the radius of the inscribed circle; and let

$$s = \frac{1}{2}(a + b + c).$$

Then

$$S = rs = \frac{1}{2}ab, \quad r + s = a + b, \quad c = s - r.$$ 

Solving the first two equations, we have

$$\frac{a}{b} = \frac{1}{2}[r + s \mp \sqrt{(r + s)^2 - 8rs}],$$

and this formula is actually used in the problem. The
the perpendicular and the base together, [getting 420], of which the half is 210 feet; and the three sides comprising the perimeter amount to 70 feet; add them to the area, getting 280 feet.\textsuperscript{a}

method is to take the sum of the area and the perimeter \( S + 2s \), separated into its two obvious factors \( s(r + 2) \), to put \( s(r + 2) = A \) (the given number), and then to separate \( A \) into suitable factors to which \( s \) and \( r + 2 \) may be equated. They must obviously be such that \( sr \), the area, is divisible by 6.

In the given problem \( A = 280 \), and the suitable factors are \( r + 2 = 8, s = 35 \), because \( r \) is then equal to 6 and \( rs \) is a multiple of 6. Then

\[
\begin{align*}
  a &= \frac{1}{2}(6 + 35 - \sqrt{(6 + 35)^2 - 8 \cdot 6 \cdot 35}) = \frac{1}{2}(41 - 1) = 20, \\
  b &= \frac{1}{2}(41 + 1) = 21, \\
  c &= 35 - 6 = 29.
\end{align*}
\]

This problem is followed by three more of the same type.
XXIII. ALGEBRA: DIOPHANTUS
XXIII. ALGEBRA: DIOPHANTUS

(a) General

Anthem. Palat. xiv. 126, The Greek Anthology, ed. Paton (L.C.L.) v. 92-93

Ouqos toj Diofavon ekev tamos. ax mega theuma:
akai tamos ev tekynis metra byioo legei.

Exetn kouroxein bitoun theos upase moiryn.

dodekata eipideis, mela porven xnoaev.


There are in the Anthology 46 epigrams which are algebraical problems. Most of them (xiv. 116-146) were collected by Metrodorus, a grammarian who lived about A.D. 500, but their origin is obviously much earlier and many belong to a type described by Plato and the scholiast to the Charmides (v. vol. i. pp. 16, 20).

Problems in indeterminate analysis solved before the time of Diophantus include the Pythagorean and Platonic methods of finding numbers representing the sides of right-angled triangles (v. vol. i. pp. 90-95), the methods (also Pythagorean) of finding "side- and diameter-numbers" (vol. i. pp. 132-139), Archimedes' Cattle Problem (v. supra, pp. 202-205) and Heron's problems (v. supra, pp. 504-509).
This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.\footnote{If $x$ was his age at death, then $x = 84$.}

Diophantus's surviving works and ancillary material are admirably edited by Tannery in two volumes of the Teubner series (Leipzig, 1895). There is a French translation by Paul Ver Eecke, Diophante d'Alexandre (Bruges, 1926). The history of Greek algebra as a whole is well treated by G. F. Nesselmann, Die Algebra der Griechen, and by T. L. Heath, Diophantus of Alexandria: A Study in the History of Greek Algebra, 2nd ed. 1910.
GREEK MATHEMATICS

Theon Alex. in Ptol. Math. Syn. Comm. i. 10, ed. Rome, Studi e Testi, lxxii. (1936), 453. 4-6

Καθ' α καὶ Διόφαντός φησι: "τῆς γὰρ μονάδος ἀμεταθέτου ὁμός καὶ ἑστώσης πάντοτε, τὸ πολλαπλασιάζόμενον εἴδος ἐπ' αὐτὴν αὐτὸ τὸ εἴδος ἑσται."

Dioph. De polyg. num. [5], Dioph. ed. Tannery i. 470. 27–472. 4

Καὶ ἀπεδείχθη τὸ παρὰ 'Ὑψικλεῖ ἐν ὀρῷ λεγόμενον, ὦτι, "ἐὰν ὅσιν ἀριθμοὶ ἀπὸ μονάδος ἐν ἴσῃ ύπεροχῇ ὁποσοιοῦν, μονάδος μενούσης τῆς ύπεροχῆς, ὁ σύμπας ἑστὶν (τρίγωνος, δυάδος δὲ), τετράγωνος, τριάδος δὲ, πεντάγωνος. λέγεται δὲ τὸ πλῆθος τῶν γωνιῶν κατὰ τὸν δυάδι μείζονα τῆς ύπεροχῆς, πλευραὶ δὲ αὐτῶν τὸ πλῆθος τῶν ἑκτεθέντων σὺν τῇ μονάδι."


Περὶ δὲ τῆς Ἀἰγυπτιακῆς μεθόδου ταύτης Διόφαντος μὲν διέλαβεν ἀκριβέστερον, δὲ λογιστατὸς Ἀνατόλιος τὰ συνεκτικῶτα μέρη τῆς κατ’ ἑξίσου, δὲ add. Bachet.

* Cf. Dioph. ed. Tannery i. 8. 13–15. The word εἴδος, as will be seen in due course, is regularly used by Diophantus for a term of an equation.
ALGEBRA: DIOPHANTUS

Theon of Alexandria, Commentary on Ptolemy's Syntaxis i. 10, ed. Rome, Studi e Testi, lxxii. (1936), 453. 4-6

As Diophantus says: "The unit being without dimensions and everywhere the same, a term that is multiplied by it will remain the same term."a

Diophantus, On Polygonal Numbers [5], Dioph. ed. Tannery i. 470. 27-472. 4

There has also been proved what was stated by Hypsicles in a definition, namely, that "if there be as many numbers as we please beginning from 1 and increasing by the same common difference, then, when the common difference is 1, the sum of all the numbers is a triangular number; when 2, a square number; when 3, a pentagonal number [; and so on]. The number of angles is called after the number which exceeds the common difference by 2, and the sides after the number of terms including 1."b

Michael Psellus, A Letter, Dioph. ed. Tannery ii. 38. 22-39. 1

Diophantus dealt more accurately with this Egyptian method, but the most learned Anatolius collected the most essential parts of the theory as stated by

\[ \frac{1}{2} n \left[ 2 + (n - 1)(a - 2) \right] ; \ v. \ vol. \ i. \ p. \ 98 \ n. \ a. \]

Michael Psellus, "first of philosophers" in a barren age, flourished in the latter part of the eleventh century A.D. There has survived a book purporting to be by Psellus on arithmetic, music, geometry and astronomy, but it is clearly not all his own work. In the geometrical section it is observed that the most favoured method of finding the area of a circle is to take the mean between the inscribed and circumscribed squares, which would give \( \pi = \sqrt{8} = 2.8284271 \).
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κείμενον ἐπιστήμης ἀπολεξάμενος ἐτέρως¹ Διοφάντῳ συνοπτικῶτατα προσεφώνησε.


Νῦν δ' ἐπὶ τὰς προτάσεις χωρῆσωμεν ὡδόν, πλείστην ἔχοντες τὴν ἐπ' αὐτοῖς τοῖς εἴδεις συνθροισμένην ὥλην. πλείστων δ' ὄντων τῷ ἀριθμῷ καὶ μεγίστων τῷ ὄγκῳ, καὶ διὰ τούτο βραδεῶς βεβαιουμένων ὑπὸ τῶν παραλαμβανόντων αὐτά καὶ ὄντων ἐν αὐτοῖς δυσμνημονεύτων, ἐδοκίμασα τά ἐν αὐτοῖς ἐπιδεχόμενα διαρείν, καὶ μάλιστα τά ἐν ἀρχῇ ἔχοντα στοιχεῖώδες ἀπὸ ἀπλουστέρων ἐπὶ σκολιώτερα διελεῖν ὡς προσήκεν. οὕτως γὰρ εὐδευτα γενήσεται τοῖς ἀρχομένοις, καὶ ἡ ἁγωγὴ αὐτῶν μνημονευθήσεται, τῆς πραγματείας αὐτῶν ἐν τρισκαίδεκα βιβλίοις. γεγενημένης.

Ibid. v. 3, Dioph. ed. Tannery i. 316. 6

"Ἐχομεν ἐν τοῖς Πορίσμασιν.

¹ ἐτέρως Tannery, ἐτέρῳ codd.

* The two passages cited before this one allow us to infer that Diophantus must have lived between Hypsicles and Theon, say 150 B.C. to A.D. 350. Before Tannery edited Michael Psellus's letter, there was no further evidence, but it is reasonable to infer from this letter that Diophantus was a contemporary of Anatolius, bishop of Laodicea about A.D. 280 (v. vol. i. pp. 2-3). For references by Plato and a scholiast to the Egyptian methods of reckoning, v. vol. i. pp. 16, 20.

* Of these thirteen books in the Arithmetica, only six

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him in a different way and in the most concise form, and dedicated his work to Diophantus.\(^a\)


Now let us tread the path to the propositions themselves, which contain a great mass of material compressed into the several species. As they are both numerous and very complex to express, they are only slowly grasped by those into whose hands they are put, and include things hard to remember; for this reason I have tried to divide them up according to their subject-matter, and especially to place, as is fitting, the elementary propositions at the beginning in order that passage may be made from the simpler to the more complex. For thus the way will be made easy for beginners and what they learn will be fixed in their memory; the treatise is divided into thirteen books.\(^b\)

*Ibid.* v. 3, Dioph. ed. Tannery i. 316. 6

We have it in the Porisms.\(^c\)

...have survived. Tannery suggests that the commentary on it written by Hypatia, daughter of Theon of Alexandria, extended only to these first six books, and that consequently little notice was taken of the remaining seven. There would be a parallel in Eutocius’s commentaries on Apollonius’s *Conics*. Nesselmann argues that the lost books came in the middle, but Tannery (Dioph. ii. xix-xxi) gives strong reasons for thinking it is the last and most difficult books which have been lost.

\(^a\) Whether this collection of propositions in the Theory of Numbers, several times referred to in the *Arithmetica*, formed a separate treatise from, or was included in, that work is disputed; Hultsch and Heath take the former view, in my opinion judiciously, but Tannery takes the latter.
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(b) Notation

Ibid. i., Praef., Dioph. ed. Tannery 1. 2. 3–6. 21

Τὴν εὑρεσιν τῶν ἐν τοῖς ἀριθμοῖς προβλημάτων, τιμιώτατε μοι Διονύσιε, γυνώσκων σε σπουδαίως ἔχοντα μαθεῖν, [ὁργανώσαι τὴν μέθοδον]¹ ἐπειράθην, ἀρξάμενος ἀφ’ ὧν συνέστηκε τὰ πράγματα θεμελίων, ὑποστήσαι τὴν ἐν τοῖς ἀριθμοῖς φύσιν τε καὶ δύναμιν.

'Ἰσως μὲν οὖν δοκεῖ τὸ πρᾶγμα δυσχερέστερον, ἐπειδὴ μήτω γυνώριμον ἐστιν, δυσέλπιστοι γὰρ εἰς κατόρθωσιν εἰσιν αἱ τῶν ἀρχιμένων ψυχαί, ὅμως δ’ εὐκατάληπτον σοι γενήσεται, διὰ τε τὴν σὴν προθυμίαν καὶ τὴν ἐμὴν ἀπόδειξιν· ταχεία γὰρ εἰς μάθησιν ἐπιθυμία προσλαβοῦσα διδαχήν.

'Αλλὰ καὶ πρὸς τούτῳ διαφορμάζοντι σοι πάντας τοὺς ἀριθμοὺς συγκειμένους ἐκ μονάδων πλῆθος τῶν, φανερὸν καθεστηκένει εἰς ἀπειρὸν ἔχειν τὴν ὑπάρξειν. τυγχανοῦντων δὴ οὖν ἐν τούτοις

ἀν μὲν τετραγώνων, οἱ εἰσιν ἐκ ἀριθμοῦ τῶν ἐφ’ ἑαυτῶν πολυπλασιασθέντων· οὕτως δὲ ὁ ἀριθμὸς καλεῖται πλευρὰ τοῦ τετραγώνου·

ἀν δὲ κύβων, οἱ εἰσὶν ἐκ τετραγώνων ἐπὶ τὰς αὐτῶν πλευρὰς πολυπλασιασθέντων,

ἀν δὲ δυναμοδυνάμεων, οἱ εἰσὶν ἐκ τετραγώνων ἐφ’ ἑαυτοὺς πολυπλασιασθέντων,

ἀν δὲ δυναμοκύβων, οἱ εἰσίν ἐκ τετραγώνων ἐπὶ

¹ ὁργανώσαι τὴν μέθοδον om. Tannery, following the most ancient ms.

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Knowing that you are anxious, my most esteemed Dionysius, to learn how to solve problems in numbers, I have tried, beginning from the foundations on which the subject is built, to set forth the nature and power in numbers.

Perhaps the subject will appear to you rather difficult, as it is not yet common knowledge, and the minds of beginners are apt to be discouraged by mistakes; but it will be easy for you to grasp, with your enthusiasm and my teaching; for keenness backed by teaching is a swift road to knowledge.

As you know, in addition to these things, that all numbers are made up of some multitude of units, it is clear that their formation has no limit. Among them are—

squares, which are formed when any number is multiplied by itself; the number itself is called the side of the square b;

cubes, which are formed when squares are multiplied by their sides,

square-squares, which are formed when squares are multiplied by themselves;

square-cubes, which are formed when squares are

This subject is admirably treated, with two original contributions, by Heath, *Diophantus of Alexandria*, 2nd ed., pp. 34-53. Diophantus's method of representing large numbers and fractions has already been discussed (vol. i. pp. 44-45). Among other abbreviations used by Diophantus are □⁰⁵, declined throughout its cases, for τετράγωνος; and ισο. (apparently ισο in the archetype) for the sign =, connecting two sides of an equation.

b Or "square root."
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tous apò tis autis autois pleurâs kúbous poluplasiasõntwn,

óv òe kubokúbwn, oi eìsw èk kúbwn èf' èautous
poluplasiasõntwn,

èk te tis toûtw õtoi súnthèsews õ ùperoxhês
ì poluplasiasmuì õ lógou toû pròs allhlos õ
kai èkástwn pròs tás iðias pleurâs sümvaìnei
plékeðhain pleîsta prôblhmatà arîtmhnikà. lûetai
de badìzontos sou tìn ùpodeîkhsomènì ðdòn.

'Eðokimásì th osù ìkástos toûtw tòwn arîmìwn
sûntomwteráwn èpownmìán kthsmámenos sòtochéion tìs
arîmhnkís thewrías èvnav. kaleítaí oûn ò mé

tetragwnon dûnamìs kai èstîn autìs sêmieuò to

èpísìmou èxon Ý, ðv dûnàmìs.

ò òe kúbos kai èstîn autòu sêmieuò K èpísìmou
èxon Ý, ðv kúbos.

ò òe èk tetragwnon èf' èautòn poluplasias-
sthntos dûnàmodûnàmìs kai èstîn autòu sêmieuò

délta dúo èpísìmou èxonata Ý, ðvð dûnàmodûnàmìs.

ò òe èk tetragwnon èpì tòv apò tìs autìs autìf
pleurâs kúbou poluplasiassthntos dûnàmòkubos
kai èstîn autòu sêmieuò tìa ðK èpísìmou èxonata

Ý, ðKð dûnàmòkubos.

ò òe èk kúbou èautòn poluplasiasántos kúbó-
kubos kai èstîn autòu sêmieuò dúo kâppa èpísìmou
èxonata Ý, ðvKð kúbókubos.

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multiplied by the cubes formed from the same side;
cube-cubes, which are formed when cubes are multiplied by themselves;
and it is from the addition, subtraction, or multiplication of these numbers or from the ratio which they bear one to another or to their own sides that most arithmetical problems are formed; you will be able to solve them if you follow the method shown below.

Now each of these numbers, which have been given abbreviated names, is recognized as an element in arithmetical science; the square [of the unknown quantity] is called \( \text{dynamis} \) and its sign is \( \Delta \) with the index \( \text{Y} \), that is \( \Delta^\text{Y} \);
the cube is called \( \text{cubus} \) and has for its sign \( \text{K} \) with the index \( \text{Y} \), that is \( \text{K}^\text{Y} \);
the square multiplied by itself is called dynamodynamis and its sign is two deltas with the index \( \text{Y} \), that is \( \Delta^\text{Y}\Delta \);
the square multiplied by the cube formed from the same root is called dynamocubus and its sign is \( \Delta\text{K} \) with the index \( \text{Y} \), that is \( \Delta\text{K}^\text{Y} \);
the cube multiplied by itself is called cubocubus and its sign is two kappas with the index \( \text{Y}, \text{K}^\text{Y}\text{K} \).

\( ^a \) It is not here stated in so many words, but becomes obvious as the argument proceeds that \( \text{dynamis} \) and its abbreviation are restricted to the square of the unknown quantity; the square of a determinate number is \( \text{tetragonos} \). There is only one term, \( \text{kubos} \), for the cube both of a determinate and of the unknown quantity. The higher terms, when written in full as \( \text{dynamodynamis}, \text{dynamokubos} \) and \( \text{kubokubos} \), are used respectively for the fourth, fifth and sixth powers both of determinate quantities and of the unknown, but their abbreviations, and that for \( \text{kubos} \), are used to denote powers of the unknown only.
I am entirely convinced by Heath's argument, based on the Bodleian ms. of Diophantus and general considerations, that this symbol is really the first two letters of ἀριθμός; this suggestion brings the symbol into line with Diophantus's abbreviations for δύναμις, κύβος, and so on. It may be declined throughout its cases, e.g., 5ων for the genitive plural, infra p. 552, line 5.

Diophantus has only one symbol for an unknown quantity, but his problems often lead to subsidiary equations involving other unknowns. He shows great ingenuity in isolating these subsidiary unknowns. In the translation I shall use 522
The number which has none of these characteristics, but merely has in it an undetermined multitude of units, is called arithmos, and its sign is $\mathfrak{a} [x]$.\(^a\)

There is also another sign denoting the invariable element in determinate numbers, the unit, and its sign is $\mathfrak{m}$ with the index $\mathfrak{o}$, that is $\mathfrak{m}$.

As in the case of numbers the corresponding fractions are called after the numbers, a third being called after 3 and a fourth after 4, so the functions named above will have reciprocals called after them:

\[
\begin{align*}
\text{arithmos } [x] & \quad \text{arithmoston } \left[ \frac{1}{x} \right], \\
\text{dynamis } [x^2] & \quad \text{dynamoston } \left[ \frac{1}{x^2} \right], \\
\text{cubus } [x^3] & \quad \text{cuboston } \left[ \frac{1}{x^3} \right], \\
\text{dynamodynamis } [x^4] & \quad \text{dynamodynamoston } \left[ \frac{1}{x^4} \right], \\
\text{dynamocubus } [x^5] & \quad \text{dynamocuboston } \left[ \frac{1}{x^5} \right], \\
\text{cubocubus } [x^6] & \quad \text{cubocuboston } \left[ \frac{1}{x^6} \right].
\end{align*}
\]

And each of these will have the same sign as the corresponding process, but with the mark $\mathfrak{y}$ to distinguish its nature.\(^b\)

different letters for the different unknowns as they occur, for example, $x, z, m$.

Diophantus does not admit negative or zero values of the unknown, but positive fractional values are admitted.

\(^a\) So the symbol is printed by Tannery, but there are many variants in the mss.
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Ibid. i., Praef., Dioph. ed. Tannery i. 12. 19-21

Λεῖψις ἐπὶ λεῖψιν πολλαπλασιασθεῖσα ποιεῖ ὑπαρξιν, λεῖψις δὲ ἐπὶ ὑπαρξιν ποιεῖ λεῖψιν, καὶ τῆς λεῖψεως σημεῖον Ψ' ἐλλιπές κατώ νεόν, Λ.

(c) DETERMINATE EQUATIONS

(i.) Pure Determinate Equations

Ibid. i., Praef., Dioph. ed. Tannery i. 14. 11-20

Μετὰ δὲ ταῦτα ἐὰν ἀπὸ προβλήματος τινος γένηται εἰδὴ τινὰ ίσα εἴδει τοῖς αὐτοῖς, μὴ ὀμοπληθῆ δὲ, ἀπὸ ἐκατέρω τῶν μερῶν δεῖσαι ἀφαιρεῖν τὰ ὀμοια ἀπὸ τῶν ὀμοίων, ἐως ἂν ἐν εἶδος ἐνὶ εἴδει ἵσον γένηται. ἐὰν δὲ πως ἐν ὀποτέρῳ ἐνυπάρχῃ ἢ ἐν ἀμφοτέροις ἐν ἐλλεῖψει τινα εἰδή, δεῖσαι προσθείναι τὰ λεῖποντα εἰδὴ ἐν ἀμφοτέροις τοῖς μέρεσιν, ἐως ἂν ἐκατέρω τῶν μερῶν τὰ εἰδή ἐνυπάρχοντα γένηται, καὶ πάλιν ἀφελεῖν τὰ ὀμοια ἀπὸ τῶν ὀμοίων, ἐως ἂν ἐκατέρω τῶν μερῶν ἐν εἶδος καταλειφθῇ.

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* Lit. "a deficiency multiplied by a deficiency makes a forthcoming."

b The sign has nothing to do with Ψ, but I see no reason why Diophantus should not have described it by means of Ψ,
ALGEBRA: DIOPHANTUS

Ibid. i., Preface, Dioph. ed. Tannery i. 12. 19-21

A minus multiplied by a minus makes a plus, and a minus multiplied by a plus makes a minus, and the sign of a minus is a truncated Ψ turned upside down, that is Λ.

(c) Determinate Equations

(i.) Pure Determinate Equations

Ibid. i., Preface, Dioph. ed. Tannery i. 14. 11-20

Next, if there result from a problem an equation in which certain terms are equal to terms of the same species, but with different coefficients, it will be necessary to subtract like from like on both sides until one term is found equal to one term. If perchance there be on either side or on both sides any negative terms, it will be necessary to add the negative terms on both sides, until the terms on both sides become positive, and again to subtract like from like until on each side one term only is left.

and cannot agree with Heath (H.G.M. ii. 459) that "the description is evidently interpolated." But Heath seems right in his conjecture, first made in 1885, that the sign Λ is a compendium for the root of the verb λέπτευ, and is, in fact, a Λ with an I placed in the middle. When the sign is resolved in the manuscripts into a word, the dative λέπτει is generally used, but there is no conclusive proof that Diophantus himself used this non-classical form.

c A pure equation is one containing only one power of the unknown, whatever its degree; a mixed equation contains more than one power of the unknown.

d In modern notation, Diophantus manipulates the equation until it is of the form Αx^n = B; as he recognizes only one value of x satisfying this equation, it is then considered solved.
(ii.) Quadratic Equations

Ibid. iv. 39, Dioph. ed. Tannery i. 298. 7–306. 8

Εὑρεῖν τρεῖς ἀριθμοὺς ὡς ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου πρὸς τὴν ὑπεροχὴν τοῦ μέσου καὶ τοῦ ἐλάσσονος λόγον ἔχῃ δεδομένον, ἐτὶ δὲ καὶ σὺν δύο λαμβανόμενοι, ποιώσι τετράγωνον.

Ἐπιτετάχθω δὴ τὴν ὑπεροχὴν τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου εἶναι γ"πλ.

Ἐπεὶ δὲ συναμφότερος ὁ μέσος καὶ ὁ ἐλάσσων ποιεῖ □ον, ποιεῖτω ᾿Μδ. ὁ ἄρα μέσος μείζων ἐστὶ διάδοσ. ἔστω 5αῤ𝐌β. ὁ ἄρα ἐλάχιστος ἐσται ᾿Μβ Λ ἔ ἂ.

Καὶ ἐπειδὴ ἡ ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου γ"πλ. ἐστὶ,1 καὶ ἡ ὑπεροχὴ τοῦ μέσου καὶ τοῦ ἐλαχίστου 5β, ἡ ἄρα ὑπεροχὴ τοῦ μείζονος καὶ τοῦ μέσου ἐσται 5ζ, καὶ ὁ μείζων ἄρα ἐσται 5ζ ᾿Μβ.

Δοιγὸν ἐστὶ δύο ἐπιτάγματα, τὸ τε συναμφότερον (τὸν μείζονα καὶ τὸν ἐλάχιστον ποιεῖν □ον, καὶ τὸ τὸν μείζονα)2 καὶ τὸν μέσον ποιεῖν □ον. καὶ γίνεται μοι διπλῆ ἡ ἴσοτης:

5η ᾿Μδίς. □ν, καὶ 5ζ ᾿Μδίς. □ν.

καὶ διὰ τὸ τὰς ᾿Μ εἶναι τετραγωνικάς, εὐχερῆς ἐστὶν ἡ ἴσωσις.

1 ἐστὶ add. Bachet.
2 τὸν μείζονα . . . τὸν μείζονα add. Tannery.
To find three numbers such that the difference of the greatest and the middle has to the difference of the middle and the least a given ratio, and further such that the sum of any two is a square.

Let it be laid down that the difference of the greatest and the middle has to the difference of the middle and the least the ratio $3:1$.

Since the sum of the middle term and the least makes a square, let it be 4. Then the middle term > 2. Let it be $x + 2$. Then the least term = $2 - x$.

And since the difference of the greatest and the middle has to the difference of the middle and the least the ratio $3:1$, and the difference of the middle and the least is $2x$, therefore the difference of the greatest and the middle is $6x$, and therefore the greatest will be $7x + 2$.

There remain two conditions, that the sum of the greatest and the least make a square and the sum of the greatest and the middle make a square. And I am left with the double equation:

$$8x + 4 = \text{a square},$$
$$6x + 4 = \text{a square}.$$

And as the units are squares, the equation is convenient to solve.

* The quadratic equation takes up only a small part of this problem, but the whole problem will give an excellent illustration of Diophantus's methods, and especially of his ingenuity in passing from one unknown to another. The geometrical solution of quadratic equations by the application of areas is treated in vol. i. pp. 192-215, and Heron's algebraical formula for solving quadratics, supra, pp. 502-505.

b For double equations, v. infra p. 543 n. b.
If we put
\[ 8x + 4 = (p + q)^2, \]
\[ 6x + 4 = (p - q)^2, \]
on subtracting, \[ 2x = 4pq. \]
Substituting \[ 2p = \frac{1}{2}x, \]
\[ 2q = 4 \]
(i.e., \( p = \frac{1}{4}x, \ q = 2 \)) in the first equation we get
\[ 8x + 4 = (\frac{1}{4}x + 2)^2, \]
or
\[ 112x = x^2, \]
whence
\[ x = 112. \]
I form two numbers whose product is $2x$, according to what we know about a double equation; let them be $\frac{1}{2}x$ and 4; and therefore $x = 112$. But, returning to the conditions, I cannot subtract $x$, that is 112, from 2; I desire, then, that $x$ be found $< 2$, so that $6x + 4 < 16$. For $2 \cdot 6 + 4 = 16$.

Then since I seek to make $8x + 4 = a$ square, and $6x + 4 = a$ square, while $2 \cdot 2 = 4$ is a square, there are three squares, $8x + 4$, $6x + 4$, and 4, and the difference of the greatest and the middle is one-third $b$ of the difference of the middle and least. My problem therefore resolves itself into finding three squares such that the difference of the greatest and the middle is one-third of the difference of the middle and least, and further such that the least $= 4$ and the middle $< 16$.

This method of solving such equations is explicitly given by Diophantus in ii. 11, Dioph. ed. Tannery i. 96. 8-14: ἕσται ᾧ ὅ μὲν ἈΔΜβ, ὅ δὲ ἈΔΜγ, ἵνα. ὧ καὶ τοῦτο τοῦ εἴδους καλεῖται διπλοῖοντῆς: εἰστὶ δὲ τῶν τρόπων τούτων. ἵδιὸν τὴν ὑπεροχήν, ἐκτείνεται ἀριθμοὺς ἵνα τὸ ὑπ' αὐτῶν ποίη τὴν ὑπεροχὴν· εἰςὶ δὲ Ἄδ καὶ Ἄδδος ἑκ. τούτων ἦται τῆς ὑπεροχῆς τοῦ Λ' ἐφ' ἐαντὸ ἵσον ἢτι τῶ· ἐλάσσονι, ἡ τῆς συνθέσεως τοῦ Λ' ἐφ' ἐαντὸ ἵσον τῶ· μεῖζόνι—"The equations will then be $x + 2 = a$ square, $x + 3 = a$ square; and this species is called a double equation. It is solved in this manner: observe the difference, and seek two [suitable] numbers whose product is equal to the difference; they are 4 and $\frac{1}{2}$. Then, either the square of half the difference of these numbers is equated to the lesser, or the square of half the sum to the greater."

The ratio of the differences in this subordinate problem has, of course, nothing to do with the ratio of the differences in the main problem; the fact that they are reciprocals may lead the casual reader to suspect an error.
Τετάγθω δ’ μὲν ἐλάχιστος Ἔδ., ἡ δὲ τοῦ μέσου πλευράς ἄρα ἐσται ποσοστὸς, Δ′ ἀπὸ δομῆς Ἐδ.

Ἐπεὶ οὖν ἡ ὑπεροχή τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου γκα μέρος ἐστίν, καὶ ἐστὶν ἡ ὑπεροχή τοῦ μέσου καὶ τοῦ ἐλαχίστου Δ′ ἀπὸ δομῆς, ὥστε ἡ ὑπεροχή τοῦ μεγίστου καὶ τοῦ μέσου ἐσται Δ′ γκα δομῆς Ἐδ.

ἐστιν ὁ μέσος Δ′ ἀπὸ δομῆς Ἐδ. ὁ ἄρα μέγιστος ἐσται Δ′ ἀπὸ γκα δομῆς Ἐδ. ἦσος. ἦσος: πάντα θεῖος. Δ′ ἄρα ἰδέας ἡ Ἐδ. ἦσος ἡ Ἐδ. ἦσος καὶ τὸ δομῆς Ἐδ. ἦσος Ἐδ. ἦσος Ἐδ.

Εἰπεὶ δὲ θέλω τὸν μέσον τετράγωνον ἐλάσσονα εἶναι Ἐδ., καὶ τὴν πλευρᾶς ἐλάσσονος Ἐδ. ἦν δὲ πλευρᾶ τοῦ μέσου ἐστὶν Ἐδ. ἦσος Ἐδ. ἦσος Ἐδ. ἦσος καὶ κοινῶν ἀφαιρεθεῖσῶν τῶν Ἐδ., ὁ δὲ ἐσται ἐλάσσονος Ἐδ.

Γέγονεν οὖν μοι Δ′ γκα δομῆς Ἐδ. ἦσος. ποιήσαι ἦσος: πλάσσω δ’ ὑποκείμενων των τινων ἤ μερισθέντος εἰς τὴν ὑπεροχήν ἢ ὑπερέχει ὁ ἀπὸ τοῦ ἀριθμοῦ δομῆς τῶν Δ′ τῶν ἐν τῇ ἱσόωσις ἤ. ἀπήκτηται οὖν μοι εἰς τὸ εὑρεῖν τινα ἀριθμόν, δος δομῆς γενόμενος καὶ προσλαβών Ἐδ. καὶ μερισθέντος εἰς τὴν ὑπεροχήν ἢ ὑπερέχει ὁ ἀπὸ τοῦ αὐτοῦ δομῆς τριάδος, ποιεῖ τὴν παραβολὴν ἐλάσσονος Ἐδ.
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Let the least be taken as 4, and the side of the middle as $z + 2$; then the square is $z^2 + 4z + 4$.

Then since the difference of the greatest and the middle is one-third of the difference of the middle and the least, and the difference of the middle and the least is $z^2 + 4z$, so that the difference of the greatest and the least is $\frac{1}{3}z^2 + \frac{1}{3}z$, while the middle term is $z^2 + 4z + 4$, therefore the greatest term $= \frac{1}{3}z^2 + 5\frac{1}{2}z + 4$ = a square. Multiply throughout by 9:

$$12z^2 + 48z + 36 = \text{a square};$$

and take the fourth part:

$$3z^2 + 12z + 9 = \text{a square.}$$

Further, I desire that the middle square <16, whence clearly its side <4. But the side of the middle square is $z + 2$, and so $z + 2 < 4$. Take away 2 from each side, and $z < 2$.

My equation is now

$$3z^2 + 12z + 9 = \text{a square.}$$

$$= (mx - 3)^2, \text{say.}$$

Then

$$z = \frac{6m + 12}{m^2 - 3},$$

and the equation to which my problem is now resolved is

$$\frac{6m + 12}{m^2 - 3} < 2,$$

i.e.,

$$\frac{2}{1} < \frac{2}{1}.$$

* As a literal translation of the Greek at this point would be intolerably prolix, I have made free use of modern notation.
"Εστώ ὁ ζητούμενος δ' οὕτως εἰς γενόμενος καὶ προσλαβῶν Ἔ ἰδ' ποιεῖ εἰς Ἐ ἰδ' ὁ δὲ ἀπ' αὐτοῦ ὁμοί, Λ Ἔ γ', ποιεῖ Δ' α' ἐπὶ Λ Ἔ γ' καὶ ποιεῖν τὴν παραβολὴν ἐλάσσονος Ἔ β'. ἀλλὰ καὶ ὁ β' μεριζόμενος εἰς Ἐ α', ποιεῖ τὴν παραβολὴν β'. ὡστε Ἐ ἰδ' ἐπὶ Δ' α' ἐπὶ Λ Ἔ γ' ἐλάσσονα λόγων ἑχουσιν ἑπερ ἐπὶ Δ' πρὸς α.

Καὶ χωρίων χωρίω ἁνίσον. ὁ αρα ὑπὸ Ἐ ἰδ' καὶ Λ ἐλάσσων ἐστὶν τοῦ ὑπὸ δυάδος καὶ Δ' α' ἐπὶ Λ ἐλάσσονες εἰσὶν Δ' β' ἐπὶ Λ ἐλάσσονες Δ' β'.

"Οταν δὲ τοιαύτην ἱσώσων ἱσώσωμεν, ποιοῦμεν τῶν ᾧ τὸ Λ' ἐφ' ἐαυτῷ, γίνεται θ', καὶ τὰς Δ' β' ἐπὶ τὰς Ἐ της, γίνονται Λε' πρὸσθες τοῖς θ', γίνονται με', ὅν πλ' οὐκ ἐλαττῶν ἐστὶ Λ ε' πρὸσθες τὸ ἣμισευμα τῶν ᾧ (γίνεται οὐκ ἐλαττῶν Λ ε' καὶ μέρισον εἰς τὰς Δ' ε') γίνεται οὐκ ἐλαττῶν Λ ε'.

Γέγονεν οὖν μοι Δ' γ' εἰσ' Λ θ' ἵσιν. ὁμοίως οὕτω ἀπὸ πλ' Λ γ' Λ ε', καὶ γίνονται δὲ ἐπὶ Λ κβ' τοῦτον ἓκα.

Τέταχα δὲ τὴν τοῦ μέσου οὖν πλ' Ἐ α' Ἐ β'.

1 γίνεται . . . τὰς Δ' add. Tannery.

* This is not strictly true. But since \(\sqrt{45}\) lies between 6 and 7, no smaller integral value than 7 will satisfy the conditions of the problem.

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The inequality will be preserved when the terms are cross-multiplied,

\[ (6m + 12) \cdot 1 < 2 \cdot (m^2 - 3); \]

\[ i.e., \quad 6m + 12 < 2m^2 - 6. \]

By adding 6 to both sides,

\[ 6m + 18 < 2m^2. \]

When we solve such an equation, we multiply half the coefficient of \( x \) [or \( m \)] into itself—getting 9; then multiply the coefficient of \( x^2 \) into the units —2 \cdot 18 = 36; add this last number to the 9—getting 45; take the square root—which is \( \sqrt{45} \); add half the coefficient of \( x \)—making a number \( \sqrt{45} \); and divide the result by the coefficient of \( x^2 \)—getting a number \( \sqrt{9} \).

My equation is therefore

\[ 3z^2 + 12z + 9 = a \text{ square on side } (3 - 5z), \]

and

\[ z = \frac{42}{22} = \frac{21}{11}. \]

I have made the side of the middle square to be

This shows that Diophantus had a perfectly general formula for solving the equation

\[ ax^2 = bx + c, \]

namely

\[ x = \frac{b + \sqrt{b^2 + 4ac}}{2a}. \]

From vi. 6 it becomes clear that he had a similar general formula for solving \( ax^2 + bx = c \), and from v. 10 and vi. 22 it may be inferred that he had a general solution for \( 2x^2 + c = bx \).
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ἐσται ἡ τοῦ □οῦ πλ. Ἔλμην. αὐτὸς δὲ ὁ □οῦ Ἔλμην ἁγιόν.

"Ερχομαι οὖν ἐπὶ τὸ εἶ ἀρχήν καὶ τάσσω Ἐλμην ἁγιόν, ὡς τοῖς Τῆς Ἐλμην ἁγιόν καὶ πάντα εἰς ἁγιόν· καὶ γίνεται ὁ 5 ψηφιστικοῦ βτζέ, καὶ ἔστιν ἐλάσσων δυάδος.

Ἐπὶ τὰς υποστάσεις τοῦ προβλήματος τοῦ εἶ ἀρχήν· ὑπέστημεν ἡ τὸν μὲν μέσον Σ.Α.Μ.Β., τὸν δὲ ἐλάχιστον Μ.Β. Δ.Σ.Α., τὸν δὲ μέγιστον Σ.Ζ.Μ.Β.

ἔσται ὁ μὲν μέγιστος α.αζ., ὁ δὲ βος βωιζ., ὁ δὲ ἐλάχιστος ὁ γος πτζ. καὶ ἐπεὶ τὸ μόριον, ἔστι τὸ ψηφιστικοῦ βτζέ, οὐκ ἔστιν □οῦ, 5οῦ δὲ ἔστιν αὐτοῦ, ἔδω λάβωτεν ἁγιόν, ὁ ἐστὶ □οῦ, πάντων οὖν τὸ 5οῦ, καὶ ὁμοίως ἔσται ὁ μὲν αος ἁγιόν ἁγιόν, ἁγιόν □', ὁ δὲ βος ὑφτζ Ζ', ὁ δὲ γος ἰδ Ζ'.

Καὶ ἔδω ἐν ὁλοκλήροις θέλης ἣν μὴ τὸ Ζ' ἐπιτρέχῃ, εἰς δο εἴμβαλε. καὶ ἔσται ὁ αος ὑπόδω καταληκτικόν, ὁ δὲ βος ὑπόδω ἁγιόν, ὁ δὲ γος ὑπόδω. καὶ ἡ ἀπόδειξις φανερά. 534
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$s + 2$; therefore the side will be $\frac{43}{11}$ and the square itself $\frac{1849}{121}$.

I return now to the original problem and make $\frac{1849}{121}$, which is a square, $= 6x + 4$. Multiplying by $121$ throughout, I get $x = \frac{1365}{726}$, which is $< 2$.

In the conditions of the original problem we made the middle term $= x + 2$, the least $= 2 - x$, and the greatest $7x + 2$. Therefore

the greatest $= \frac{11007}{726}$,

the middle $= \frac{2817}{726}$,

the least $= \frac{87}{726}$.

Since the denominator, 726, is not a square, but its sixth part is, if we take 121, which is a square, and divide throughout by 6, then similarly the numbers are $\frac{1834\frac{1}{2}}{121}$, $\frac{469\frac{1}{2}}{121}$, $\frac{14\frac{1}{2}}{121}$.

And if you prefer to use integers only, avoiding the $\frac{1}{2}$, multiply throughout by 4. Then the numbers will be $\frac{7338}{484}$, $\frac{1878}{484}$, $\frac{58}{484}$.

And the proof is obvious.
(iii.) Simultaneous Equations Leading to a Quadratic

\[ \begin{align*}
\text{Ibid. i. 28, Dioph. ed. Tannery i. 62. 20–64. 10} \\
\quad \text{Εύρειν δύο ἀριθμοὺς ὡς καὶ ή σύνθεσις αὐτῶν \\
\quad \text{kai ή σύνθεσις τῶν ἀπ’ αὐτῶν τετραγώνων ποιή} \\
\quad \text{δοθέντας ἀριθμοὺς.} \\
\quad Δεὶ δὴ τοὺς δίς ἀπ’ αὐτῶν τετραγώνους τοῦ ἀπὸ \\
\quad \text{συναμφοτέρου αὐτῶν τετραγώνου ὑπερέχειν τετρα-
\quad \text{γώνως. ἔστι δὲ καὶ τοῦτο πλασματικόν.} \\
\quad ’Επιτετάξθω δὴ τὴν μὲν σύνθεσιν αὐτῶν ποιεῖν \\
\quad \text{Μί, τὴν δὲ σύνθεσιν τῶν ἀπ’ αὐτῶν τετραγώνων} \\
\quad \text{ποιεῖν \text{Μ} \text{ση}.} \\
\quad \text{Tετάξθω δὴ ή ύπεροχὴ αὐτῶν \text{Σβ}. καὶ ἐστώ} \\
\quad \text{ὁ μείζων \text{Σα} καὶ Μί, τῶν ἡμίσεων πάλιν τοῦ} \\
\quad \text{συνθέματος, ὁ δὲ ἐλάσσων Μί \text{λσά}. καὶ μένει} \\
\quad \text{πάλιν τὸ μὲν σύνθεμα αὐτῶν Μί, ή δὲ ύπεροχῇ} \\
\quad \text{Σβ.} \\
\quad \text{Λοιπὸν ἔστι καὶ τὸ σύνθεμα τῶν ἀπ’ αὐτῶν} \\
\quad \text{τετραγώνων ποιεῖν \text{Μ} \text{ση}. ἀλλὰ τὸ σύνθεμα τῶν} \\
\quad \text{ἀπ’ αὐτῶν τετραγώνων ποιεῖ \text{Δβ \text{Μ} σ.} \\
\quad \text{ταῦτα ἤς Μση, καὶ γίνεται ὁ \text{Σβ}.} \\
\quad \text{ʼΕπὶ τὰς υποστάσεις. ἔσται ὁ μὲν μείζων \text{Μ β},} \\
\quad \text{ὁ δὲ ἐλάσσων Μη. καὶ ποιοῦσι τὰ τῆς προτάσεως.} \\
\end{align*} \]

\[ \begin{align*}
\text{In general terms, Diophantus's problem is to solve the} \\
\text{simultaneous equations} \\
\xi + \eta &= 2a \\
\xi^2 + \eta^2 &= A.
\end{align*} \]

He says, in effect, let \[ \xi - \eta = 2x; \]
then \[ \xi = a + x, \eta = a - x. \]
(iii.) Simultaneous Equations Leading to a Quadratic

To find two numbers such that their sum and the sum of their squares are given numbers.\(^a\)

It is a necessary condition that double the sum of their squares exceed the square of their sum by a square. This is of the nature of a formula.\(^b\)

Let it be required to make their sum 20 and the sum of their squares 208.

Let their difference be \(2x\), and let the greater \(= x + 10\) (again adding half the sum) and the lesser \(= 10 - x\).

Then again their sum is 20 and their difference \(2x\).

It remains to make the sum of their squares 208. But the sum of their squares is \(2x^2 + 200\).

Therefore \(2x^2 + 200 = 208\), and \(x = 2\).

To return to the hypotheses—the greater = 12 and the lesser = 8. And these satisfy the conditions of the problem.

\[ (a + x)^2 + (a - x)^2 = \Lambda, \]
\[ i.e., \]
\[ 2(a^2 + x^2) = \Lambda. \]

A procedure equivalent to the solution of the pair of simultaneous equations \(\xi + \eta = 2\alpha, \xi\eta = \Lambda\), is given in i. 27, and a procedure equivalent to the solution of \(\xi - \eta = 2\alpha, \xi\eta = \Lambda\), in i. 30.

\(^a\) In other words, \(2(\xi^2 + \eta^2) - (\xi + \eta)^2 = \alpha\) a square; it is, in fact, \((\xi - \eta)^2\). I have followed Heath in translating \(\xi\sigma\tau\iota \delta\iota\ καὶ τὸ ἄλλο πλασματικὸν\) as "this is of the nature of a formula." Tannery evades the difficulty by translating "est et hoc formativum," but Bachet came nearer the mark with his "effectum aliunde." The meaning of \(πλασματικὸν\) should be "easy to form a mould," i.e. the formula is easy to discover.
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(iv.) Cubic Equation

Ibid. vi. 17, Dioph. ed. Tannery i. 432. 19–434. 22

Εὑρεῖν τριγώνον ὀρθογώνιον ὡς ὁ ἐν τῷ ἐμβαδῷ αὐτοῦ, προσλαβῶν τὸν ἐν τῇ ὑποτευνοῦσῃ, ποιῇ τετράγωνον, ὃ δὲ ἐν τῇ περιμέτρῳ αὐτοῦ ἢ κύβος.

Τετάχθω ὃ ἐν τῷ ἐμβαδῷ αὐτοῦ ἵνα, ὃ δὲ ἐν τῇ ὑποτευνοῦσῃ αὐτοῦ Μ τινῶν τετραγωνικῶν ΛΞΣΑ, ἔστω ΜΙΣΛΣΑ.

'Ἀλλ' ἐπεὶ ὑπεθέμεθα τὸν ἐν τῷ ἐμβαδῷ αὐτοῦ εἶναι ΣΑ, ὃ ἀρα ὑπὸ τῶν περὶ τὴν ὀρθήν αὐτοῦ γίνεται ΣΒ. ἀλλὰ ἩΒ περιέχονται ὑπὸ ΞΑ καὶ ΜΒ· ἐὰν οὖν τὰξωμεν μίαν τῶν ὀρθῶν ΜΒ, ἔσται ἡ ἐτέρα ΞΑ.

Καὶ γίνεται ἡ περιμέτρος ΜΗ καὶ οὐκ ἔστι κύβος· ὃ δὲ Η γέγονεν ἐκ τινος ΔΟτ καὶ ΜΒ· δεήσει ἄρα εὑρεῖν ΔΟτ τινα, ὃς, προσλαβῶν ΜΒ, ποιεὶ κύβον, ἀὕτε κύβον ΔΥ ὑπερέχειν ΜΒ.

Τετάχθω οὖν ἡ μὲν τοῦ ΔΟτ πΑ· ΞΑΜΑ, ἡ δὲ τοῦ κύβου ΞΑΛΜΑ. γίνεται ὃ μὲν ΔΟτ, ΔΑΖΒΜΑ, ὃ δὲ κύβος, ΚΑΖΥΛΔΥΓΜΑ. θέλω οὖν τὸν κύβον τὸν ΔΟτ ὑπερέχειν δυάδι· ὃ ἄρα ΔΟτ μετὰ δυάδος, τουτέστιν ΔΑΖΒΜΥ, ἔστιν ἵσος ΚΑΖΥΛΔΥΓΜΑ, ὡθεν ὃ ἦν εὑρίσκεται ΜΔ.

'Ἔσται οὖν ἡ μὲν τοῦ ΔΟτ πΑ· ΜΕ, ἡ δὲ τοῦ
To find a right-angled triangle such that its area, added to one of the perpendiculars, makes a square, while its perimeter is a cube.

Let its area = \( x \), and let its hypotenuse be some square number minus \( x \), say \( 16 - x \).

But since we supposed the area = \( x \), therefore the product of the sides about the right angle = \( 2x \). But \( 2x \) can be factorized into \( x \) and 2; if, then, we make one of the sides about the right angle = 2, the other = \( x \).

The perimeter then becomes 18, which is not a cube; but 18 is made up of a square \([16]+2\). It shall be required, therefore, to find a square number which, when 2 is added, shall make a cube, so that the cube shall exceed the square by 2.

Let the side of the square = \( m + 1 \) and that of the cube \( m - 1 \). Then the square = \( m^2 + 2m + 1 \) and the cube = \( m^3 + 3m - 3m^2 - 1 \). Now I want the cube to exceed the square by 2. Therefore, by adding 2 to the square,

\[
m^2 + 2m + 3 = m^3 + 3m - 3m^2 - 1,
\]

whence \( m = 4 \).

Therefore the side of the square = 5 and that of

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* This is the only example of a cubic equation solved by Diophantus. For Archimedes' geometrical solution of a cubic equation, v. supra, pp. 126-163.
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κύβον Ἔν. αυτοὶ ἄρα ὁ μὲν □ος Ἔν κε, ὁ δὲ κύβος Ἔκες.
Μεθυφίσταμαι οὖν τὸ ὀρθογώνιον, καὶ τάξις αυτοῦ τὸ ἐμβαδὸν 5 ἀ, τάσσω τὴν ὑποτεύνουσαν Ἔκε λχζ ἀ. μένει δὲ καὶ ἡ βάσις Ἔμ, ἡ δὲ κάθετος 5 ἀ.
Λοιπὸν ἐστιν τὸν ἀπὸ τὴς ὑποτευνούσης ἴσον εἶναι τοὺς ἀπὸ τῶν περὶ τὴν ὀρθὴν γίνεται δὲ . ⌜Δγ Ἔκε λχζ ἀ. 5 ἀ. ἐσται ἴση Δγ Ἔκε λχζ. ἀ. δὴν ὁ ⌜5 Ἔκε λχζ. ἀ. ἐστὶν.⌟
Ἐπὶ τὰς ὑποστάσεις καὶ μένει.

(d) Indeterminate Equations

(i.) Indeterminate Equations of the Second Degree

(a) Single Equations

_Ibid._ ii. 20, Dioph. ed. Tannery i. 114. 11-22

Εὑρεῖν δύο ἀριθμοὺς ὅπως ὁ ἀπὸ τοῦ ἐκατέρου αὐτῶν τετράγωνος, προσλαβῶν τὸν λοιπὸν, ποιή τετράγωνον.

Τετάχθω ὁ ους 5 ἀ, ὁ δὲ βος Ἔκε β, ἢν ὁ ἀπὸ τοῦ ους □ος, προσλαβῶν τὸν βο, ποιή □ος. λοιπὸν ἐστι καὶ τὸν ἀπὸ τοῦ βο □ος, προσλαβοῦτα τὸν ου, ποιεῖ □ος. ἄλλ' ὁ ἀπὸ τοῦ βο □ος, προσλαβῶν τὸν ου, ποιεῖ Δγ 5 ἐ Ἔκε β. ταῦτα ἵσα □ος.

* Diophantus makes no mention of indeterminate equations of the first degree, presumably because he admits 540
the cube = 3; and hence the square is 25 and the cube 27.

I now transform the right-angled [triangle], and, assuming its area to be \( x \), I make the hypotenuse = 25 - \( x \); the base remains = 2 and the perpendicular = \( x \).

The condition is still left that the square on the hypotenuse is equal to the sum of the squares on the sides about the right angle;

\[ i.e., \quad x^2 + 625 - 50x = x^2 + 4, \]

whence \[ x = \frac{621}{50}. \]

This satisfies the conditions.

(d) Indeterminate Equations

(i.) Indeterminate Equations of the Second Degree

(a) Single Equations

Ibid. ii. 20, Dioph. ed. Tannery i. 114. 11-22

To find two numbers such that the square of either, added to the other, shall make a square.

Let the first be \( x \), and the second \( 2x + 1 \), in order that the square on the first, added to the second, may make a square. There remains to be satisfied the condition that the square on the second, added to the first, shall make a square. But the square on the second, added to the first, is \( 4x^2 + 5x + 1 \); and therefore this must be a square.

rational fractional solutions, and the whole point of solving an indeterminate equation of the first degree is to get a solution in integers.
GREEK MATHEMATICS

Πλάσσω τὸν □ον ἀπὸ ζβ Λ Μ β· αὐτὸς ἀρα ἐσται Δγ δ Μ δ Λ ζ διή· καὶ γίνεται δ ἕρ γ.

"Εσται ὁ μὲν αος γ', ὁ δὲ βος ιθ', καὶ ποιοῦν τὸ πρόβλημα.

(β) Double Equations

Ibid. iv. 32, Dioph. ed. Tannery 268. 18–272. 15

Δοθέντα ἀριθμὸν διελείν εἰς τρεῖς ἀριθμοὺς ὡς ὁ ὑπὸ τοῦ πρώτου καὶ τοῦ δεύτερου, ἐάν τε προσ-
λάβη τὸν τρίτον, ἐάν τε λείψη, ποιή τετράγωνον.

Τετάχθω ὁ γος ζ α, καὶ ὁ βος Μ ἕλασσονων τοῦ

The problem, in its most general terms, is to solve the equation

\[ Ax^2 + Bx + C = y^2. \]

Diophantus does not give a general solution, but takes a number of special cases. In this case A is a square number (\(=a^2\), say), and in the equation

\[ a^2x^2 + Bx + C = y^2 \]

he apparently puts

\[ y^2 = (ax - m)^2, \]

where \(m\) is some integer,

whence

\[ x = \frac{m^2 - C}{2am + B}. \]
I form the square from \(2x - 2\); it will be \(4x^2 + 4 - 8x\); and \(x = \frac{3}{13}\).

The first number will be \(\frac{3}{13}\), the second \(\frac{19}{13}\), and they satisfy the conditions of the problem.

\((\beta)\) Double Equations

*Ibid. iv. 32, Dioph. ed. Tannery 268. 18–272. 15*

To divide a given number into three parts such that the product of the first and second ± the third shall make a square.

Let the given number be 6.

Let the third part be \(x\), and the second part any number <6, say 2; then the first part = \(4 - x\); and the two remaining conditions are that the product of the first and second ± the third = a square. There results the double equation

\[8 - x = \text{a square},\]
\[8 - 3x = \text{a square}.\]

And this does not give a rational result since the ratio

\(\frac{2}{3}\)

Diophantus’s term for a double equation is \(\delta\piλοι\sigma\omega\xi\), \(\delta\iota\nu\lambda\nu\ ι\sigma\omega\nu\ς\) or \(\delta\iota\nu\lambda\nu\ ι\sigma\omega\nu\ς\). It always means with him that two different functions of the unknown have to be made simultaneously equal to two squares. The general equations are therefore

\[A_1x^2 + B_1x + C_1 = u_1^2,\]
\[A_2x^2 + B_2x + C_2 = u_2^2.\]

Diophantus solves several examples in which the terms in \(x^2\) are missing, and also several forms of the general equation.

Ἀλλὰ ὁ 5 ὁ ἄ μονάδι ἐλάσσων τοῦ β, οἴ δὲ 5 γ ομοίως μείζονες Ἄτο τοῦ β. ἀπήκταί οὖν μοι εἰς τὸ εὐρεῖν ἀριθμόν τινα, ὡς τὸν β, ἢν ὁ Ἄτο αὐτοῦ μείζον, πρὸς τὸν Ἄτο <ἀυτοῦ> ἐλάσσωνα, λόγον ἐξή διὸν [ον] ἀριθμὸς πρὸς [ον] ἀριθμόν.

"Εστω ὁ ζητούμενος 5 ἁ, καὶ ὁ Ἄτο αὐτοῦ μείζον ἐσται 5 Ἄτο ἁ, οἴ δὲ Ἄτο αὐτοῦ ἐλάσσων 5 Ἄτο Λ Ἄτο ἁ. θέλομεν οὖν αὐτοῦ πρὸς ἄλληλους λόγον ἐχεῖν ὀν [ον] ἀριθμὸς πρὸς [ον] ἀριθμὸν. ἐστω ὃν δ πρὸς ἁ. ἀοτε 5 Ἄτο Λ Ἄτο ἁ ἐπὶ Ἄτο δ γίνονται 5 δ Ἄτο Λ Ἄτο δ. καὶ 5 Ἄτο ἁ ἐπὶ την Ἄτο ἁ <γίνονται 5 Ἄτο ἁ> καὶ εἰσὶν οὕτως οἱ ἐκκείμενοι ἀριθμοὶ λόγον ἐχοντες πρὸς ἄλληλους ὃν ἐχει [ον] ἀριθμὸς πρὸς [ον] ἀριθμόν. νῦν 5 δ Ἄτο Λ Ἄτο δ ἢσ. 5 Ἄτο ἁ, καὶ γίνεται ὁ 5 Ἄτο ἁ γ

Τάσσω οὖν τὸν ἑρ Ψ Ἄτο δ ἡ γὰρ γ οτ στὶν 5 ἁ. ο ἂ ῃ οτ στὶν Ἄτο γ Λ 5 ἁ.

Λοιπὸν δεῖ εἶναι τὸ ἐπίταγμα, ἐστω τὸν ὑπὸ ἂ οτ καὶ ἑρ οτ, προσλαβόντα τὸν γ οτ, ποιεῖν [οτ] καὶ λεύσαντα τὸν γ οτ, ποιεῖν [οτ] ἀλλ' ὁ ὑπὸ ἂ οτ καὶ ἑρ οτ, προσλαβὼν τὸν γ οτ, ποιεῖ Ἅθ θ Λ 5 ἁ' ἢσ. [οτ]. Ἄτο δὲ τὸν γ οτ, ποιεῖ Ἅθ θ Λ 5 ἃ' ἢσ. [οτ]. καὶ 544
of the coefficients of $x$ is not the ratio of a square to a square.

But the coefficient 1 of $x$ is $2 - 1$ and the coefficient 3 of $x$ likewise is $2 + 1$; therefore my problem resolves itself into finding a number to take the place of 2 such that (the number +1) bears to (the number – 1) the same ratio as a square to a square.

Let the number sought be $y$; then (the number +1) = $y + 1$, and (the number – 1) = $y - 1$. We require these to have the ratio of a square to a square, say $4 : 1$. Now $(y - 1) \cdot 4 = 4y - 4$ and $(y + 1) \cdot 1 = y + 1$. And these are the numbers having the ratio of a square to a square. Now I put

$$4y - 4 = y + 1,$$

giving

$$y = \frac{5}{3}.$$

Therefore I make the second part $\frac{5}{3}$, for the third = $x$; and therefore the first = $\frac{13}{3} - x$.

There remains the condition, that the product of the first and second ± the third = a square. But the product of the first and second + the third =

$$\frac{65}{9} - \frac{2}{3}x = \text{a square},$$

and the product of the first and second – the third =

$$\frac{65}{9} - 2\frac{2}{3}x = \text{a square}.$$

---

1 $\alpha\nu\rho\omicron\omicron\upsilon \ldots \pi\rho\omicron\sigma$ add. Bachet.

2 $\gamma\iota\nu\omicron\omicron\rho\omicron\omicron\tau\alpha\iota \delta \dot{\alpha} \overline{M} \ddot{\alpha}$ add. Tannery.
These are a pair of equations of the form
\[
\begin{align*}
am^2x + a &= u^2, \\
an^2x + b &= v^2.
\end{align*}
\]
Multiply by \(n^2, m^2\) respectively, getting, say
\[
\begin{align*}
am^2n^2x + an^2 &= u'^2, \\
am^2n^2x + bm^2 &= v'^2.
\end{align*}
\]

Let
\[
\begin{align*}
an^2 - bm^2 &= u'^2 - v'^2, \\
an^2 - bm^2 &= pq,
\end{align*}
\]
and put
\[
\begin{align*}
u' + v' &= p, \\
u' - v' &= q;
\end{align*}
\]

and so
\[
\begin{align*}
u'^2 &= \frac{1}{4}(p + q)^2, \\
v'^2 &= \frac{1}{4}(p - q)^2,
\end{align*}
\]
whence, from either,
\[
x = \frac{1}{4}(p^2 + q^2) - \frac{1}{2}(an^2 + bm^2).
\]
Multiply throughout by 9, getting

\[ 65 - 6x = \text{a square} \]

and

\[ 65 - 24x = \text{a square.}^4 \]

Equating the coefficients of \( x \) by multiplying the first equation by 4, I get

\[ 260 - 24x = \text{a square} \]

and

\[ 65 - 24x = \text{a square} \]

Now I take their difference, which is 195, and split it into the two factors 15 and 13. Squaring the half of their difference, and equating the result to the lesser square, I get \( x = \frac{8}{3} \).

Returning to the conditions—the first part will be \( \frac{5}{3} \), the second \( \frac{5}{3} \), and the third \( \frac{8}{3} \). And the proof is obvious.

This is the procedure indicated by Diophantus. In his example,

\[ p = 15, \; q = 13, \]

and

\[ \frac{1}{4}(15 - 13)^2 = 65 - 24x, \]

whence

\[ 24x = 64, \; \text{and} \; x = \frac{8}{3} \]
(ii.) Indeterminate Equations of Higher Degree

Ibid. iv. 18, Dioph. ed. Tannery i. 226. 2–228. 5

Εὑρεῖν δύο ἀριθμοὺς, ὥστε δὲ ἀπὸ τοῦ πρώτου κύβος προσλαβῶν τὸν δεύτερον ποιῇ κύβον, δὲ δὲ ἀπὸ τοῦ δεύτερου τετράγωνος προσλαβῶν τὸν πρῶτον ποιῇ τετράγωνον.

Τετάχθω δὲ ἀριθμοὺς, ὥστε δὲ ἀριθμοὺς ἐσται Μ κυβικαί η Λ ΚΥ α. καὶ γίνεται δὲ ἀπὸ τοῦ αου κύβος, προσλαβῶν τὸν βον, κύβοις.

Λοιπὸν ἐστι καὶ τὸν ἀπὸ τοῦ βον, προσλαβόντα τὸν αου, ποιεῖν βον. ἄλλθ δὲ ἀπὸ τοῦ βον, ποιεῖν τὸν αου, ποιεῖ

ΚΥ K α Σ α ῶ Μ ξδ Λ ΚΥ τς. (ταῦτα ἕσοι τὸ ἀπὸ πω. ΚΥ α Μ η, τουτεστι ΚΥ K α ΚΥ τς ῶ Μ ξδ η)

καὶ κοινῶν προστιθεμένων τῶν λειπομένων καὶ ἀφαιρομένων τῶν ὁμοίων ἀπὸ ὀμοίων, λοιπὸι

ΚΥ λΒ ἢσοι Ξ α. καὶ πάντα παρὰ Σ. ΔΥ λΒ ἢσαι Μ α.

Καὶ ἐστιν ἡ Μ τον, καὶ ΔΥ λΒ εἰ ἢσαν τον, λειπομένη ἀν μοι ἢν ἡ ἴσωσις. ἄλλθ αἱ ΔΥ λΒ εἰσοῦ ἐκ τῶν δις ΚΥ τς. οἱ δὲ ΚΥ τς εἰσοῦ ὑπὸ τῶν δις Μ η

1 ταῦτα ... ῶ Μ ξδ add. Bachet.

* As with equations of the second degree, these may be single or double. Single equations always take the form that an expression in \(x\), of a degree not exceeding the sixth, is to be made equal to a square or cube. The general form is therefore

\[ A_0x^6 + A_1x^5 + \ldots + A_6 = y^2 \text{ or } y^3. \]

Diophantus solves a number of special cases of different degrees.

In double equations, one expression is made equal to a
(ii.) Indeterminate Equations of Higher Degree

Ibid. iv. 18, Dioph. ed. Tannery i. 226. 2–228. 5

To find two numbers such that the cube of the first added to the second shall make a cube, and the square of the second added to the first shall make a square.

Let the first number be $x$. Then the second will be a cube number less $x^3$, say $8 - x^3$. And the cube of the first, added to the second, makes a cube.

There remains the condition that the square on the second, added to the first, shall make a square. But the square on the second, added to the first, is $x^6 + x + 64 - 16x^3$. Let this be equal to $(x^3 + 8)^2$, that is to $x^6 + 16x^3 + 64$. Then, by adding or subtracting like terms,

$$32x^3 = x;$$

and, after dividing by $x$,

$$32x^2 = 1.$$ 

Now 1 is a square, and if $32x^2$ were a square, my equation would be soluble. But $32x^2$ is formed from $2 \cdot 16x^3$, and $16x^3$ is $(2 \cdot 8)(x^3)$, that is, it is formed cube and the other to a square, but only a few simple cases are solved by Diophantus.

The general type of the equation is

$$x^6 - Ax^3 + Bx + c^2 = y^2.$$ 

Put $y = x^3 + c$, then

$$x^2 = \frac{B}{A + 2c},$$

and if the right-hand expression is a square, there is a rational solution.

In the case of the equation $x^6 - 16x^3 + x + 64 = y^2$ it is not a square, and Diophantus replaces the equation by another, $x^6 - 128x^3 + x + 4096 = y^2$, in which it is a square.
καὶ τοῦ \(K^2\) ἂ, τούτου δὴ τῶν \(M\) ἦ. ὡστε αἱ \(\lambda\beta\ \Delta^y\) ἐκ \(\delta^x\) τῶν \(\eta\ M\). γέγονεν οὖν μοι εὐρεῖν κύβον ὄς \(\delta^x\) γενόμενος ποιεῖ \(\Box^x\).

*Εστώ ὁ \(ξητούμενος \(K^2\) ἂ. οὗτος \(\delta^x\) γενόμενος ποιεῖ \(K^2\) δ ἵσ. \(\Box^x\). \(\varepsilon\τω \(\Delta^y\) \(i\tau^\nu\). καὶ γίνεται \(\delta\ \ast\ M\ \delta\). ἐπὶ τὰς ὑποστάσεις ἐσταὶ \(K^2\) \(M\) \(\xi\) \(\delta\).

Τάσοι ἀρα τῶν \(\beta^0\) \(M\) \(\xi\) \(\lambda\) \(K^2\) ἂ. καὶ λοιπὸν ἐστι τῶν ἀπὸ τοῦ \(\beta^0\) \(\Box^x\) προσλαβόντα τῶν \(\alpha^0\) ποιεῖ \(\Box^x\). ἀλλὰ ὁ ἀπὸ τοῦ \(\beta^0\) προσλαβὼν τῶν \(\alpha^0\) ποιεῖ \(K^2\) \(K\) \(\alpha\) \(M\) \(\delta\zeta\) \(\varepsilon\) \(\delta\Lambda\) \(K^x\) \(\rho\kappa\eta\) ἵσ. \(\Box^x\) τῶ ἀπὸ \(\pi\). \(K^2\) \(\alpha\) \(M\) \(\xi\) \(\delta\). καὶ γίνεται \(\Box^x\) \(K^2\) \(K\) \(\alpha\) \(M\) \(\delta\zeta\) \(K^y\) \(\rho\kappa\eta\). καὶ γίνονται λοιποὶ \(K^x\) \(\sigma\nu\) \(\iota\). \(\Box\) \(\alpha\). καὶ γίνεται \(\delta\ \ast\ \epsilon\ν\) \(\iota\) \(\nu\).

*Επὶ τὰς ὑποστάσεις ἐσταὶ \(\alpha^0\) \(\epsilon\ν\) \(\iota\) \(\nu\), \(\delta\ \delta\) \(\beta^0\) \(\kappa^\nu\), \(\beta\mu\nu\).

(e) Theory of Numbers: Sums of Squares

Ibid. ii. 8, Dioph. ed. Tannery i. 90. 9-21

Τὸν ἐπιταχθέντα τετράγωνον διελεῖν εἰς δύο τετραγώνους.

* It was on this proposition that Fermat wrote a famous note: "On the other hand, it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly marvellous proof of this, which, however, the margin is not large enough
from 2.8. Therefore $32x^2$ is formed from 4.8. My problem therefore becomes to find a cube which, when multiplied by 4, makes a square.

Let the number sought be $y^3$. Then $4y^3= a$ square

$=16y^2$ say; whence $y = 4$. Returning to the conditions—the cube will be 64.

I therefore take the second number as $64 - x^3$. There remains the condition that the square on the second added to the first shall make a square. But the square on the second added to the first $= x^6 + 4096 + x - 128x^3 = a$ square

$= (x^3 + 64)^2$, say,

$= x^6 + 4096 + 128x^3$.

On taking away the common terms,

$$256x^3 = x,$$

and

$$x = \frac{1}{16}.$$

Returning to the conditions—

first number $= \frac{1}{16}$, second number $= \frac{262143}{4096}$.

(e) Theory of Numbers: Sums of Squares

Ibid. ii. 8, Dioph. ed. Tannery i. 90. 9-21

To divide a given square number into two squares. Fermat claimed, in other words, to have proved that $x^m + y^m = z^m$ cannot be solved in rational numbers if $m > 2$. Despite the efforts of many great mathematicians, a proof of this general theorem is still lacking.

Fermat's notes, which established the modern Theory of Numbers, were published in 1670 in Bachet's second edition of the works of Diophantus.
'Επιτετάχθω δὴ τὸν ἵς διελεῖν εἰς δύο τετραγώνους.
Καὶ τετάχθω ὁ αὸς Δυ ἀ., ὁ ἄρα ἐτερος ἐσται Ἔζ. 
Πλάσσω τὸν ὡς ἀνὸ δεῦσε ἀρα Ἔζ. ΛΔΥ ἀ. ἤς ἐλαιαν ἔτι ὥ. 
Τοσοῦτων Ἔζ. ὡσων ἐστὶν ἤ τῶν ἐς. Ἔζ. ἔλευρα. ἐστιν Ἔζ. ΛΔΥ. 
αὐτὸς ἂρα ὁ ὡς ἐσται ΔΥ Ἔζ. Ἔζ. ἔται ταῦτα ἤς Ἔζ. ΛΔΥ. ἆ. 
κοινὴ προσκείσθω ἦ λεῦσε καὶ ἄν ὀμοίων ὄμοια.
ΔΥ ἂρα ἔ ἤς ἐς. ἔτι, καὶ γίνεται ὁ ἐς. ἐς. πέμπτων.
"Εσται ὁ μὲν κες, ὁ δὲ κες ὑν, καὶ οἱ δύο συντεθέντες ποιοῦσιν κες, ἦ τοῦ Ἔζ. καὶ ἐστὶν ἐκάτερος τετράγωνων.

Ibid. v. 11, Dioph. ed. Tannery I. 342. 13–346. 12

Μονάδα διελεῖν εἰς τρεῖς ἀριθμοὺς καὶ προσθεῖναι ἐκάστω αὐτῶν πρότερον τὸν αὐτὸν δοθέντα καὶ ποιεῖν ἐκαστὸν τετράγωνον.

Δεῖ δὴ τὸν διδόμενον ἀριθμὸν μήτε δυάδα εἶναι μήτε τινὰ τῶν ἀπὸ δυάδος ὀκτάδι παραπλανομένων.

'Επιτετάχθω δὴ τὴν Ἔζ. διελεῖν εἰς τρεῖς ἀριθμοὺς καὶ προσθεῖναι ἐκάστω Ἔζ. ἕ καὶ ποιεῖν ἐκαστὸν ἔτι.

* Lit. “I take the square from any number of ἀριθμοι minus as many units as there are in the side of 16.”
* i.e., a number of the form 3(8n + 2) + 1 or 24n + 7 cannot be the sum of three squares. In fact, a number of the form 8n + 7 cannot be the sum of three squares, but there are other 552
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Let it be required to divide 16 into two squares.
And let the first square $= x^2$; then the other will
be $16 - x^2$; it shall be required therefore to make
$$16 - x^2 = \text{a square}.$$  I take a square of the form $a (mx - 4)^2$, $m$ being any integer and $4$ the root of 16; for example, let the side be $2x - 4$, and the square itself $4x^2 + 16 - 16x$. Then
$$4x^2 + 16 - 16x = 16 - x^2.$$  Add to both sides the negative terms and take like from like. Then
$$5x^2 = 16x,$$
and
$$x = \frac{16}{5}.$$  One number will therefore be $\frac{256}{25}$, the other $\frac{144}{25}$, and their sum is $\frac{400}{25}$ or 16, and each is a square.

Ibid. v. 11, Dioph. ed. Tannery i. 342-346. 12

To divide unity into three parts such that, if we add the same number to each of the parts, the results shall all be squares.

It is necessary that the given number be neither 2 nor any multiple of 8 increased by 2.\textsuperscript{b}

Let it be required to divide unity into three parts such that, when 3 is added to each, the results shall all be squares.

numbers not of this form which also are not the sum of three squares. Fermat showed that, if $3a + 1$ is the sum of three squares, then it cannot be of the form $4^n (24k + 7)$ or $4^n (8k + 7)$, where $k = 0$ or any integer.
Πάλιν δεῖ τὸν ἰ διελείν εἰς τρεῖς \( \square^{ους} \) ὁπως ἑκατοστὸς αὐτῶν μεῖζων ἦ \( \hat{M} \hat{y} \). Ἐὰν οὖν πάλιν τὸν ἰ διελωμεν εἰς τρεῖς \( \square^{ους} \), τῇ τῆς παρουσίας ἀγωγῇ, ἔσται ἑκατοστὸς αὐτῶν μεῖζων τριάδος καὶ δυνησόμεθα, ἀφ’ ἑκάστου αὐτῶν ἀφελόντες \( \hat{M} \hat{y} \), ἔχειν εἰς οὖς ἦ \( \hat{M} \) διαιρεῖται.

Λαμβάνομεν ἁρτὶ τοῦ ἰ τὸ \( \gamma^{ου} \), γ. \( \gamma \gamma^{x} \), καὶ ἐτοῦμεν τῇ προστιθέντες μόριοιν τετραγώνικον ταῖς \( \hat{M} \gamma \gamma^{x} \), ποὺσομεν \( \square^{ου} \). πάντα θαίς. δεὶ καὶ τῷ \( \lambda \) προσθεῖναι τῇ μόριοιν τετραγώνικον καὶ πολειν τὸν ὁλον \( \square^{ου} \).

"Εστω τὸ προστιθέμενον μόριον \( \Delta^{\times} \hat{a} \) καὶ πάντα ἐπὶ \( \Delta^{\times} \). γίνονται \( \Delta^{\times} \hat{\lambda} \hat{M} \hat{a} \). \( \square^{\cdot} \). τῷ ἀπὸ πλευρᾶς \( \varepsilon \hat{M} \hat{α} \). γίνεται \( \sigma \) \( \square^{ος} \) \( \Delta^{\times} \varepsilon \hat{\varepsilon} \hat{M} \hat{α} \). \( \\theta \)θεν \( \sigma \) \( \varepsilon \hat{M} \beta \), ἦ \( \Delta^{\times} \hat{M} \delta \), τῷ \( \Delta^{\times} \hat{M} \delta^{x} \).

Εἰ οὖν ταῖς \( \hat{M} \lambda \) προστίθεται \( \hat{M} \delta^{x} \), ταῖς \( \hat{M} \gamma \gamma^{x} \) προστηθήσεται \( \lambda^{x} \) καὶ γίνεται \( \lambda \). δεὶ οὖν τὸν ἰ διελεῖν εἰς τρεῖς \( \square^{ους} \) ὁπως ἑκάστου \( \square^{ου} \) ἡ πλευρά πάρισος ἦ \( \hat{M} ^{\cdot} \).

'Αλλὰ καὶ ὁ ἰ σύγκειται ἐκ δύο \( \square^{ωυ} \), τοῦ τῇ \( \theta \) καὶ τῆς \( \hat{M} \). διαιροῦμεν τῇ \( \hat{M} \) εἰς δύο \( \square^{ους} \) τά τῇ \( \theta \) καὶ τά \( \ke^{\cdot} \), ὡστε τὸν ἰ σύγκεισθαι ἐκ τριῶν \( \square^{ωυ} \).

* The method has been explained in v. 19, where it is proposed to divide 13 into two squares each > 6. It will be sufficiently obvious from this example. The method is also used in v. 10, 12, 13, 14.

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Then it is required to divide 10 into three squares such that each of them > 3. If then we divide 10 into three squares, according to the method of approximation, each of them will be > 3 and, by taking 3 from each, we shall be able to obtain the parts into which unity is to be divided.

We take, therefore, the third part of 10, which is 3\(\frac{1}{3}\), and try by adding some square part to 3\(\frac{1}{3}\) to make a square. On multiplying throughout by 9, it is required to add to 30 some square part which will make the whole a square.

Let the added part be \(\frac{1}{x^2}\); multiply throughout by \(x^2\); then

\[30x^2 + 1 = \text{a square}.
\]

Let the root be 5\(x + 1\); then, squaring,

\[25x^2 + 10x + 1 = 30x^2 + 1;
\]

whence

\[x = 2, \ x^2 = 4, \ \frac{1}{x^2} = \frac{1}{4}.
\]

If, then, to 30 there be added \(\frac{1}{4}\), to 3\(\frac{1}{3}\) there is added \(\frac{1}{36}\), and the result is \(\frac{121}{36}\). It is therefore required to divide 10 into three squares such that the side of each shall approximate to \(\frac{11}{6}\).

But 10 is composed of two squares, 9 and 1. We divide 1 into two squares, \(\frac{9}{25}\) and \(\frac{16}{25}\), so that 10 is composed of three squares, 9, \(\frac{9}{25}\) and \(\frac{16}{25}\). It is there-
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The sides are, in fact, \( \frac{1321}{711} \), \( \frac{1288}{711} \), \( \frac{1285}{711} \), and the squares are \( 1745041 \), \( 1658944 \), \( 1651225 \).
fore required to make each of the sides approximate to $\frac{11}{6}$.

But their sides are $3, \frac{4}{5}$ and $\frac{3}{5}$. Multiply throughout by $30$, getting $90, 24$ and $18$; and $\frac{11}{6}$ [when multiplied by $30$] becomes $55$. It is therefore required to make each side approximate to $55$.

[Now $3 > \frac{55}{30}$ by $\frac{35}{30}, \frac{4}{5} < \frac{55}{30}$ by $\frac{31}{30}$, and $\frac{3}{5} < \frac{55}{30}$ by $\frac{37}{30}$.]

If, then, we took the sides of the squares as $3 - \frac{35}{30}$, $\frac{4}{5} + \frac{31}{30}$, $\frac{3}{5} + \frac{37}{30}$, the sum of the squares would be $3 \cdot \left(\frac{11}{6}\right)^2$ or $\frac{363}{36}$, which $> 10$.

Therefore we take the side of the first square as $3 - 35x$, of the second as $\frac{4}{5} + 31x$, and of the third as $\frac{3}{5} + 37x$. The sum of the aforesaid squares

$$3555x^2 + 10 - 116x = 10;$$

whence

$$x = \frac{116}{3555}.$$

Returning to the conditions—as the sides of the squares are given, the squares themselves are also given. The rest is obvious.\(^a\)


*To find four square numbers such that their sum added to the sum of their sides shall make a given number.*
"Εστώ δὴ τὸν Ἴβ.

"Επεὶ πᾶς □ος προσλαβῶν τὴν ἰδίαν πλ. καὶ Μ ἔκτ., ποιεῖ □ού, οὔ ἢ πλ. Λ Μ ἔκτ. ποιεῖ ἀριθμὸν τινα, ὃς ἐστι τοῦ ἐξ ἀρχῆς □οῦ πλευρά, οἱ τέσσαρες ἀριθμοὶ ἀρα, προσλαβόντες μὲν τὰς ἰδίας πλ. ποιοῦσι Μ Ἴβ, προσλαβόντες δὲ καὶ δ δ′, ποιοῦσι τέσσαρας □οὺς· εἰσὶ δὲ καὶ αἱ Μ Ἴβ μετὰ δ δων, ὃ ἐστι Μ α, Μ ἕγ. τὰς ἐγ. ἀρα Μ διαρεῖν δεῖ εἰς τέσσαρας □οὺς, καὶ ἀπὸ τῶν πλευρῶν, ἀφελῶν ἀπὸ ἐκάστης πλ. Μ ἔκτ., ἔξω τῶν δ □οῦ τὰς πλ.

Διαρεῖται δὲ ὁ ἐγ. εἰς δύο □οὺς, τόν τε δ καὶ θ. καὶ πάλιν ἐκάτερος τούτων διαρεῖται εἰς δύο □οὺς, εἰς καὶ καὶ λ, καὶ πα. λαβῶν τούτων ἐκάστου τὴν πλευράν, ε ε ε ε καὶ αἱρω ἀπὸ ἐκάστου τούτων πλευρᾶς Μ ἔκτ., καὶ ἐσονται αἱ πλ. τῶν ἐστομένων □οῦ, ι ι ι ι. αὐτοὶ ἄρα οἱ □οῦ, ὃς μὲν ῥκα, ὃς δὲ μ, ὃς δὲ τξα, ὃς δὲ ρξθ.
Let it be 12.

Since any square added to its own side and \( \frac{1}{4} \) makes a square, whose side *minus* \( \frac{1}{2} \) is the number which is the side of the original square, and the four numbers added to their own sides make 12, then if we add \( 4 + \frac{1}{4} \) they will make four squares. But

\[ 12 + 4 \cdot \frac{1}{4} \text{ (or 1)} = 13. \]

Therefore it is required to divide 13 into four squares, and then, if I subtract \( \frac{1}{2} \) from each of their sides, I shall have the sides of the four squares.

Now 13 may be divided into two squares, 4 and 9. And again, each of these may be divided into two squares, \( \frac{64}{25} \) and \( \frac{36}{25} \), and \( \frac{144}{25} \) and \( \frac{81}{25} \). I take the side of each \( \frac{8}{5}, \frac{6}{5}, \frac{12}{5}, \frac{9}{5} \), and subtract half from each side, and the sides of the required squares will be

\[ \frac{11}{10}, \frac{7}{10}, \frac{19}{10}, \frac{13}{10}. \]

The squares themselves are therefore respectively

\[ \frac{121}{100} \quad \frac{49}{100} \quad \frac{361}{100} \quad \frac{169}{100}. \]

* i.e., \( x^2 + x + \frac{1}{4} = (x + \frac{1}{2})^2. \)

* In IV. 30 and V. 14 it is also required to divide a number into four squares. As every number is either a square or the sum of two, three or four squares (a theorem stated by Fermat and proved by Lagrange), and a square can always be divided into two squares, it follows that any number can be divided into four squares. It is not known whether Diophantus was aware of this.
GREEK MATHEMATICS

(f) Polygonal Numbers

Dioph. De polyg. num., Praef., Dioph. ed. Tannery
i. 450. 3-19

"Εκαστὸς τῶν ἀπὸ τὴς τριάδος ἀριθμῶν αὐξο-μένων μονάδι, πολύγωνός ἐστι πρῶτος\(^1\) ἀπὸ τῆς μονάδος, καὶ ἔχει γωνίας τοσαύτας ὡσον ἔστιν τὸ πλήθος τῶν ἐν αὐτῷ μονάδων· πλευρά τε αὐτοῦ ἐστὶν ὁ ἕξις τῆς μονάδος ἀριθμός, ὁ β. ἔσται δὲ ὁ μὲν ἵπ τρίγωνον, ὁ δὲ ὁ τετράγωνον, ὁ δὲ ὁ πεντάγωνον, καὶ τοῦτο ἐξῆς.

Τῶν δὲ τετράγωνων προδήλων ὄντων ὃτι καθ-εστήκασι τετράγωνοι διὰ τὸ γεγονέας αὐτοὺς ἐξ ἀριθμοῦ τινος ἐφ' ἐαυτὸν πολλαπλασιασθέντος, ἐδοκιμάσθη ἐκαστὸν τῶν πολυγώνων, πολυπλασια-ζόμενον ἐπὶ τινα ἀριθμὸν κατὰ τὴν ἀναλογίαν τοῦ πλῆθους τῶν γωνιῶν αὐτοῦ, καὶ προσλαβόντα τετράγωνον τινα πάλιν κατὰ τὴν ἀναλογίαν τοῦ πλῆθους τῶν γωνιῶν αὐτῶν, φαίνεσθαι τετρά-γωνον· ὁ δὲ παραστήσομεν ὑποδείγαστε πῶς ἀπὸ δοθείσης πλευρᾶς ὁ ἔπιταχθεὶς πολύγωνὸς εὐρί-σκεται, καὶ πῶς δοθέντι πολυγώνῳ ἡ πλευρὰ λαμβάνεται.

\(^1\) πρῶτος Bachet, πρῶτον codd.

\(^a\) A fragment of the tract On Polygonal Numbers is the only work by Diophantus to have survived with the Arithmetica. The main fact established in it is that stated in Hypsicles' definition, that the \(a\)-gonal number of side \(n\) is
ALGEBRA: DIOPHANTUS

(f) POLYGONAL NUMBERS

Diophantus, On Polygonal Numbers, Preface, Dioph. ed. Tannery i. 450. 3-19

From 3 onwards, every member of the series of natural numbers increasing by unity is the first (after unity) of a particular species of polygon, and it has as many angles as there are units in it; its side is the number next in order after the unit, that is, 2. Thus 3 will be a triangle, 4 a square, 5 a pentagon, and so on in order.

In the case of squares, it is clear that they are squares because they are formed by the multiplication of a number into itself. Similarly it was thought that any polygon, when multiplied by a certain number depending on the number of its angles, with the addition of a certain square also depending on the number of its angles, would also be a square. This we shall establish, showing how any assigned polygonal number may be found from a given side, and how the side may be calculated from a given polygonal number.

\[ \frac{1}{2} n \{2 + (n - 1)(a - 2)\} \] (v. supra, p. 396 n. a, and vol. i. p. 98 n. a). The method of proof contrasts with that of the Arithmetica in being geometrical. For polygonal numbers, v. vol. i. pp. 86-99.

The meaning is explained in vol. i. p. 86 n. a, especially in the diagram on p. 89. In the example there given, 5 is the first (after unity) of the series of pentagonal numbers 1, 5, 12, 22 ... It has 5 angles, and each side joins 2 units.

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XXIV. REVIVAL OF GEOMETRY:
PAPPUS OF ALEXANDRIA
XXIV. REVIVAL OF GEOMETRY: PAPPUS OF ALEXANDRIA

(a) General

Suidas, s.v. Πάππος

Πάππος, Ἀλεξανδρεύς, φιλόσοφος, γεγονὼς κατὰ τὸν πρεσβύτερον Θεοδόσιον τὸν βασιλέα, ὅτε καὶ Θέων ὁ φιλόσοφος ἠκμαζεν, ὁ γράφας εἰς τὸν Πτολεμαίου Κανώνα. βιβλία δὲ αὐτοῦ Χωρογραφία οἰκουμενική, Εἰς τὰ δ ἑκατότρια τῆς Πτολεμαίου

Theodosius I reigned from A.D. 379 to 395, but Suidas may have made a mistake over the date. A marginal note opposite the entry Diocletian in a Leyden ms. of chronological tables by Theon of Alexandria says, "In his time Pappus wrote"; Diocletian reigned from A.D. 284 to 305. In Rome's edition of Pappus's commentary on Ptolemy's Syntaxis (Studi e Testi, liv. pp. x-xiii), a cogent argument is given for believing that Pappus actually wrote his Collection about A.D. 320.

Suidas obviously had a most imperfect knowledge of Pappus, as he does not mention his greatest work, the Synagoge or Collection. It is a handbook to the whole of Greek geometry, and is now our sole source for much of the history of that science. The first book and half of the second are missing. The remainder of the second book gives an account of Apollonius's method of working with large numbers (v. supra, pp. 352-357). The nature of the remaining books to the eighth will be indicated by the passages here cited. There is some evidence (v. infra, p. 607 n. a) that the work was originally in twelve books.

The edition of the Collection with ancillary material published in three volumes by Friedrich Hultsch (Berlin, 1876–564)
Pappus of Alexandria (a) General

Suidas, s.v. Pappus

Pappus, an Alexandrian, a philosopher, born in the time of the Emperor Theodosius I, when Theon the philosopher also flourished, who commented on Ptolemy's Table. His works include a Universal Geography, a Commentary on the Four Books of

1878) was a notable event in the revival of Greek mathematical studies. The editor's only major fault is one which he shares with his generation, a tendency to condemn on slender grounds passages as interpolated.

Pappus also wrote a commentary on Euclid's Elements; fragments on Book x. are believed to survive in Arabic (v. vol. i. p. 456 n. a). A commentary by Pappus on Euclid's Data is referred to in Marinus's commentary on that work. Pappus (v. vol. i. p. 301) himself refers to his commentary on the Analemma of Diodorus. The Arabic Fihrist says that he commented on Ptolemy's Planisphaerium.

The separate books of the Collection were divided by Pappus himself into numbered sections, generally preceded by a preface, and the editors have also divided the books into chapters. References to the Collection in the selections here given (e.g., Coll. iii. 11. 28, ed. Hultsch 68. 17-70. 8) are first to the book, then to the number or preface in Pappus's division, then to the chapter in the editors' division, and finally to the page and line of Hultsch's edition. In the selections from Book vii. Pappus's own divisions are omitted as they are too complicated, but in the collection of lemmas the numbers of the propositions in Hultsch's edition are added as these are often cited.
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Мегάλης συντάξεως υπόμνημα, Ποταμίος τούς ἐν Διβύῃ, Ὀνειροκριτικά.

(b) PROBLEMS AND THEOREMS

Papp. Coll. iii., Praef. 1, ed. Hultsch 30. 8-32. 3

Οἱ τὰ ἐν γεωμετρία ᾠδούμενα βουλόμενοι τεχνικώτερον διακρίνειν, ὡς κράτιστε Πανδροσίων, πρόβλημα μὲν ἄξιονι ταῦτα ἐν' οὗ προβάλλεται τι ποιήσαι καὶ κατασκευάσαι, θεώρημα δὲ ἐν ὧν τινών ὑποκειμένων τὸ ἐπόμενον αὐτοῖς καὶ πάντως ἐπισυμβαίνων θεωρεῖται, τῶν παλαιῶν τῶν μὲν προβλήματα πάντα, τῶν δὲ θεωρήματα εἶναι φασκότων. Ὁ μὲν οὖν τὸ θεώρημα προτείνων, συνιδὼν ὑπεροχὴν τρόπον, τὸ ἀκόλουθον τούτω ἄξιον ἄντειν καὶ οὐκ ἄν ἄλλως ἤγιος προτείνοι, ὅ δὲ τὸ πρόβλημα προτείνων [ἂν μὲν ἀμαθῆς ἢ καὶ παντάπασιν ἰδιώτης], κἂν ἀδύνατον πως κατασκευασθήναι προστάξῃ, σύγγνωστός ἐστιν καὶ ἀνυπεύθυνος. τοῦ γὰρ ἄναλοντος ἐργον καὶ τοῦτο διορίσαι, τὸ τε δυνατὸν καὶ τὸ ἀδύνατον, κἂν ἢ δυνατόν, πότε καὶ πῶς καὶ ποσακχῶς δυνατόν.

ἐὰν δὲ προσποιούμενος ἢ τὰ μαθηματά πως ἀπείρως προβάλλων, οὐκ ἐστὶν αἰτίας ἔξω. πρῶτη γοῦν τινες τῶν τὰ μαθημάτων προσποιομένων εἰδέναι διὰ σοῦ τὰς τῶν προβλημάτων προτάσεις ἀμαθῶς ἠκὼν ὑμιᾶν. περὶ δὲν ἔδει καὶ τῶν

Suidas seems to be confusing Ptolemy's Μαθηματική τετράβιβλος σύνταξις (Tetrabiblos or Quadripartitum) which was in four books but on which Pappus did not comment, with the Μαθηματική σύνταξις (Syntaxis or Almagest), which was the subject of a commentary by Pappus but extended to 566.
REVIVAL OF GEOMETRY: PAPPUS

Ptolemy’s Great Collection,* The Rivers of Libya, On the Interpretation of Dreams.

(b) PROBLEMS AND THEOREMS

Pappus, Collection iii., Preface 1, ed. Hultsch 30. 3–32. 3

Those who favour a more exact terminology in the subjects studied in geometry, most excellent Pandrosion, use the term problem to mean an inquiry in which it is proposed to do or to construct something, and the term theorem an inquiry in which the consequences and necessary implications of certain hypotheses are investigated, but among the ancients some described them all as problems, some as theorems. Therefore he who propounds a theorem, no matter how he has become aware of it, must set for investigation the conclusion inherent in the premises, and in no other way would he correctly propound the theorem; but he who propounds a problem, even though he may require us to construct something which is in some way impossible, is free from blame and criticism. For it is part of the investigator’s task to determine the conditions under which a problem is possible and impossible, and, if possible, when, how and in how many ways it is possible. But when a man professing to know mathematics sets an investigation wrongly he is not free from censure. For example, some persons professing to have learnt mathematics from you lately gave me a wrong enunciation of problems. It is desirable that I should state some of the proofs of thirteen books. Pappus’s commentary now survives only for Books v. and vi., which have been edited by A. Rome, Studi e Testi, liv., but it certainly covered the first six books and possibly all thirteen.

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The method, as described by Pappus, but not reproduced here, does not actually solve the problem, but it does furnish a series of successive approximations to the solution, and deserves more kindly treatment than it receives from him.
these and of matters akin to them, for the benefit both of yourself and of other lovers of this science, in this third book of the *Collection*. Now the first of these problems was set wrongly by a person who was thought to be a great geometer. For, given two straight lines, he claimed to know how to find by plane methods two means in continuous proportion, and he even asked that I should look into the matter and comment on his construction, which is after this manner:

\[(c)\] **The Theory of Means**

*Ibid.* iii. 11. 28, ed. Hultsch 68. 17–70. 8

The second of the problems was this:

A certain other [geometer] set the problem of exhibiting the three means in a semicircle. Describing a semicircle \(AB\Gamma\), with centre \(E\), and taking any point \(\Delta\) on \(\Delta\Gamma\), and from it drawing \(\Delta B\) perpendicular to \(\Delta\Gamma\), and joining \(EB\), and from \(\Delta\) drawing \(\Delta Z\) perpendicular to it, he claimed simply that the three means had been set out in the semicircle, \(\Delta \Gamma\) being the arithmetic mean, \(\Delta B\) the geometric mean and \(BZ\) the harmonic mean.

That \(B\Delta\) is a mean between \(\Delta\Delta\), \(\Delta\Gamma\) in geometrical
τὴ γεωμετρικὴ ἀναλογία, ἢ δὲ ΕΓ τῶν ΑΔ, ΔΓ ἐν τῇ ἀριθμητικῇ μεσότητι, φανερὸν. ἔστι γὰρ ὡς μὲν ἡ ΑΔ πρὸς ΔΒ, ἡ ΔΒ πρὸς ΔΓ, ὡς δὲ ἡ ΑΔ πρὸς ἐαυτὴν, οὕτως ἡ τῶν ΑΔ, ΑΕ ύπεροχὴ, τούτητιν ἡ τῶν ΑΔ, ΕΓ, πρὸς τὴν τῶν ΕΓ, ΓΔ. πῶς δὲ καὶ ἡ ΖΒ μέση ἐστὶν τῆς ἀρμονικῆς μεσότητος, ἡ ποίων εὐθειῶν, οὐκ εἶπεν, μόνον δὲ ὁτι τρίτη ἀναλογὸν ἐστὶν τῶν ΕΒ, ΒΔ, ἀγνοῶν ὁτι ἀπὸ τῶν ΕΒ, ΒΔ, ΖΖ ἐν τῇ γεωμετρικῇ ἀναλογίᾳ οὐσῶν πλάσσεται ἡ ἀρμονικὴ μεσότητις. δειχθῆσεται γὰρ ὅτι ἡ μιᾶς ὑπέροχον ὁτι δύο αἱ ΕΒ καὶ τρεῖς αἱ ΔΒ καὶ μία ἡ ΒΖ ὡς μία συντεθεῖσαι ποιοῦσι τὴν μεῖζον ἀκραν τῆς ἀρμονικῆς μεσότητος, δύο δὲ αἱ ΒΔ καὶ μία ἡ ΒΖ τὴν μέσην, μία δὲ ἡ ΒΔ καὶ μία ἡ ΒΖ τὴν ἑλαχίστην.

(d) The Paradoxes of Erycinus

Ibid. iii. 24. 58, ed. Hultsch 104. 14–106. 9

Τὸ δὲ τρίτον τῶν προβλημάτων ἣν τόδε. "Εστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν
REVIVAL OF GEOMETRY: PAPPUS

proportion, and $E\Gamma$ between $A\Delta, \Delta\Gamma$ in arithmetical proportion, is clear. For

$$A\Delta : \Delta B = \Delta B : \Delta\Gamma,$$  
$[\text{Eucl. iii. 31, vi. 8 Por.}]$

and

$$A\Delta : A\Delta = (A\Delta - AE) : (E\Gamma - \Gamma\Delta) = (A\Delta - E\Gamma) : (E\Gamma - \Gamma\Delta).$$

But how $ZB$ is a harmonic mean, or between what kind of lines, he did not say, but only that it is a third proportional to $EB, B\Delta$, not knowing that from $EB, B\Delta, BZ$, which are in geometrical proportion, the harmonic mean is formed. For it will be proved by me later that a harmonic proportion can thus be formed—

greater extreme $= 2EB + 3\DeltaB + BZ$,
mean term $= 2B\Delta + BZ$,
lesser extreme $= B\Delta + BZ$.

(d) THE PARADOXES OF ERYCINUS

Ibid. iii. 24. 58, ed. Hultsch 104. 14–106. 9

The third of the problems was this:

Let $AB\Gamma$ be a right-angled triangle having the

• It is Pappus, in fact, who seems to have erred, for $BZ$ is a harmonic mean between $A\Delta, \Delta\Gamma$, as can thus be proved:

Since $B\Delta E$ is a right-angled triangle in which $\Delta Z$ is perpendicular to $BE$,

.$\therefore$$BZ : B\Delta = B\Delta : BE,$

$i.e.$$BZ \cdot BE = B\Delta^2 = A\Delta \cdot \Delta\Gamma.$

But

$BE = \frac{1}{2}(A\Delta + \Delta\Gamma);$  

.$\therefore$$BZ(A\Delta + \Delta\Gamma) = 2A\Delta \cdot \Delta\Gamma.$

.$\therefore$$A\Delta(BZ - \Delta\Gamma) = \Delta\Gamma(A\Delta - BZ),$  

$i.e.$$A\Delta : \Delta\Gamma = (A\Delta - BZ) : (BZ - \Delta\Gamma).$

and $:. BZ$ is a harmonic mean between $A\Delta, \Delta\Gamma$.

The three means and the several extremes have thus been
GREEK MATHEMATICS

Εξον την Β γωνίαν, και διήχω τις η ΑΔ, και κείσθω τη ΑΒ ἵση η ΔΕ, και δίχα τιμηθείσης της ΕΑ κατὰ τὸ Ζ, και ἐπιζευγθείσης τῆς ΖΓ δείξαι συναμφοτέρας τὰς ΔΖΓ δύο πλευρὰς ἐντὸς τοῦ τριγώνου μείζονας τῶν ἐκτὸς συναμφοτέρων τῶν ΒΑΓ πλευρῶν.

Καὶ ἐστὶ δήλον. ἐπεὶ γὰρ αἱ ΓΖΑ, τούτεστιν αἱ ΓΖΕ, τῆς ΓΑ μεῖζονες εἰσον, ἵση δὲ ἡ ΔΕ τῆς ΑΒ, αἱ ΓΖΔ ἄρα δύο τῶν ΓΑΒ μεῖζονες εἰσον... Αλλ’ ὅτι τούτο μὲν, ὅπως ἄν τις ἐθέλοι προτείνειν, ἀπειραχῶς δείκνυται δήλον, οὐκ ἄκαιρον δὲ καθολικῶτερον περὶ τῶν τοιούτων προβλημάτων διαλαβεῖν ἀπὸ τῶν φερομένων παραδόξων Ἐρυκίνου προτείνοντας οὕτως.

(e) The Regular Solids

Ibid. iii. 40. 75, ed. Hultsch 132. 1-11

Εἰς τὴν δοθεῖσαν σφαίραν ἐγγράφαι τὰ πέντε πολύεδρα, προγράφεται δὲ τάδε.

'Εστώ ἐν σφαίρα κύκλος ὁ ΑΒΓ, οὗ διάμετρος η ΑΓ καὶ κέντρον τὸ Δ, καὶ προκείσθω εἰς τὸν

represented by five straight lines (ΕΒ, ΒΖ, ΑΔ, ΔΓ, ΒΔ). Pappus takes six lines to solve the problem. He proceeds to define the seven other means and to form all ten means as linear functions of three terms in geometrical progression (v. vol. i. pp. 124-129).

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REVIVAL OF GEOMETRY: PAPPUS

angle B right, and let $\Delta \Delta$ be drawn, and let $\Delta E$ be placed equal to $AB$, then if $EA$ be bisected at $Z$, and $Z \Gamma$ be joined, to show that the sum of the two sides $\Delta Z$, $Z \Gamma$ within the triangle, is greater than the sum of the two sides $BA$, $A \Gamma$ without the triangle.

And it is obvious. For since $\Gamma Z + ZA > \Gamma A$, $[\text{Eucl. i. 20}$

i.e., $\Gamma Z + ZE > \Gamma A$,

while $\Delta E = AB$,

\[ \therefore \quad [\Gamma Z + ZE + \Delta E = \Gamma A + A] \]

i.e., $\Gamma Z + ZA > \Gamma A + AB. \ldots$

But it is clear that this type of proposition, according to the different ways in which one might wish to propound it, can take an infinite number of forms, and it is not out of place to discuss such problems more generally and [first] to propound this from the so-called paradoxes of Erycinus.\(^a\)

\(e\) The Regular Solids \(^b\)

Ibid. iii. 40. 75, ed. Hultsch 132. 1-11

In order to inscribe the five polyhedra in a sphere, these things are premised.

Let $AB\Gamma$ be a circle in a sphere, with diameter $A \Gamma$ and centre $\Delta$, and let it be proposed to insert in the

\(^a\) Nothing further is known of Erycinus. The propositions next investigated are more elaborate than the one just solved.

\(^b\) This is the fourth subject dealt with in Coll. iii. For the treatment of the subject by earlier geometers, v. vol. i. pp. 216-225, 466-479.
GREEK MATHEMATICS

κύκλον ἐμβαλεῖν εὐθείαν παράλληλον μὲν τῇ ΑΓ διαμέτρω, ἵναν δὲ τῇ δοθείσῃ μὴ μεῖζον οὐσιν τῆς ΑΓ διαμέτρου.

Κείσθω τῇ ἡμισείᾳ τῆς δοθείσης ἴσῃ ἡ ΕΔ, καὶ τῇ ΑΓ διαμέτρῳ ἕχων πρὸς ὅρθᾶς ἡ ΕΒ, τῇ δὲ ΑΓ παράλληλος ἡ ΒΖ, ἕτερη ἴσῃ ἑσται τῇ δοθείσῃ. διπλῆ γάρ ἐστιν τῆς ΕΔ, ἐπει καὶ ἴσῃ τῇ ΕΗ, παραλλήλου ἀκτείσης τῆς ΖΗ τῇ ΒΕ.

(f) Extension of Pythagoras's Theorem

Ibid. iv. 1. 1, ed. Hultsch 176. 9–178. 13

'Εαν ἢ τρίγωνον τὸ ΑΒΓ, καὶ ἀπὸ τῶν ΑΒ, ΒΓ ἀναγραφῇ τυχόντα παράλληλόγραμμα τὰ ΑΒΔΕ, ΒΓΖΗ, καὶ αἱ ΔΕ, ΖΗ ἐκβληθῶσιν ἐπὶ τὸ Θ, καὶ ἐπιζευχθῇ ἡ ΘΒ, γίνεται τὰ ΑΒΔΕ, 574
circle a chord parallel to the diameter $\Delta \Gamma$ and equal to a given straight line not greater than the diameter $\Delta \Gamma$.

Let $E\Delta$ be placed equal to half of the given straight line, and let $EB$ be drawn perpendicular to the diameter $\Delta \Gamma$, and let $BZ$ be drawn parallel to $\Delta \Gamma$; then shall this line be equal to the given straight line. For it is double of $E\Delta$, inasmuch as $ZH$, when drawn, is parallel to $BE$, and it is therefore equal to $EH$.

(f) Extension of Pythagoras’s Theorem


If $AB\Gamma$ be a triangle, and on $AB$, $B\Gamma$ there be described any parallelograms $AB\Delta E$, $B\Gamma ZH$, and $\Delta E$, $ZH$ be produced to $\Theta$, and $\Theta B$ be joined, then the

* This lemma gives the key to Pappus’s method of inscribing the regular solids, which is to find in the case of each solid certain parallel circular sections of the sphere. In the case of the cube, for example, he finds two equal and parallel circular sections, the square on whose diameter is two-thirds of the square on the diameter of the sphere. The squares inscribed in these circles are then opposite faces of the cube. In each case the method of analysis and synthesis is followed. The treatment is quite different from Euclid’s.
Εκβεβλήσθω γὰρ ἡ ΘΒ ἐπὶ τὸ Κ, καὶ διὰ τῶν Α, Γ τῆς ΘΚ παράλληλου ήχθωσαν αἱ ΑΛ, ΓΜ, καὶ ἐπεζεύχθω ἡ ΛΜ. ἐπεὶ παράλληλόγραμμόν ἐστὶν τὸ ΑΛΘΒ, αἱ ΑΛ, ΘΒ ἵσαι τὲ εἰσὶν καὶ παράλληλοι. ὅμως καὶ αἱ ΜΓ, ΘΒ ἵσαι τὲ εἰσὶν καὶ παράλληλοι, ώστε καὶ αἱ ΛΑ, ΜΓ ἵσαι τὲ εἰσὶν καὶ παράλληλοι. καὶ αἱ ΛΜ, ΑΓ ἀρὰ ἵσαι τὲ καὶ παράλληλοί εἰσιν: παράλληλόγραμμον ἄρα ἐστὶν τὸ ΑΛΜΓ ἐν γωνίᾳ τῆς ύπὸ ΛΑΓ, τουτέστιν συναμφοτέρω τῇ τε ύπὸ ΒΑΓ καὶ ύπὸ ΔΘΒ. ἵσθι γὰρ ἐστὶν ἡ ύπὸ ΔΘΒ τῇ ύπὸ ΛΑΒ. καὶ ἐπεὶ τὸ ΔΑΒΕ παράλληλόγραμμον τῷ ΛΑΒΘ ἵσον ἐστὶν (ἐπὶ τὲ γὰρ τῆς αὐτῆς βάσεως ἐστὶν 576
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parallelograms $AB\Delta E$, $BFZH$ are together equal to the parallelogram contained by $A\Gamma$, $\Theta B$ in an angle which is equal to the sum of the angles $BA\Gamma$, $\Delta \Theta B$.

For let $\Theta B$ be produced to $K$, and through $A$, $\Gamma$ let $A\Lambda$, $\Gamma M$ be drawn parallel to $\Theta K$, and let $\Lambda M$ be joined. Since $A\Lambda \Theta B$ is a parallelogram, $A\Lambda$, $\Theta B$ are equal and parallel. Similarly $M\Gamma$, $\Theta B$ are equal and parallel, so that $\Lambda A$, $M\Gamma$ are equal and parallel. And therefore $\Lambda M$, $A\Gamma$ are equal and parallel; therefore $A\Lambda M\Gamma$ is a parallelogram in the angle $A\Lambda A\Gamma$, that is an angle equal to the sum of the angles $BA\Gamma$ and $\Delta \Theta B$; for the angle $\Delta \Theta B$ = angle $\Lambda AB$. And since the parallelogram $\Delta ABE$ is equal to the parallelogram $\Lambda A\Theta B$ (for they are upon the same base $AB$ and in the
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τῆς ΑΒ καὶ ἐν ταῖς αὐταῖς παραλλήλους ταῖς ΑΒ, ΔΘ, ἀλλὰ τὸ ΔΑΒΘ τῷ ΔΑΚΝ ἴσον ἔστιν (ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως ἴσον τῆς ΛΑ καὶ ἐν ταῖς αὐταῖς παραλλήλους ταῖς ΛΑ, ΘΚ), καὶ τὸ ΑΔΕΒ ἄρα τῷ ΔΑΚΝ ἴσον ἔστιν. διὰ τὰ αὐτὰ καὶ τὸ ΒΗΖΓ τῷ ΝΚΓΜ ἴσον ἔστιν. τὰ ἄρα ΔΑΒΕ, ΒΗΖΓ παραλληλόγραμμα τῷ ΔΑΓΜ ἴσα ἔστιν, τούτου ὁπλά τῷ ὑπὸ ΑΓ, ΘΒ ἐν γωνίᾳ τῇ ὑπὸ ΔΑΓ, ἡ ἔστιν ἴση συναμφοτέραις ταῖς ὑπὸ ΒΑΓ, ΘΔ. καὶ ἔστι τούτῳ καθολικώτερον πολλῷ τοῦ ἐν τοῖς ὀρθογωνίοις ἐπὶ τῶν τετραγώνων ἐν τοῖς Στουχείοις δεδειγμένου.

(g) Circles Inscribed in the ἄρβηλος

Ibid. iv. 14. 19, ed. Hultsch 208. 9-21

Φέρεται ἐν τοῖς ἀρχαία πρότασις τοιαύτη ὑποκείσθω τρία ἡμικύκλια ἐφαπτόμενα ἀλλήλων

![Diagram of circles inscribed in a figure]

tὰ ΑΒΓ, ΑΔΕ, ΕΖΓ, καὶ εἰς τὸ μεταξὺ τῶν περιφερεῖων αὐτῶν χωρίων, δ ὁ καλοῦσιν ἄρβηλον, 578
same parallels $\Delta B\Delta \Theta$, while $\Delta A\Theta = \Delta AKN$ (for they are upon the same base $\Delta A$ and in the same parallels $\Delta A, \Theta K$), therefore $\Delta \Theta = \Delta AKN$. By the same reasoning $\Delta \Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$.

By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$.

Therefore the parallelograms $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$. By the same reasoning $\Delta A\Theta = \Delta AKN$.

And this is much more general than the theorem proved in the *Elements* about the squares on right-angled triangles.}

(g) **Circles Inscribed in the *dubetlos***


There is found in certain [books] an ancient proposition to this effect: Let $\Delta A\Theta$, $\Delta \Theta$, $\Delta \Theta$ be supposed to be three semicircles touching each other, and in the space between their circumferences, which

-- Eucl. i. 47, v. vol. i. pp. 178-185. In the case taken by Pappus, the first two parallelograms are drawn outwards and the third, equal to their sum, is drawn inwards. If the areas of parallelograms drawn outwards be regarded as of opposite sign to the areas of those drawn inwards, the theorem may be still further generalized, for the algebraic sum of the three parallelograms is equal to zero.
Three propositions (Nos. 4, 5 and 6) about the figure known as the ἀρβηλός from its resemblance to a leather-worker's knife are contained in Archimedes' Liber Assumptorum, which has survived in Arabic. They are included as particular cases in Pappus's exposition, which is unfortunately too long for reproduction here. Professor D'Arcy W. Thompson (The Classical Review, ivi, (1942), pp. 75-76) gives reasons for thinking that the ἀρβηλός was a saddler's knife rather than a shoemaker's knife, as usually translated.
is called the "leather-worker's knife," let there be inscribed any number whatever of circles touching both the semicircles and one another, as those about the centres H, Θ, K, A; to prove that the perpendicular from the centre H to ΛΓ is equal to the diameter of the circle about H, the perpendicular from Θ is double of the diameter of the circle about Θ, the perpendicular from K is triple, and the [remaining] perpendiculars in order are so many times the diameters of the proper circles according to the numbers in a series increasing by unity, the inscription of the circles proceeding without limit.*

(h) Spiral on a Sphere b

Ibid. iv. 35. 53-56, ed. Hultsch 264. 3-268. 21

Just as in a plane a spiral is conceived to be generated by the motion of a point along a straight line revolving in a circle, and in solids [such as the cylinder or cone,] c by the motion of a point along one straight line describing a certain surface, so also a corresponding spiral can be conceived as described on the sphere after this manner.

Let ΚΛΜ be a great circle in a sphere with pole Θ, and from Θ let the quadrant of a great circle ΘΝΚ be

b After leaving the δρβηλος, Pappus devotes the remainder of Book iv. to solutions of the problems of doubling the cube, squaring the circle and trisecting an angle. This part has been frequently cited already (v. vol. i. pp. 298-309, 336-363). His treatment of the spiral is noteworthy because his method of proof is often markedly different from that of Archimedes; and in the course of it he makes this interesting digression.

c Some such addition is necessary, as Commandinus, Chasles and Hultsch realized.
κύκλου τεταρτημόριον γεγράφθω τὸ ΘΝΚ, καὶ ἡ μὲν ΘΝΚ περιφέρεια, περὶ τὸ Θ μένον φερομένη κατὰ τῆς ἐπιφάνειας ὡς ἐπὶ τὰ Δ, Μ μέρη,

ἀποκαθιστάσθω πάλιν ἐπὶ τὸ αὐτὸ, σημεῖον δὲ τι φερομένον ἐπὶ αὐτῆς ἀπὸ τοῦ Θ ἐπὶ τὸ Κ παραγινέσθω· γράφει δὴ τινα ἐπὶ τῆς ἐπιφάνειας ἔλικα, οἷα ἐστὶν ἡ ΘΟΙΚ, καὶ ἦτις ἂν ἀπὸ τοῦ Θ γραφῇ μεγίστου κύκλου περιφέρεια, πρὸς τὴν ΚΛ περιφερειαν λόγον ἔχει ὡς ἡ ΛΘ πρὸς τὴν ΘΟ· λέγω δὴ ὅτι, ἂν ἐκτεθῇ τεταρτημόριον τοῦ μεγίστου ἐν τῇ σφαίρᾳ κύκλῳ τὸ ΑΒΓ περὶ κέντρου τὸ Δ, καὶ ἐπιζευγθῇ ἡ ΓΔ, γίνεται ὡς ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τὴν μεταξὺ τῆς ΘΟΙΚ ἔλικος καὶ τῆς ΚΝΘ περιφερείας ἀπολαμβανομένην ἐπιφάνειαν, οὕτως ὁ ΑΒΓΔ τομεύς πρὸς τὸ ΑΒΓ τμῆμα.

Ἡχθω γὰρ ἐφαπτομένη τῆς περιφερείας ἡ ΓΖ, καὶ περὶ κέντρου τὸ Γ διὰ τοῦ Α γεγράφθω περιφερεία ἡ ΑΕΖ· ἵσος ἀρὰ ὁ ΑΒΓΔ τομεύς τῷ 582
described, and, $\theta$ remaining stationary, let the arc $\theta NK$ revolve about the surface in the direction $\Lambda, M$

and again return to the same place, and [in the same time] let a point on it move from $\theta$ to $K$; then it will describe on the surface a certain spiral, such as $\theta OIK$, and if any arc of a great circle be drawn from $\theta$ [cutting the circle $KAM$ first in $\Lambda$ and the spiral first in $O$], its circumference $^a$ will bear to the arc $KA$ the same ratio as $\Lambda \theta$ bears to $\theta O$. I say then that if a quadrant $ABG$ of a great circle in the sphere be set out about centre $\Delta$, and $GA$ be joined, the surface of the hemisphere will bear to the portion of the surface intercepted between the spiral $\theta OIK$ and the arc $KN\theta$ the same ratio as the sector $ABG\Delta$ bears to the segment $ABG$.

For let $\Gamma Z$ be drawn to touch the circumference, and with centre $\Gamma$ let there be described through $A$ the arc $AEZ$; then the sector $ABG\Delta$ is equal to the

$^a$ Or, of course, the circumference of the circle $KAM$ to which it is equal.
Pappus's method of proof is, in the Archimedean manner, to circumscribe about the surface to be measured a figure consisting of sectors on the sphere, and to circumscribe about the segment $ABG$ a figure consisting of sectors of circles; in the same way figures can be inscribed. The divisions need, therefore, to be as numerous as possible. The conclusion can then be reached by the method of exhaustion.
sector $AEZ\Gamma$ (for \(\angle A\Delta \Gamma = 2 \cdot \angle A\Gamma Z\), and \(\Delta \Delta^2 = \frac{1}{2} \Delta \Gamma^2\)); I say, then, that the ratio of the aforesaid surfaces one towards the other is the same as the ratio of the sector $AEZ\Gamma$ to the segment $AB\Gamma$.

Let $ZE$ be the same [small] part of $ZA$ as $K\Lambda$ is of the whole circumference of the circle, and let $E\Gamma$ be joined; then the arc $B\Gamma$ will be the same part of the arc $AB\Gamma$.

But $\Theta\Omega$ is the same part of $\Theta\Omega\Lambda$ as $K\Lambda$ is of the whole circumference [by the property of the spiral]. And arc $\Theta\Omega\Lambda = \text{arc } AB\Gamma$ [ex constructione]. Therefore $\Theta\Omega = B\Gamma$. Let there be described through $O$ about the pole $\Theta$ the arc $O\Omega$, and through $B$ about centre $\Gamma$ the arc $BH$. Then since the [sector of the] spherical surface $\Lambda K\Theta$ bears to the [sector] $O\Omega\Omega$ the same ratio as the whole surface of the hemisphere bears to the surface of the segment with pole $\Theta$ and circular base $O\Omega$; while the surface of the hemisphere bears to the surface of the segment the same ratio as $\Theta\Lambda^2$ to $\Theta\Omega^2$, or $E\Gamma^2$ to $B\Gamma^2$, therefore the sector $K\Lambda\Theta$ on the surface [of the sphere] bears to $O\Omega\Omega$ the same ratio as the sector $EZ\Gamma$ [in the plane] bears to the sector $B\Omega\Gamma$.

Similarly we may show that all the sectors [on the surface of] the hemi-

\[\text{For arc } ZA : \text{arc } ZE = \text{angle } Z\Gamma A : \text{angle } Z\Gamma E.\]
\[\text{But angle } Z\Gamma A = \frac{1}{3} \cdot \text{angle } A\Delta \Gamma, \text{ and angle } Z\Gamma E = \frac{1}{3} \cdot \text{angle } B\Delta \Gamma\]

\[\text{[Eucl. iii. 32, 20].} \quad \therefore \text{arc } ZA : \text{arc } ZE = \text{arc } AB\Gamma : \text{arc } B\Gamma.\]

\[\text{Because the arc } K\Lambda \text{ is the same part of the circumference } K\Lambda M \text{ as the arc } O\Omega \text{ is of its circumference.}\]

\[\text{The square on } \Theta\Lambda \text{ is double the square on the radius of the hemisphere, and therefore half the surface of the hemisphere is equal to a circle of radius } \Theta\Lambda \text{ [Archim. De sph. et cyl. i. 33]; and the surface of the segment is equal to a circle of radius } \Theta\Omega \text{ [ibid. i. 42]; and as circles are to one another as the squares on their radii [Eucl. xii. 2], the surface of the hemisphere bears to the surface of the segment the ratio } \Theta\Lambda^2 : \Theta\Omega^2.\]
This would be proved by the method of exhaustion. It is proof of the great part played by this method in Greek geometry that Pappus can take its validity for granted.

For the surface of the hemisphere is double of the circle of radius $\Delta \Delta$ [Archim. De sph. et cyl. i. 33] and the sector $AB\Gamma \Delta$ is one-quarter of the circle of radius $\Delta \Delta$.

For the surface between the spiral and the base of the hemisphere is equal to the surface of the hemisphere less the surface cut off from the spiral in the direction $\Theta \mathrm{NK}$, \( i.e. \) Surface in question = surface of hemisphere -

\[ 8 \text{ segment } AB\Gamma, \]
\[ = 8 \text{ sector } AB\Gamma \Delta - 8 \text{ segment } AB\Gamma \]

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sphere equal to $K\Lambda \Theta$, together making up the whole surface of the hemisphere, bear to the sectors described about the spiral similar to $O\Theta N$ the same ratio as the sectors in $AZ\Gamma$ equal to $EZ\Gamma$, that is the whole sector $AZ\Gamma$, bear to the sectors described about the segment $AB\Gamma$ similar to $\Gamma BH$. In the same manner it may be shown that the surface of the hemisphere bears to the [sum of the] sectors inscribed in the spiral the same ratio as the sector $AZ\Gamma$ bears to the [sum of the] sectors inscribed in the segment $AB\Gamma$, so that the surface of the hemisphere bears to the surface cut off by the spiral the same ratio as the sector $AZ\Gamma$, that is the quadrant $AB\Gamma\Delta$, bears to the segment $AB\Gamma$. From this it may be deduced that the surface cut off from the spiral in the direction of the arc $\Theta NK$ is eight times the segment $AB\Gamma$ (since the surface of the hemisphere is eight times the sector $AB\Gamma\Delta$), while the surface between the spiral and the base of the hemisphere is eight times the triangle $A\Gamma\Delta$, that is, it is equal to the square on the diameter of the sphere.

\[
\begin{align*}
&= 8 \text{ triangle } A\Gamma\Delta \\
&= 4A\Delta^2 \\
&= (2A\Delta)^2,
\end{align*}
\]

and $2A\Delta$ is the diameter of the sphere.

Heath (H.G.M. ii. 384-385) gives for this elegant proposition an analytical equivalent, which I have adapted to the Greek lettering. If $\rho, \omega$ are the spherical co-ordinates of $O$ with reference to $\Theta$ as pole and the arc $\Theta NK$ as polar axis, the equation of the spiral is $\omega = 4\rho$. If $A$ is the area of the spiral to be measured, and the radius of the sphere is taken as unity, we have as the element of area

\[dA = d\omega(1 - \cos \rho) = 4d\rho(1 - \cos \rho).\]
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(2) Isoperimetric Figures

Ibid. v., Praef. 1-3, ed. Hultsch 304. 5-308. 5

Σοφίας καὶ μαθημάτων ἐννοιαν ἀρίστην μὲν καὶ
tελειοτάτην ἀνθρώποις θεὸς ἔδωκεν, ὡς κράτιστε
Μεγεθίον, ἐκ μέρους δὲ που καὶ τῶν ἁλόγων ζῴων
μοῖραν ἀπένειμέν τισιν. ἀνθρώποις μὲν οὖν ἀτε
λογικοὶς οὗσι τὸ μετὰ λόγου καὶ ἀποδείξεως
παρέσχεν ἔκαστα ποιεῖν, τοὺς δὲ λοιποῖς ζῷοις
ἀνευ λόγου τὸ χρῆσιμον καὶ βιωφελές αὐτὸ μόνον
κατὰ τινα φυσικὴν πρόνοιαν ἐκάστοις ἔχειν ἐδωρή-
σατο. τούτο δὲ μάθοι τις ἃν ὑπάρχον καὶ ἐν ἑτέροις
μὲν πλείστοις γένεσιν τῶν ζῴων, οὐχ ἥκιστα δὲ
καὶ ταῖς μελίσσαις. ἦ τε γάρ εὐταξία καὶ πρὸς
τὰς ἠγουμένας τῆς ἐν αὐτᾶς πολυτείας εὑπείθεα
θαυμαστὴ τις, ἢ τε φιλοτιμία καὶ καθαριότης ἢ
περὶ τὴν τοῦ μέλιτος συναγωγὴν καὶ ἢ περὶ τὴν
φυλακῆς αὐτοῦ πρόνοια καὶ οἰκονομία πολὺ μᾶλλον
θαυμασιωτέρα. πεπιστευμέναι γάρ, ὡς εἰκός,
παρὰ θεῶν κομίζεων τοῖς τῶν ἀνθρώπων μουσικοῖς

\[ A = \int_0^{\frac{1}{2}\pi} 4d\rho(1 - \cos \rho) \]

\[ = 2\pi - 4. \]

\[ \frac{A}{\text{surface of hemisphere}} = \frac{2\pi - 4}{2\pi} \]

\[ = \frac{\frac{1}{2}\pi - \frac{1}{2}}{\frac{1}{2}\pi} \]

\[ = \frac{\text{segment } \overline{AB}\Gamma}{\text{sector } \overline{AB}\Gamma\Delta}. \]

* The whole of Book v. in Pappus's Collection is devoted
to isoperimetry. The first section follows closely the exposition
of Zenodorus as given by Theon (v. supra, pp. 386-395), 588
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(i) ISOPERIMETRIC FIGURES

Ibid. v., Preface 1-3, ed. Hultsch 304. 5–308. 5

Though God has given to men, most excellent Megethion, the best and most perfect understanding of wisdom and mathematics, He has allotted a partial share to some of the unreasoning creatures as well. To men, as being endowed with reason, He granted that they should do everything in the light of reason and demonstration, but to the other unreasoning creatures He gave only this gift, that each of them should, in accordance with a certain natural forethought, obtain so much as is needful for supporting life. This instinct may be observed to exist in many other species of creatures, but it is specially marked among bees. Their good order and their obedience to the queens who rule in their commonwealths are truly admirable, but much more admirable still is their emulation, their cleanliness in the gathering of honey, and the forethought and domestic care they give to its protection. Believing themselves, no doubt, to be entrusted with the task of bringing from the gods to the more cultured part of mankind a share of

except that Pappus includes the proposition that of all circular segments having the same circumference the semicircle is the greatest. The second section compares the volumes of solids whose surfaces are equal, and is followed by a digression, already quoted (supra, pp. 194-197) on the semi-regular solids discovered by Archimedes. After some propositions on the lines of Archimedes' De sph. et cyl., Pappus finally proves that of regular solids having equal surfaces, that is greatest which has most faces.

The introduction, here cited, on the sagacity of bees is rightly praised by Heath (H.G.M. ii. 389) as an example of the good style of the Greek mathematicians when freed from the restraints of technical language.
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τής ἀμβροσίας ἀπόμοιραν τινα ταύτην οὐ μάτην ἐκχειν εἰς γῆν καὶ ξύλον ἢ τινα ἐτέραν ἀσχήμονα καὶ ἄτακτον ὑλὴν ἥξιωσαν, ἀλλ' ἐκ τῶν ήδίστων ἐπὶ γῆς φυσικών ἀνθέων συνάγουσι τὰ κάλλιστα κατασκευάζουσι ἐκ τούτων εἰς τὴν τοῦ μέλιτος ὑποδοχὴν ἀγγεία τὰ καλούμενα κηρία πάντα μὲν ἀλλήλοις ἵσα καὶ ὄμοια καὶ παρακείμενα, τῷ δὲ τχήματι ἐξάγωνα.

Τούτο δ' ὅτι κατά τινα γεωμετρικήν μηχανῆν ται πρόνοιαν οὔτως ἀν μάθομεν. πάντως μὲν γὰρ ὄντος δεῖν τὰ σχῆματα παρακείσθαι τε ἀλλήλοις καὶ κοινωνεῖν κατὰ τὰς πλευρὰς, ἵνα μὴ τοὺς μεταξὺ παραπληρώμασιν ἐμπίπτοντά τινα ἔτερα λυμήνηται αὐτῶν τὰ ἔργα· τρία δὲ σχῆματα εὐθύγραμμα τὸ προκείμενον ἐπιτελεῖν ἔδυνατο, λέγω δὲ τεταγμένα τὰ ἰσοπλευρά τε καὶ ἵσογυνία, τὰ δ' ἀνόμοια ταῖς μελίσσαις οὐκ ἦρεσεν. τὰ μὲν οὖν ἰσοπλευρα τρίγωνα καὶ τετράγωνα καὶ τὰ ἐξάγωνα χωρίς ἀνομοίων παραπληρωμάτων ἀλλήλοις δύναται παρακείμενα τὰς πλευρὰς κοινός ἔχειν [ταύτα] γὰρ δύναται συμπληρωθῶν ἐς αὐτῶν τὸν περὶ τὸ αὐτὸ σημεῖον τόπον, ἐτέρω δὲ τεταγμένῳ σχῆματι τούτῳ ποιεῖν ἀδύνατον]. 1 δ' γὰρ περὶ τὸ αὐτὸ σημεῖον τόπος ὑπὸ ἤ μὲν τριγώνων ἰσοπλεύρων καὶ διὰ ἤ γυνἰῶν, ὅν ἔκαστῃ διμοιρίου ἔστιν ὀρθὴς, συμπληρωθῇ, τεσσάρων δὲ τετραγώνων καὶ δ' ὀρθῶν γυνἰῶν [αὐτοῖν], 2 τριῶν δὲ ἐξαγώνων καὶ ἐξαγώνου γυνἰῶν τριῶν, ὅν ἔκαστῃ ἢ γ' ἔστιν ὀρθής. πεντάγωνα δὲ τὰ τρία μὲν οὐ φθάνει συμπληρώσας τὸν περὶ τὸ αὐτὸ σημεῖον τόπον, ὑπερβάλλει δὲ τὰ τέσσαρα· τρεῖς μὲν γὰρ τοῦ πενταγώνου γυνίαι δ' ὀρθῶν ἐλάσσονες εἰσιν 590
ambrosia in this form, they do not think it proper to
pour it carelessly into earth or wood or any other
unseemly and irregular material, but, collecting the
fairest parts of the sweetest flowers growing on the
earth, from them they prepare for the reception of
the honey the vessels called honeycombs, [with cells]
all equal, similar and adjacent, and hexagonal in form.

That they have contrived this in accordance with a
certain geometrical forethought we may thus infer.
They would necessarily think that the figures must
all be adjacent one to another and have their sides
common, in order that nothing else might fall into the
interstices and so defile their work. Now there are
only three rectilinéal figures which would satisfy the
condition, I mean regular figures which are equilateral
and equiangular, inasmuch as irregular figures would
be displeasing to the bees. For equilateral triangles
and squares and hexagons can lie adjacent to one
another and have their sides in common without
irregular interstices. For the space about the same
point can be filled by six equilateral triangles and six
angles, of which each is \( \frac{2}{3} \) right angle, or by four
squares and four right angles, or by three hexagons
and three angles of a hexagon, of which each is
\( 1 \frac{1}{3} \) right angle. But three pentagons would not
suffice to fill the space about the same point, and four
would be more than sufficient; for three angles of
the pentagon are less than four right angles (inasmuch

\[ \text{1 } \tau\alpha\nu\rho\alpha \ldots \ \delta\delta\nu\alpha\tau\alpha \textrm{ om. Hultsch.} \]

\[ \text{2 } \text{"} \alpha\nu\tau\omicron\omicron\ \text{spurium, nisi forte a} \nu\tau\omicron\nu \text{ dedit scriptor"—} \]

\[ \text{Hultsch.} \]
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(ἐκάστη γὰρ γωνία μιᾶς καὶ ε’ ἔστιν ὀρθὴς),
tέσσαρες δὲ γωνίαι μεῖζον τῶν τεσσάρων ὀρθῶν.
ἐπτάγωνα δὲ οὖδὲ τρία περὶ τὸ αὐτὸ σημεῖον
dύναται τίθεσθαι κατὰ τὰς πλευρὰς ἀλλήλους παρα-
κείμενα· τρεῖς γὰρ ἐπταγώνου γωνίαι τετράγων ὀρθῶν μεῖζονες (ἐκάστη γὰρ ἔστιν μιᾶς ὀρθῆς καὶ
τριῶν ἐβδομῶν). ἔτι δὲ μάλλον ἐπὶ τῶν πολυγωνο-
tέρων ὁ αὐτὸς ἐφαρμόσαι δυνήσεται λόγος. ὀντων
dὲ οὖν τριῶν σχημάτων τῶν ἔξι αὐτῶν δυναμένων
συμπληρώσαι τὸν περὶ τὸ αὐτὸ σημεῖον τόπον,
τριγώνον τε καὶ τετραγώνον καὶ ἔξαγωγον, τὸ
πολυγωνότερον εἰλαντο διὰ τὴν σοφίαν αἰ μέλισσαι
πρὸς τὴν παρασκευὴν, ἀτε καὶ πλείον ἐκατέρου
τῶν λοιπῶν αὐτὸ χωρεῖν ὑπολαμβάνουσα τοῖς.
Καὶ αἰ μέλισσαι μὲν τὸ χρῆσιμον αὑταῖς ἐπί-
στανται μόνον τοῦ ὅτι τὸ ἐξάγωνον τοῦ τετρα-
γώνου καὶ τοῦ τριγώνου μεῖζον ἔστιν καὶ χωρῆσαι
dύναται πλεῖον μέλι τῆς ἱσης εἰς τὴν ἐκάστου
κατασκευὴν ἀναλυσκομένης ὕλης, ἥμεις δὲ πλέον
tῶν μελισσῶν σοφίας μέρος ἔχειν ὑποσχούμενον
ζητήσομεν τι καὶ περιμέτρον. τῶν γὰρ ἱσην
ἐχόντων περίμετρον ἰσοπλεύρων τε καὶ ἰσογωνίων
ἐπιπέδων σχημάτων μεῖζον ἔστιν ἀεί τὸ πολυ-
γωνότερον, μέγιστος δ’ ἐν πᾶσιν ο κύκλος, ὅταν
ἱσην αὐτοῖς περίμετρον ἔχῃ.

(5) Apparent Form of a Circle

Ibid. vi. 48. 90-91, ed. Hultsch 580. 12-27

"Εστώ κύκλος ὁ ΑΒΓ, οὗ κέντρον τὸ Ε, καὶ
ἀπὸ τοῦ Ε πρὸς ὀρθὰς ἔστω τῷ τοῦ κύκλου ἐπί-
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as each angle is $1\frac{1}{2}$ right angle), and four angles are greater than four right angles. Nor can three heptagons be placed about the same point so as to have their sides adjacent to each other; for three angles of a heptagon are greater than four right angles (inasmuch as each is $1\frac{1}{2}$ right angle). And the same argument can be applied even more to polygons with a greater number of angles. There being, then, three figures capable by themselves of filling up the space around the same point, the triangle, the square and the hexagon, the bees in their wisdom chose for their work that which has the most angles, perceiving that it would hold more honey than either of the two others.

Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. But we, claiming a greater share in wisdom than the bees, will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest of them all is the circle having its perimeter equal to them.

(j) Apparent Form of a Circle

_Ibid. vi._ 48. 90-91, ed. Hultsch 580. 12-27

Let ABΓ be a circle with centre E, and from E let EZ be drawn perpendicular to the plane of the circle;

* Most of Book vi. is astronomical, covering the treatises in the _Little Astronomy_ (v. _supra_, p. 408 n. b). The proposition here cited comes from a section on Euclid’s _Optics_.

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Τούτο δὲ δῆλον· ἀπασαι γὰρ αἱ ἀπὸ τοῦ Ζ πρὸς τὴν τοῦ κύκλου περιφέρειαν προσπίπτουσαι εὐθεῖαι ἵσαι εἰς ἀλλήλας καὶ ἵσαι γωνίας περιέχουσιν.

Μὴ ἔστω δὲ ἡ ΕΖ πρὸς ὀρθὰς τῷ τοῦ κύκλου ἐπιπέδως, ἵσῃ δὲ ἐστῶ τῇ ἐκ τοῦ κέντρου τοῦ κύκλου· λέγω, ὅτι τοῦ ὀμματος ὄντος πρὸς τῷ Ζ σημείῳ καὶ ὀυτῶς αἱ διάμετροι ἵσαι ὀρθώνται.

"Ηχθωσαν γὰρ δύο διάμετροι αἱ ΑΓ, ΒΔ, καὶ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΒ, ΖΓ, ΖΔ. ἐπεὶ αἱ
I say that, if the eye be placed on EZ, the diameters of the circle appear equal.

This is obvious; for all the straight lines falling from Z on the circumference of the circle are equal one to another and contain equal angles.

Now let EZ be not perpendicular to the plane of the circle, but equal to the radius of the circle; I say that, if the eye be at the point Z, in this case also the diameters appear equal.

For let two diameters AG, BG be drawn, and let ZA, ZB, ZG, ZG be joined. Since the three straight

- As they will do if they subtend an equal angle at the eye.

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τρεῖς αἳ ΕΑ, ΕΓ, ΕΖ ἵσαι εἰσὶν, ὄρθη ἀρα ἡ ὑπὸ ΑΖΓ γωνία. διὰ τὰ αὕτα δὴ καὶ ἡ ὑπὸ ΒΖΔ ὄρθη ἐστὶν· ἵσαι ἀρα φανῆσονται αἳ ΑΓ, ΒΔ διάμετροι. ὁμοίως δὴ δείξομεν ὅτι καὶ πᾶσαι.

(k) The “Treasury of Analysis”

Ibid. vii., Praef. 1-3, ed. Hultsch 634. 3–636. 30

'Ο καλοῦμενος ἀναλυόμενος, ἕρμοδωρε τέκνων, κατὰ σύλληψιν ἱδίᾳ τίς ἔστιν ὑλὴ παρεσκευασμένη μετὰ τὴν τῶν κοινῶν στοιχείων ποίησιν τοῖς βουλομένοις ἀναλαμβάνειν ἐν γραμμαῖς δύναμιν εὔρετρὴν τῶν προτεινομένων αὐτοῖς προβλημάτων, καὶ εἰς τοῦτο μόνον χρησίμη καθεστώσα. γεγρα-πται δὲ ὑπὸ τριῶν ἀνδρῶν, Εὐκλείδου τε τοῦ Στοιχειωτοῦ καὶ Ἀπολλωνίου τοῦ Περγαῖον καὶ Ἀρισταῖον τοῦ πρεσβυτέρου, κατὰ ἀνάλυσιν καὶ σύνθεσιν ἔχουσα τὴν ἔφοδον.

'Ανάλυσις τοῖνυν ἐστὶν ὁδὸς ἀπὸ τοῦ ξητουμένου ὡς ὀμολογουμένου διὰ τῶν ἔξης ἀκολούθων ἐπὶ τι ὀμολογουμένων συνθέσεις ἐν μὲν γὰρ τῇ ἀνάλυσις τοῦ ξητουμένου ὡς γεγονός υποθέμενοι τὸ ἐξ οὐ τοῦτο συμβαίνει σκοπούμεθα καὶ πάλιν ἐκεῖνον τὸ προηγούμενον, ἐως ἃν οὕτως ἀναποδίζοντες καταντήσωμεν εἰς τι τῶν ἡδη γνωριζομένων ἡ τάξιν ἀρχής ἐχόντων καὶ τὴν τοιαύτην ἔφοδον ἀνάλυσιν καλοῦμεν, οἷον ἀνάπαλιν λύσιν.

'Εν δὲ τῇ συνθέσει ἐξ ὑποστροφῆς τὸ ἐν τῇ ἀνάλυσις καταληφθέν υστατον ὑποστησάμενον γεγονός ἡδη, καὶ ἐπόμενα τὰ ἐκεῖ [ἐνταῦθα]1 προ-

1 ἐνταῦθα om. Hultsch.
lines EA, EG, EZ are equal, therefore the angle AZG is right. And by the same reasoning the angle BZΔ is right; therefore the diameters AG, BΔ appear equal. Similarly we may show that all are equal.

(k) The "Treasury of Analysis"


The so-called *Treasury of Analysis*, my dear Hermodorus, is, in short, a special body of doctrine furnished for the use of those who, after going through the usual elements, wish to obtain power to solve problems set to them involving curves, and for this purpose only is it useful. It is the work of three men, Euclid the writer of the *Elements*, Apollonius of Perga and Aristaeus the elder, and proceeds by the method of analysis and synthesis.

Now analysis is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a reverse solution.

But in synthesis, proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural

*a Or, perhaps, “to give a complete theoretical solution of problems set to them”; *v. supra*, p. 414 n. a.*
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ηγούμενα κατὰ φύσιν τάξαντες καὶ ἄλληλοις ἐπισυνήθεντες, εἰς τέλος ἀφικνούμεθα τῆς τοῦ ζητουμένου κατασκευῆς· καὶ τούτο καλοῦμεν σύνθεσιν.

Διὸ τὸ εἶστιν ἀναλύσεως γένος τὸ μὲν ζητητικὸν τάληθοὺς, δὲ καλεῖται θεωρητικόν, τὸ δὲ ποριστικὸν τοῦ προταθέντος [λέγειν], δὲ καλεῖται προβληματικόν. ἐπὶ μὲν οὖν τοῦ θεωρητικοῦ γένους τὸ ζητουμένον ὡς ὅν υποθέμεναι καὶ ὡς ἀληθὲς, εἶτα διὰ τῶν ἐξῆς ἀκολούθων ὡς ἀληθῶν καὶ ὡς ἐστὶν καθ' ὑπόθεσιν προελθόντες ἐπὶ τινὶ ὁμολογούμενον, ἐάν μὲν ἀληθὲς ἢ ἐκεῖνο τὸ ὁμολογούμενον, ἀληθὲς ἐσται καὶ τὸ ζητουμένον, καὶ ἡ ἀπόδειξις ἀντίστροφος τῇ ἀναλύσει, ἐάν δὲ πρεύδει ὁμολογούμενον ἐντύχομεν, πρεύδος ἐσται καὶ τὸ ζητουμένον. ἐπὶ δὲ τοῦ προβληματικοῦ γένους τὸ προταθὲν ὡς γνωσθὲν υποθέμεναι, εἶτα διὰ τῶν ἐξῆς ἀκολούθων ὡς ἀληθῶν προελθόντες ἐπὶ τινὶ ὁμολογούμενον, ἐάν μὲν τὸ ὁμολογούμενον δυνατὸν ἢ καὶ ποριστόν, δὲ καλοῦσιν οἱ ἀπὸ τῶν μαθηματῶν δοθέν, δυνατὸν ἔσται καὶ τὸ προταθὲν, καὶ πάλιν ἡ ἀπόδειξις ἀντίστροφος τῇ ἀναλύσει, ἐάν δὲ ἀδυνάτω ὁμολογούμενῳ ἐντύχωμεν, ἀδύνατον ἐσται καὶ τὸ πρόβλημα.

Τοσαῦτα μὲν οὖν περὶ ἀναλύσεως καὶ συνθέσεως.

Τῶν δὲ προειρημένων τοῦ ἀναλυμένου βιβλίων ἢ τάξις εἶστιν τοιαύτη. Εὐκλείδου Δεδομένων βιβλίων ἃ, Ἀπολλωνίου Λόγου ἀποτομῆς β, Χωρίου ἀποτομῆς β, Διωρισμένης τομῆς δύο, Ἐπαφῶν δύο, Εὐκλείδου Πορισμάτων τρία, Ἀπολλωνίου Νεύσεων δύο, τοῦ αὐτοῦ Τόπων ἐπιπέδων δύο,

1 λέγει om. Hultsch.
order as consequents what were formerly antecedents
and linking them one with another, we finally arrive
at the construction of what was sought; and this we
call synthesis.

Now analysis is of two kinds, one, whose object is
to seek the truth, being called *theoretical*, and the
other, whose object is to find something set for find-
ing, being called *problematical*. In the theoretical
kind we suppose the subject of the inquiry to exist
and to be true, and then we pass through its con-
sequences in order, as though they also were true and
established by our hypothesis, to something which is
admitted; then, if that which is admitted be true,
that which is sought will also be true, and the proof
will be the reverse of the analysis, but if we come
upon something admitted to be false, that which is
sought will also be false. In the problematical kind
we suppose that which is set as already known, and
then we pass through its consequences in order, as
though they were true, up to something admitted;
then, if what is admitted be possible and can be done,
that is, if it be what the mathematicians call *given*,
what was originally set will also be possible, and the
proof will again be the reverse of the analysis, but if
we come upon something admitted to be impossible,
the problem will also be impossible.

So much for analysis and synthesis.

This is the order of the books in the aforesaid
*Treasury of Analysis*. Euclid’s *Data*, one book,
Apollonius’s *Cutting-off of a Ratio*, two books, *Cutting-
off of an Area*, two books, *Determinate Section*, two
books, *Contacts*, two books, Euclid’s *Porisms*, three
books, Apollonius’s *Vergings*, two books, his *Plane
Loci*, two books, *Conics*, eight books, Aristaeus’s
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Κωνικῶν ἦ, Ἀρισταῖον Τόπων στερεῶν πέντε, Εὐκλείδου Τόπων τῶν πρὸς ἐπιφανείᾳ δύο, Ἐρατοσθένους Περὶ μεσοτήτων δύο. γίνεται βιβλία λγ, ὥς τὰς περιοχὰς μέχρι τῶν Ἀπολλωνίου Κωνικῶν ἐξεθέμην οὐ πρὸς ἐπίσκεψιν, καὶ τὸ πλῆθος τῶν τόπων καὶ τῶν διορισμῶν καὶ τῶν πτώσεων καθ' ἐκαστον βιβλίον, ἀλλὰ καὶ τὰ λήμματα τὰ ζητούμενα, καὶ οὐδεμίαν ἐν τῇ πραγματείᾳ τῶν βιβλίων καταλέλοιπα ζητήσων, ὡς ἐνόμιζον.

(I) Locus with Respect to Five or Six Lines

Ibid. vii. 38-40, ed. Hultsch 680. 2-30

Ἐὰν ἀπὸ τινος σημείου ἐπὶ θέσει δεδομένας εὐθείας πέντε καταχθῶσιν εὐθείαι ἐν δεδομέναις γωνίαις, καὶ λόγος ἡ δεδομένος τοῦ ὑπὸ τριῶν κατηγομένων περιεχομένου στερεοῦ παραλληλεπιπέδου ὀρθογώνιον πρὸς τὸ ὑπὸ τῶν λοιπῶν δύο κατηγομένων καὶ δοθέσις τινὸς περιεχόμενον παραλληλεπιπέδου ὀρθογώνιον, ἀψεται τὸ σημεῖον θέσει δεδομένης γραμμῆς. ἐὰν τε ἐπὶ Salir, καὶ λόγος ἡ δοθεῖς τοῦ ὑπὸ τῶν τριῶν περιεχομένου προερχομένου στερεοῦ πρὸς τὸ ὑπὸ τῶν λοιπῶν τριῶν, πάλιν τὸ σημεῖον ἀψεται θέσει δεδομένης. ἐὰν δὲ ἐπὶ πλεύονας τῶν Salir, οὐκετί μὲν ἔχουσι λέγειν, "ἔὰν λόγος ἡ δοθεῖς τοῦ ὑπὸ τῶν δὲ περιεχομένου τινὸς πρὸς τὸ ὑπὸ τῶν λοιπῶν," ἐπεὶ οὐκ ἐστι τι

* These propositions follow a passage on the locus with respect to three or four lines which has already been quoted (v. vol. i. pp. 486-489). The passages come from Pappus’s 600
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Solid Loci, five books, Euclid's Surface Loci, two books, Eratosthenes' On Means, two books. In all there are thirty-three books, whose contents as far as Apollonius's Conics I have set out for your examination, including not only the number of the propositions, the conditions of possibility and the cases dealt with in each book, but also the lemmas which are required; indeed, I believe that I have not omitted any inquiry arising in the study of these books.

(l) Locus with Respect to Five or Six Lines

Ibid. vii. 38-40, ed. Hultsch 680. 2-30

If from any point straight lines be drawn to meet at given angles five straight lines given in position, and the ratio be given between the volume of the rectangular parallelepiped contained by three of them to the volume of the rectangular parallelepiped contained by the remaining two and a given straight line, the point will lie on a curve given in position. If there be six straight lines, and the ratio be given between the volume of the aforesaid solid formed by three of them to the volume of the solid formed by the remaining three, the point will again lie on a curve given in position. If there be more than six straight lines, it is no longer permissible to say "if the ratio be given between some figure contained by four of them to some figure contained by the remainder," since no figure can be contained in more account of the Conics of Apollonius, who had worked out the locus with respect to three or four lines. It was by reflection on this passage that Descartes evolved the system of co-ordinates described in his Géométrie.
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περιεχόμενον ύπο πλειόνων ἢ τριῶν διαστάσεων.
συγκεχωρήκας δὲ έαυτοῖς οί βραχὺ πρὸ ἡμῶν
ἐρμηνεύειν τὰ τοιαῦτα, μηδὲ ἐν μηδαμῶς διάλυστον
σημαίνοντες, τὸ ύπὸ τῶν δὲ περιεχόμενον λέγοντες
ἐπὶ τὸ ἀπὸ τῆς τετράγωνος ἡ ἐπὶ τὸ ύπὸ τῶν.
παρὴν δὲ διὰ τῶν συνημμένων λόγων ταῦτα καὶ
λέγειν καὶ δεικνύναι καθόλου καὶ ἐπὶ τῶν προειρη-
μένων προτάσεων καὶ ἐπὶ τούτων τῶν τρόπων
τούτων ἐὰν ἀπὸ τίνος σημείου ἐπὶ θέσει δεδομένας
εὐθεῖας καταχθῶσιν εὐθεῖαι ἐν δεδομέναις γωνίαις,
καὶ δεδομένος ἡ λόγος οἱ συνημμένος ἐξ οὗ ἔχει
μία κατηγομένη πρὸς μίαν καὶ ἔτερα πρὸς ἔτεραν,
καὶ ἄλλη πρὸς ἄλλην, καὶ ἡ λοιπὴ πρὸς λοιπὴν,
ἐὰν ὅσιν ζ, ἐὰν δὲ η, καὶ ἡ λοιπὴ πρὸς λοιπὴν,
τὸ σημεῖον ἄφησει θέσει δεδομένης γραμμῆς καὶ
ὅμως ὅσιν ἀν ὅσιν περισσαὶ ἡ ἄρτια τὸ πλῆθος.
τούτων, ὡς ἐφην, ἐπομένων τῷ ἐπὶ τέσσαρας
τόπων οὐδὲ ἐν συντεθείκασιν, ὡστε τὴν γραμμῆν
εἶδέναι.

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than three dimensions. It is true that some recent writers have agreed among themselves to use such expressions, but they have no clear meaning when they multiply the rectangle contained by these straight lines with the square on that or the rectangle contained by those. They might, however, have expressed such matters by means of the composition of ratios, and have given a general proof both for the aforesaid propositions and for further propositions after this manner: If from any point straight lines be drawn to meet at given angles straight lines given in position, and there be given the ratio compounded of that which one straight line so drawn bears to another, that which a second bears to a second, that which a third bears to a third, and that which the fourth bears to a given straight line—if there be seven, or, if there be eight, that which the fourth bears to the fourth—the point will lie on a curve given in position; and similarly, however many the straight lines be, and whether odd or even. Though, as I said, these propositions follow the locus on four lines, [geometers] have by no means solved them to the extent that the curve can be recognized.\footnote{As Heron in his formula for the area of a triangle, given the sides (supra, pp. 476-477).}

\footnote{The general proposition can thus be stated: If $p_1$, $p_2$, $p_3$ \ldots $p_n$ be the lengths of straight lines drawn to meet $n$ given straight lines at given angles (where $n$ is odd), and $a$ be a given straight line, then if
\[ \frac{p_1}{p_2} \cdot \frac{p_3}{p_4} \cdot \ldots \cdot \frac{p_n}{a} = \lambda, \]
where $\lambda$ is a constant, the point will lie on a curve given in position. This will also be true if $n$ is even and}

\[ \frac{p_1}{p_2} \cdot \frac{p_3}{p_4} \cdot \ldots \cdot \frac{p_{n-1}}{p_n} = \lambda. \]
(m) Anticipation of Guldin's Theorem

_Ibid._ vii. 41-42, ed. Hultsch 680. 30–682. 20

Ταῦθ' οἱ βλέποντες ήκιστα ἐπαίρονται, καθάπερ οἱ πάλαι καὶ τῶν τὰ κρείττονα γραψάντων ἐκαστοῦ· ἐγὼ δὲ καὶ πρὸς ἄρχαὶς ἔτι τῶν μαθημάτων καὶ τῆς ύπο ψύσεως προκειμένης ζητημάτων ἕλθε κινομένους ὀρῶν ἀπαντας, αἰδούμενος ἐγὼ καὶ δεῖξας γε πολλῷ κρεῖσσονα καὶ πολλὴν προφερό-μενα ὑφέλειαν ... ἵνα δὲ μὴ κεναῖς χερῶ τούτο φθεγξάμενος ὅδε χωρισθῶ τοῦ λόγου, ταῦτα δῶσο ταῖς ἀναγνώσσον· ὁ μὲν τῶν τελείων ἀμφοι-στικῶν λόγος συνήπται ἐκ τε τῶν ἀμφοισμάτων καὶ τῶν ἐπὶ τοὺς ἀξονας ὁμοίως κατηγμένων εὑθεῖων ἀπὸ τῶν ἐν αὐτοῖς κεντροβαρικῶν σημείων, ὁ δὲ τῶν ἀτελῶν ἐκ τε τῶν ἀμφοισμάτων καὶ τῶν περιφερειῶν, ὅσα ἐποίησεν τὰ ἐν τούτω τε κεντρο-βαρικὰ σημεῖα, ὁ δὲ τούτων τῶν περιφερειῶν λόγος συνήπται δὴλον ὡς ἐκ τε τῶν κατηγμένων καὶ ὧν περιέχοντον αἱ τούτων ἀκραί, εἰ καὶ εἰσὶν πρὸς τοὺς ἀξονα ἀμφοιστικῶν, γωνίῶν. 

* Paul Guldin (1577–1643), or Guldinus, is generally credited with the discovery of the celebrated theorem here enunciated by Pappus. It may be stated: If any plane figure revolve about an external axis in its plane, the volume of the solid figure so generated is equal to the product of the area of the figure and the distance travelled by the centre of gravity of the figure. There is a corresponding theorem for the area.

b The whole passage is ascribed to an interpolator by Hultsch, but without justice; and, as Heath observes (H.G.M. ii. 403), it is difficult to think of any Greek mathematician after Pappus's time who could have discovered such an advanced proposition.

Though the meaning is clear enough, an exact translation
The men who study these matters are not of the same quality as the ancients and the best writers. Seeing that all geometers are occupied with the first principles of mathematics and the natural origin of the subject matter of investigation, and being ashamed to pursue such topics myself, I have proved propositions of much greater importance and utility... and in order not to make such a statement with empty hands, before leaving the argument I will give these enunciations to my readers.

**Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded (a) of the ratio \([\text{of the areas}]\) of the figures, and (b) of the ratio of the straight lines similarly drawn to the axes of rotation from the respective centres of gravity.**

**Figures generated by incomplete revolutions are in a ratio compounded (a) of the ratio \([\text{of the areas}]\) of the figures, and (b) of the ratio of the arcs described by the centres of gravity of the respective figures, the ratio of the arcs being itself compounded (1) of the ratio of the straight lines similarly drawn from the respective centres of gravity to the axes of rotation and (2) of the ratio of the angles contained about the axes of revolution by the extremities of these straight lines.**

These propositions, which are practicable; I have drawn on the translations made by Halley (v. Papp. Coll., ed. Hultsch 683 n. 2) and Heath (H.G.M. ii. 402-403). The obscurity of the language is presumably the only reason why Hultsch brackets the passage, as he says: "exciderunt autem in eodem loco pauciora plurave genuina Pappi verba."

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* i.e., drawn to meet at the same angles.
* The extremities are the centres of gravity.
(n) **Lemmas to the Treatises**

(i.) *To the "Determinate Section" of Apollonius*

*Ibid.* vii. 115, ed. Hultsch, Prop. 61, 756. 28–760. 4

Τριών δοθεισῶν εὐθειῶν τῶν ΑΒ, ΒΓ, ΓΔ, εὰν γένηται ὡς τὸ ὑπὸ ΑΒΔ πρὸς τὸ ὑπὸ ΑΓΔ,

\[\text{οὕτως τὸ ἀπὸ ΒΕ πρὸς τὸ ἀπὸ ΕΓ, μοναχὸς λόγος καὶ ἐλάχιστος ἐστὶν ὁ τοῦ ὑπὸ ΑΕΔ πρὸς τὸ ὑπὸ} \]

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cally one, include a large number of theorems of all sorts about curves, surfaces and solids, all of which are proved simultaneously by one demonstration, and include propositions never before proved as well as those already proved, such as those in the twelfth book of these elements.

(n) Lemmas to the Treatises

(i.) To the "Determinate Section" of Apollonius

_Ibid._ vii. 115, ed. Hultsch, Prop. 61, 756. 28–760. 4

_Given three straight lines_ $AB$, $BΓ$, $ΓΔ$, if $AB \cdot BΔ : AΓ \cdot ΓΔ = BE^2 : EG^2$, _then the ratio_ $AE : EΔ : BE : EG$

* If the passage be genuine, which there seems little reason to doubt, this is evidence that Pappus's work ran to twelve books at least.

* The greater part of Book vii. is devoted to lemmas required for the books in the _Treasury of Analysis_ as far as Apollonius's _Conics_, with the exception of Euclid's _Data_ and with the addition of two isolated lemmas to Euclid's _Surface-Loci_. The lemmas are numerous and often highly interesting from the mathematical point of view. The two here cited are given only as samples of this important collection: the first lemma to the _Surface-Loci_, one of the two passages in Greek referring to the focus-directrix property of a conic, has already been given (vol. i. pp. 492–503).

* It is left to be understood that they are in one straight line $AΔ$. 607
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$\text{BE} \Gamma$. λέγω δὴ ότι ό αὐτὸς ἐστὶν τῷ τοῦ ἀπὸ τῆς $\Delta \pi ρὸς τὸ ἀπὸ τῆς ὑπεροχῆς ὑ ὑπερέχει ἡ δυναμένη τὸ ύπὸ $\Delta$, $\Delta \pi ρὸς τῆς δυναμένης τὸ ύπὸ $\Delta$, $\Gamma \Delta$.

Γεγράφθω περὶ τῆς $\Delta$ κύκλος, καὶ ἡχθωσαν ὀρθαὶ αἱ $\Delta$, $\Gamma$. ἐπεὶ οὖν ἐστὶν ὡς τὸ ύπὸ $\Delta \pi ρὸς τὸ ύπὸ $\Delta \Gamma \Delta$, τούτεστιν ὡς τὸ ἀπὸ $\Delta \pi ρὸς τὸ ἀπὸ $\Gamma$, οὗτως τὸ ἀπὸ $\Delta \pi ρὸς τὸ $\Gamma$ $\Sigma$ $\Gamma$, καὶ μὴ ἰκέ ἐστὶν ὡς ἡ $\Delta$, $\Sigma$ $\Gamma$ πρὸς τὴν $\Gamma$, οὗτως ἡ $\Delta$ πρὸς τὴν $\Gamma$. ἐνθεία ἀρα ἐστὶν ἡ διὰ τῶν $\Sigma$, $\Gamma$. ἐστω ἡ $\Sigma$ $\Theta$, καὶ ἐκβεβλήσθω ἡ μὲν $\Gamma$ ἐπὶ τὸ $\Theta$, ἐπιζευγθείσα δὲ ἡ $\Sigma$ $\Theta$ ἐκβεβλήσθω ἐπὶ τὸ $\Theta$, καὶ ἐπ' αὐτὴν κάθετος ἡχθω ἡ $\Delta$. καὶ διὰ δὴ τὸ προγεγραμμένον λήμμα γίνεται τὸ μὲν ύπὸ $\Delta$, $\Sigma$ $\Gamma$, τὸ δὲ ύπὸ $\Delta$, $\Gamma$, $\Sigma$ ἀπὸ $\Theta$ $\Sigma$. λοιπὴ ἀρα ἡ $\Sigma$ $\Theta$ ἐστὶν ἡ ὑπεροχὴ ὑ ὑπερέχει ἡ δυναμένη τὸ ύπὸ $\Delta$, $\Sigma$ $\Gamma$ τῆς δυναμένης τὸ ύπὸ $\Delta$, $\Gamma$. ἡχθω οὖν διὰ τοῦ κέντρου ἡ $\Sigma$, καὶ ἐπιζευγθω ἡ $\Theta$. ἐπεὶ οὖν ὀρθὴ ἡ ύπὸ $\Sigma$ $\Theta$ ὀρθή τῇ ύπὸ $\Gamma$ $\Sigma$ ἐστὶν Ὑ Ὑ, ἐστὶν δὲ καὶ ἡ πρὸς τῷ $\Delta$ τῇ πρὸς τῷ $\Gamma$ γνωρία ὑ Ὑ, ἴσογωνα ἀρα τὰ τρίγωνα. ἐστὶν ἀρα ὡς ἡ $\Delta$ πρὸς τὴν $\Theta$, τούτεστιν ὡς ἡ $\Delta$ πρὸς τὴν $\Theta$, $\Sigma$, οὗτως ἡ $\Theta$ πρὸς τὴν $\Gamma$. καὶ ἐστὶν ἀρα τὸ ἀπὸ $\Delta$ πρὸς τὸ ἀπὸ $\Theta$, οὗτως τὸ ἀπὸ $\Theta$ πρὸς τὸ ἀπὸ $\Gamma$, καὶ τὸ ύπὸ $\Sigma$ $\Theta$, $\Gamma$ $\Delta$ πρὸς τὸ ύπὸ $\Delta$, $\Gamma$, $\Sigma$, πρὸς τὸ ύπὸ $\Delta$, $\Gamma$, $\Sigma$ πρὸς τὸ ύπὸ $\Delta$, $\Gamma$.

* For, because $\Delta$ : $\Gamma$ = $\Delta$ : $\Gamma$, the triangles $\Sigma$ $\Theta$, $\Theta$ $\Gamma$ are similar, and angle $\Sigma$ $\Theta$ $\Gamma$ = angle $\Theta$ $\Gamma$; . . . $\Gamma$ is in the same straight line with $\Delta$, $\Sigma$ [Eucl. 1. 13, Conv.].

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is singular and a minimum; and I say that this ratio is equal to $A\Delta^2: (\sqrt{A\Gamma \cdot B\Delta} - \sqrt{A\Gamma \cdot}\Gamma\Delta)^2$.

Let a circle be described about $A\Delta$, and let $BZ, \Gamma H$ be drawn perpendicular [to $A\Delta$]. Then since

$AB \cdot B\Delta : A\Gamma \cdot \Gamma\Delta = BE^2 : E\Gamma^2$, \[ex \ hyp.\]

i.e.,

$BZ^2 : \Gamma H^2 = BE^2 : E\Gamma^2$,

[Eucl. x. 33, Lemma]

\[\therefore \quad BZ : \Gamma H = BE : E\Gamma.\]

Therefore $Z, E, H$ lie on a straight line.² Let it be $ZEH$, and let $H\Gamma$ be produced to $\Theta$, and let $Z\Theta$ be joined and produced to $K$, and let $\Delta K$ be drawn perpendicular to it. Then by the lemma just proved [Lemma 19]

$A\Gamma \cdot B\Delta = ZK^2,$

$AB \cdot \Gamma\Delta = \Theta K^2$;

[on taking the roots and] subtracting,

$[ZK - \Theta K = ]Z\Theta = \sqrt{A\Gamma \cdot B\Delta} - \sqrt{A\Gamma \cdot}\Gamma\Delta.$

Let $Z\Lambda$ be drawn through the centre, and let $\Theta \Lambda$ be joined. Then since the right angle $Z\Theta \Lambda = $ the right angle $E\Gamma H$, and the angle at $\Lambda = $ the angle at $H$, therefore the triangles $[Z\Theta \Lambda, E\Gamma H]$ are equiangular;

\[\therefore \quad \Lambda Z : \Theta Z = EH : E\Gamma, \]

i.e.,

$A\Delta : Z\Theta = EH : E\Gamma$;

\[\therefore \quad A\Delta^2 : Z\Theta^2 = EH^2 : E\Gamma^2 = HE \cdot EZ : BE \cdot E\Gamma \]

$= AE \cdot E\Delta : BE \cdot E\Gamma.$

[Eucl. iii. 35]

And [therefore] the ratio $AE \cdot E\Delta : BE \cdot E\Gamma$ is

² Because, on account of the similarity of the triangles $H\Gamma E, ZBE$, we have $HE : E\Gamma = EZ : EB.$
ΓΕΓ μοναχὸς καὶ ἐλάσσων λόγος, ἡ δὲ ΖΘ ἡ ὑπεροχή ἢ ὑπερέχει ἡ δυναμένη τὸ ὑπὸ τῶν ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ [τουτέστω τὸ ἀπὸ τῆς ΖΚ τοῦ ἀπὸ τῆς ΘΚ], ὥστε οἱ μοναχὸς καὶ ἐλάσσων λόγος ὁ αὐτός ἐστιν τῶ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ὑπεροχῆς ἢ ὑπερέχει ἡ δυναμένη τὸ ὑπὸ ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ, ὅπερ ἢ

(ii.) To the “Porisms” of Euclid

Ibid. vii. 198, ed. Hultsch, Prop. 130, 872. 23-874. 27

Κατάγραφῃ ἡ ΑΒΓΔΕΖΗΘΚΛ, ἔστω δὲ ὡς τὸ ὑπὸ ΑΖ, ΒΓ πρὸς τὸ ὑπὸ ΑΒ, ΓΖ, οὕτως τὸ ὑπὸ ΑΖ, ΔΕ πρὸς τὸ ὑπὸ ΑΔ, ΕΖ. οτι εὐθεὶα ἔστω ἡ διὰ τῶν Θ, Η, Ζ σημειών.

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singular and a minimum, while [as proved above,]

\[ Z \theta = \sqrt{AB \cdot \overline{BD}} - \sqrt{AB \cdot \overline{CF}}, \]

so that the same singular and minimum ratio is

\[ \Delta \Delta^2 : (\sqrt{AB \cdot \overline{BD}} - \sqrt{AB \cdot \overline{CF}})^2. \]

Q.E.D. \(^a\)

(ii.) To the "Porisms" of Euclid \(^b\)

*ibid. *vii. 198, ed. Hultsch, Prop. 130, 872. 23-874. 27

Let \( AB \Delta \Theta \overline{K} \) be a figure, \(^c\) and let \( AZ \cdot \overline{BL} : AB \cdot \overline{LC} = AZ \cdot \Delta E : A \Delta \cdot EZ; \) [I say] that the line through the points \( \Theta, H, Z \) is a straight line.

\(* \) Notice the sign  used in the Greek for "eSei eSei^at. In all Pappus proves this property for three different positions of the points, and it supports the view (v. supra, p. 341 n. a) that Apollonius's work formed a complete treatise on involution.

\(^b\) v. vol. i. pp. 478-485.

\(^c\) Following Breton de Champ and Hultsch I reproduce the second of the eight figures in the mss., which vary according to the disposition of the points.

\(^a\) toutéòtw . . . τῆς ΘК om. Hultsch.
It is not perhaps obvious, but is easily proved, and is in fact proved by Pappus in the course of iv. 21, ed. Hultsch 212. 4-13, by drawing an auxiliary parallelogram.

Conversely, if \( \Theta\Theta\Delta \) be any quadrilateral, and any
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Since $AZ \cdot \beta \Gamma : AB \cdot \Gamma Z = AZ \cdot \Delta E : A \Delta \cdot EZ$, permutando

$$AZ \cdot \beta \Gamma : AZ \cdot \Delta E = AB \cdot \Gamma Z : A \Delta \cdot EZ,$$

i.e.,

$$\beta \Gamma : \Delta E = AB \cdot \Gamma Z : A \Delta \cdot EZ.$$

But, if $KM$ be drawn through $K$ parallel to $AZ$,

$$\beta \Gamma : \Delta E = (\beta \Gamma : KN) \cdot (KN : KM) \cdot (KM : \Delta E),$$

and

$$AB \cdot \Gamma Z : A \Delta \cdot EZ = (BA : A \Delta) \cdot (\Gamma Z : ZE).$$

Let the equal ratios $BA : A \Delta$ and $NK : KM$ be eliminated;

then the remaining ratio

$$\Gamma Z : ZE = (\beta \Gamma : KN) \cdot (KM : \Delta E),$$

i.e.,

$$\Gamma Z : ZE = (\Theta \Gamma : K \Theta) \cdot (KH : HE);$$

then shall the line through $\Theta, H, Z$ be a straight line.

For if through $E$ I draw $E \Xi$ parallel to $\Theta \Gamma$, and if $\Theta H$ be joined and produced to $\Xi$,

$$KH : HE = K \Theta : E \Xi,$$

and

$$(\Theta \Theta : \Theta K) \cdot (\Theta K : E \Xi) = \Theta \Gamma : E \Xi,$$

..$

$$\Gamma Z : ZE = \Theta \Theta : E \Xi,$$

and since $\Theta \Theta$ is parallel to $E \Xi$, the line through $\Theta$, $\Xi, Z$ is a straight line (for this is obvious a), and therefore the line through $\Theta, H, Z$ is a straight line.b

transversal cut pairs of opposite sides and the diagonals in the points $A, Z, \Delta, \Gamma, B, E$, then $\beta \Gamma : \Delta E = AB \cdot \Gamma Z : A \Delta \cdot EZ$. This is one of the ways of expressing the proposition enunciated by Desargues: The three pairs of opposite sides of a complete quadrilateral are cut by any transversal in three pairs of conjugate points of an involution (v. L. Cremona, Elements of Projective Geometry, tr. by C. Leudesdorf, 1885, pp. 106-108). A number of special cases are also proved by Pappus.
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(o) Mechanics

_Ibid._ viii., Praef. 1-3, ed. Hultsch 1022. 3-1028. 3

‘Ἡ μηχανικὴ θεωρία, τέκνον Ἑρμόδωρε, πρὸς πολλὰ καὶ μεγάλα τῶν ἐν τῷ βίῳ χρῆσιμος ὑπ- ἁρχουσα πλεῖστης εἰκότως ἀποδοχὴς ἡξίωται πρὸς τῶν φιλοσόφων καὶ πάσι τοῖς ἀπὸ τῶν μαθημάτων περισσοῦδαστός ἐστιν, ἐπειδὴ σχέδον πρώτῃ τῆς περὶ τὴν ὑλὴν τῶν ἐν τῷ κόσμῳ στοιχείων φυσιολογίας ἀπτεται. στάσεως γὰρ καὶ φορᾶς σωμάτων καὶ τῆς κατὰ τόπον κινήσεως ἐν τοῖς ὀλούς θεωρηματικῆς τυχάνουσα τὰ μὲν κινοῦμενα κατὰ φύσιν αὐτιολογεῖ, τὰ δ’ ἀναγκάζουσα παρὰ φύσιν ἐξω τῶν οἰκειῶν τῶν εἰς ἐναντίας κινήσεις μεθίστησιν ἐπιμηχανωμένη διὰ τῶν ἐξ αὐτῆς τῆς ὑλῆς ὑποπτόντων αὐτῇ θεωρημάτων. τῆς δὲ μηχανικῆς τὸ μὲν εἶναι λογικὸν τὸ δὲ χειροπυργικὸν οἷς περὶ τὸν Ἡρωνα μηχανικοὶ λέγουσιν καὶ τὸ μὲν λογικὸν συνεστάναι μέρος ἐκ τε γεωμετρίας καὶ ἀριθμητικῆς καὶ ἀστρονομίας καὶ τῶν φυσικῶν λόγων, τὸ δὲ χειροπυργικὸν ἐκ τε χαλκευτικῆς καὶ οἰκοδομικῆς καὶ τεκτονικῆς καὶ ξυγραφικῆς καὶ τῆς ἐν τούτοις κατὰ χεῖρα ἀσκήσεως τὸν μὲν οὖν ἐν ταῖς προειρημέναις ἐπιστήμαις ἐκ παιδὸς γενόμενον καὶ ταῖς προειρημέναις τέχναις ἐξών εἰληφότα πρὸς δὲ τούτοις φύσιν εὐκίνητον ἔχοντα, κράτιστων ἐσεσθαί μηχανικῶν ἐργῶν εὐρετήν καὶ ἀρχιτέκτονά φαιν. μὴ δυνατοῦ δ’ οὖντος τὸν αὐτὸν μαθημάτων

* After the historical preface here quoted, much of Book viii. is devoted to arrangements of toothed wheels, already encountered in the section on Heron (supra, pp. 488-497). A
The science of mechanics, my dear Hermodorus, has many important uses in practical life, and is held by philosophers to be worthy of the highest esteem, and is zealously studied by mathematicians, because it takes almost first place in dealing with the nature of the material elements of the universe. For it deals generally with the stability and movement of bodies [about their centres of gravity], and their motions in space, inquiring not only into the causes of those that move in virtue of their nature, but forcibly transferring [others] from their own places in a motion contrary to their nature; and it contrives to do this by using theorems appropriate to the subject matter. The mechanicians of Heron's school say that mechanics can be divided into a theoretical and a manual part; the theoretical part is composed of geometry, arithmetic, astronomy and physics, the manual of work in metals, architecture, carpentering and painting and anything involving skill with the hands. The man who had been trained from his youth in the aforesaid sciences as well as practised in the aforesaid arts, and in addition has a versatile mind, would be, they say, the best architect and inventor of mechanical devices. But as it is impossible for the same person to familiarize himself with such number of interesting theoretical problems are solved in the course of the book, including the construction of a conic through five points (viii. 13-17, ed. Hultsch 1072. 30-1084. 2).

It is made clear by Pappus later (vii., Praef. 5, ed. Hultsch 1030. 1-17) that ϕισαδ has this meaning.

With Pappus, this is practically equivalent to Heron himself: cf. vol. i. p. 184 n. b.
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τε τοσούτων περιγενέσθαι καὶ μαθεῖν ἀμα τὰς προερημένας τέχνας παραγγέλλουσι τῷ τὰ μηχανικά ἔργα μεταχειρίζεσθαι βουλομένων χρήσθαι ταῖς οἰκείαις τέχναις ὑποχειρίοις ἐν ταῖς παρ' ἑκαστα χρείαις.

Μάλιστα δὲ πάντων ἀναγκαίωταται τέχναι τυγχάνουσιν πρὸς τὴν τοῦ βίου χρείαν [μηχανικῇ προηγουμένη τῆς ἀρχιτεκτονῆς] ¹ ἢ τε τῶν μαγγαρίων, μηχανικῶν καὶ αὐτῶν κατὰ τοὺς ἀρχαίους λεγομένων (μεγάλα γὰρ οὖντι βάρη διὰ μηχανῶν παρὰ φύσιν εἰς υψός ἀνάγουσιν ἐλάττονι δυνάμει κινοῦντες), καὶ ἢ τῶν ὀργανοποιῶν τῶν πρὸς τὸν πόλεμον ἀναγκαίων, καλομένων δὲ καὶ αὐτῶν μηχανικῶν (βέλη γὰρ καὶ λίθινα καὶ σιδηρὰ καὶ τὰ παραπλήσια τούτοις ἔξαποστέλλεται εἰς μακρὸν όδοὶ μήκος τοῖς ὑπ' αὐτῶν γινομένοις ὀργάνοις καταπαλτικοῖς), πρὸς δὲ ταύταις ἢ τῶν ἰδίων πάλιν καλομένων μηχανοποιῶν (ἐκ βάθους γὰρ πολλοῦ ὑδρῶν εὐκολώτερον ἀνάγεται διὰ τῶν ἀντληματικῶν ὀργάνων ὧν αὐτοὶ κατασκευάζουσι). καλοῦμεν δὲ μηχανικοὺς οἱ παλαιοὶ καὶ τοὺς θαυμασιουργοὺς, ὡν οἱ μὲν διὰ πνευμάτων φιλοτεχνοῦσιν, ὡς Ἠρων Πνευματικοῖς, οἱ δὲ διὰ νευρῶν καὶ σπάρτων ἐμφύχων κινήσεις δοκοῦσι μιμεῖσθαι, ὡς Ἠρων Αὐτομάτως καὶ Ζυγίοις, ἄλλοι δὲ διὰ τῶν ἐφ' ὑδατος ὄχουμένων, ὡς Ἀρχιμήδης Ὀξουμένους, ἡ τῶν δὲ ὑδατῶν ὄρολογίων, ὡς Ἠρων Ἡδρείους, ἡ δὴ καὶ τῇ γνωμονικῇ

¹ μηχανικῇ ... ἀρχιτεκτονῆς om. Hultsch.

* μάγγαρον is properly the block of a pulley, as in Heron’s
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mathematical studies and at the same time to learn the above-mentioned arts, they instruct a person wishing to undertake practical tasks in mechanics to use the resources given to him by actual experience in his special art.

Of all the mechanical arts the most necessary for the purposes of practical life are: (1) that of the makers of mechanical powers, they themselves being called mechanicians by the ancients—for they lift great weights by mechanical means to a height contrary to nature, moving them by a lesser force; (2) that of the makers of engines of war, they also being called mechanicians—for they hurl to a great distance weapons made of stone and iron and such-like objects, by means of the instruments, known as catapults, constructed by them; (3) in addition, that of the men who are properly called makers of engines—for by means of instruments for drawing water which they construct water is more easily raised from a great depth; (4) the ancients also describe as mechanicians the wonder-workers, of whom some work by means of pneumatics, as Heron in his Pneumatia, some by using strings and ropes, thinking to imitate the movements of living things, as Heron in his Automata and Balancings, some by means of floating bodies, as Archimedes in his book On Floating Bodies, or by using water to tell the time, as Heron in his Hydria, which appears to have affinities with the

Belopoeica, ed. Schneider 84. 12, Greek Papyri in the British Museum iii. (ed. Kenyon and Bell) 1164 n. 8.

* v. supra, p. 466 n. a.  
* v. supra, pp. 242-257.

* This work is mentioned in the Pneumatia, under the title Περί ὑδρῶν ὅροσκοπείων, as having been in four books. Fragments are preserved in Proclus (Hypotyposis 4) and in Pappus’s commentary on Book v. of Ptolemy’s Syntaxis.
θεωρία κοινωνοῦντα φαίνεται. μηχανικούς δὲ καλοῦσιν καὶ τοὺς τὰς σφαιροποιὰς [ποιεῖν] 1 επιστημένους, ὥστε ὅν έικὼν τοῦ οὐρανοῦ κατασκευάζεται δι' ὅμαλῆς καὶ ἐγκυκλίου κινήσεως ὕδατος.

Πάντων δὲ τούτων τὴν αἰτίαν καὶ τὸν λόγον ἐπεγνωκέναι φασίν τινες τὸν Συρακόσιον Ἀρχιμήδην. μόνος γὰρ οὗτος ἐν τῷ καθ' ἡμᾶς βίω ποικίλῃ πρὸς πάντα κέχρηται τῇ φύσει καὶ τῇ ἐπινοίᾳ, καθὼς καὶ Γέμινος ὁ μαθηματικὸς ἐν τῷ Περὶ τῆς τῶν μαθημάτων τάξεώς φησιν. Κάρπος δὲ πού φησιν ὁ 'Ἀντιοχεὺς Ἀρχιμήδη τὸν Συρακόσιον ἐν μόνον βιβλίον συντεταχθεῖν μηχανικὸν τὸ κατὰ τὴν σφαιροποίαν, τῶν δὲ ἅλλων οὔδὲν ἢξιωκέναι συντάξαι. καίτοι παρὰ τοῖς πολλοῖς ἐπὶ μηχανικῆς δοξασθεῖς καὶ μεγαλοφυῆς τις γενόμενος ὁ θαυμαστὸς ἐκεῖνος, ὡστε διαμειναι παρὰ πᾶσιν ἀνθρώποις ὑπερβαλλόντως ὑμνοῦμενος, τῶν τε προηγουμένων γεωμετρικῆς καὶ ἀριθμητικῆς ἐχομένων θεωρίας τὰ βραχύτατα δοκοῦντα εἶναι σπουδαῖος συνέγραφεν ὃς φαίνεται τὰς εἰρημένας ἐπιστήμας οὕτως ἀγαπήσας ὡς μηδὲν ἐξωθεὶς ὑπομένειν αὐταῖς ἐπεισάγειν. αὐτὸς δὲ Κάρπος καὶ ἅλλοι τινὲς συνεχήσαντο γεωμετρία καὶ εἰς τέχνας τινὰς εὐλόγως· γεωμετρία γὰρ οὔδὲν βλάπτεται, σωματοποιεῖν περικύκλιο τολλὰς τέχνας, διὰ τοῦ συνείναι αὐταῖς [μῆτηρ οὖν ὡσπερ οὖσα τεχνῶν οὐ βλάπτεται διὰ τοῦ φροντίζειν ὅργανικῆς καὶ ἀρχιτεκτονικῆς· οὔδὲ γὰρ διὰ τὸ συνείναι γεωμετρία καὶ γνωμονικῆ καὶ μηχανικῆ καὶ σκηνογραφία βλάπτεται τι], 2 τοῦναντίον δὲ προάγωσα

1 ποιεῖν om. Hultsch. 2 μῆτηρ ... τι om. Hultsch.

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science of sun-dials; (5) they also describe as mechanicians the *makers of spheres*, who know how to make models of the heavens, using the uniform circular motion of water.

Archimedes of Syracuse is acknowledged by some to have understood the cause and reason of all these arts; for he alone applied his versatile mind and inventive genius to all the purposes of ordinary life, as Geminus the mathematician says in his book *On the Classification of Mathematics*. Carpus of Antioch says somewhere that Archimedes of Syracuse wrote only one book on mechanics, that on the construction of spheres, not regarding any other matters of this sort as worth describing. Yet that remarkable man is universally honoured and held in esteem, so that his praises are still loudly sung by all men, but he himself on purpose took care to write as briefly as seemed possible on the most advanced parts of geometry and subjects connected with arithmetic; and he obviously had so much affection for these sciences that he allowed nothing extraneous to mingle with them. Carpus himself and certain others also applied geometry to some arts, and with reason; for geometry is in no way injured, but is capable of giving content to many arts by being associated with them, and, so far from being injured, it is obviously, while itself

- For Geminus and this work, *v. supra*, p. 370 n. c.
- Carpus has already been encountered (*vol. i. p. 334*) as the discoverer (according to Iamblichus) of a *curve arising from a double motion* which can be used for squaring the circle. He is several times mentioned by Proclus, but his date is uncertain.
- This work is not otherwise known.
With the great figure of Pappus, these selections illustrating the history of Greek mathematics may appropriately come to an end. Mathematical works continued to be written in Greek almost to the dawn of the Renaissance, and
advancing those arts, appropriately honoured and
adorned by them.²

they serve to illustrate the continuity of Greek influence in
the intellectual life of Europe. But, after Pappus, these
works mainly take the form of comment on the classical
treatises. Some, such as those of Proclus, Theon of Alex-
andria, and Eutocius of Ascalon have often been cited
already, and others have been mentioned in the notes.
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INDEX OF GREEK TERMS

The purpose of this index is to give one or more typical examples of the use of Greek mathematical terms occurring in these volumes. Non-mathematical words, and the non-mathematical uses of words, are ignored, except occasionally where they show derivation. Greek mathematical terminology may be further studied in the Index Graecitatis at the end of the third volume of Hultsch’s edition of Pappus and in Heath’s notes and essays in his editions of Euclid, Archimedes and Apollonius. References to vol. i. are by page alone, to vol. ii. by volume and page. A few common abbreviations are used. Words should be sought under their principal part, but a few cross-references are given for the less obvious.

'Αγευ, to draw; εὐθεῖαν γραμμήν ἀγαθεῖν, to draw a straight line, 442 (Eucl.); ἔαν ἐπιψαύονοι ἀρχων, if tangents be drawn, ii. 64 (Archim.); παράλληλος ἡθὼ ἢ ΑΚ, let ΑΚ be drawn parallel, ii. 312 (Apollon.)

ἀγεωμέτρητος, οὐ, ignorant of or unversed in geometry, 386 (Tzetzes)

ἀδιάδικτος, οὐ, undivided, indivisible, 366 (Aristot.)

ἀδύνατος, οὐ, impossible, 394 (Plat.), ii. 566 (Papp.); ὅπερ ἐστὶν ἄ., often without ἐστὶν, which is impossible, a favourite conclusion to reasoning based on false premises, ii. 122 (Archim.); οἱ διὰ τοῦ ἄ. περαινοντες, those who argue per impossible, 110 (Aristot.)

ἀδρούσμα, ἄτος, τὸ, collection; ἄ. φιλοτεχνότατον, a collection most skilfully framed, 480 (Papp.)

Αἰγυπτιακός, ἤ, ὅν, Egyptian; οἱ Αἰ. καλοῦμεναι μέθοδοι ἐν πολλαπλασιασμοῖς, 16 (Schol. in Plat. Charm.)

ἀφέω, to take away, subtract, ii. 506 (Heron)

ἀντεῖν, to postulate, 442 (Eucl.), ii. 206 (Archim.)

ἀπημια, ἄτος, τὸ, postulate, 420 (Aristot.), 440 (Eucl.), ii. 366 (Procl.)

ἀκίνητος, οὐ, that cannot be moved, immobile, fixed, 394 (Aristot.), ii. 246 (Archim.)

ἀκολουθεῖν, to follow, ii. 414 (Ptol.)
INDEX OF GREEK TERMS

ἀκόλουθος, ὁν, following, consequent, corresponding, ii. 550 (Papp.); as subst., ἀκόλουθον, τό, consequence, ii. 566 (Papp.)

ἀκολούθως, adv., consistently, consequentially, in turn, 458 (Eucl.), ii. 384 (Procl.)

ἀκοουσματικός, ἡ, ὁν, eager to hear; as subst., ἄ., ὁ, hearer, exoteric member of Pythagorean school, 3 n. ἄ (Iambi.)

ἀκριβής, ἐς, exact, accurate, precise, ii. 414 (Ptol.)

ἀκρος, ἂ, ὁν, at the farthest end, extreme, ii. 270 (Cleom.); of extreme terms in a proportion, 122 (Nicom.); ἂ. καὶ μέσος λόγος, extreme and mean ratio, 472 (Eucl.), ii. 416 (Ptol.)

ἀλλως, alternatively, 356 (Papp.)

ἀλογος, ὁν, irrational, 420 (Aristot.), 452 (Eucl.), 456 (Eucl.); ἃ. ἀλόγου, by irrational means, 388 (Plut.)

ἀμβλυγάννος, ὁν, obtuse-angled; ἃ. τρίγωνον, 440 (Eucl.); ἃ. κάτως, ii. 278 (Eutoc.)

ἀμβλύς, ἕιδ, ὁ, obtuse; ἃ. γωνία, often without γωνία, obtuse angle, 438 (Eucl.)

ἀμετάθετος, ὁν, unaltered, immutable; μονάδος ὁ. οὐσία, ii. 514 (Dioph.)

ἀμήχανος, ὁν, impracticable, 298 (Eutoc.)

ἀμφισμα, ἄτο, τό, revolving figure, ii. 604 (Papp.)

ἀμφοστικός, ἡ, ὁν, described by revolution; ἀμφοστικόν, τό, figure generated by revolution, ii. 604 (Papp.)

ἀναγράφειν, to describe, construct, 180 (Eucl.), ii. 68 (Archim.)

ἀνακλάν, to bend back, incline, reflect (of light), ii. 496 (Damian.)

ἀναλημμα, ἄτο, τό, a representation of the sphere of the heavens on a plane, analemma; title of work by Diodorus, 300 (Papp.)

ἀναλογία, ἡ, proportion, 446 (Eucl.); κύριως ἃ. καὶ πράγμα, proportion par excellence and primary, i.e., the geometric proportion, 125 n. ἄ; συνεχῆς ἃ. continued proportion, 262 (Eutoc.)

ἀναλογον, adv., proportionally, but nearly always used adjectivally, 70 (Eucl.), 446 (Eucl.)

ἀναλύειν, to solve by analysis, ii. 160 (Archim.); ὁ ἀναλύομενος τόπος, the Treasury of Analysis, often without τόπος, e.g., ὁ καλούμενος ἀναλυόμενος, ii. 596 (Papp.)

ἀναλύον, ἐως, ἡ, solution of a problem by analytical methods, analysis, ii. 596 (Papp.)

ἀναλυτικός, ἡ, ὁν, analytical, 158 (Procl.)
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άναμέτρησις, ews, ἡ, measurement; Περὶ τῆς ἀ. τῆς γῆς, title of work by Eratosthenes, ii. 272 (Heron)

άνάπαλυ, adv., in a reverse direction; transformation of a ratio known as invertendo, 448 (Eucl.)

άναποδεικτικῶς, adv., independently of proof, ii. 370 (Procl.)

άναστρέφειν, to convert a ratio according to the rule of Eucl. v. Def. 16; αναστρέφαντι, lit. to one who has converted, convertingo, 466 (Eucl.)

άνάστροφή, ἡ, conversion of a ratio according to the rule of Eucl. v. Def. 16, 448 (Eucl.)

ανεπαιθῆς, ov, unperceived, imperceptible; hence, negligible, ii. 482 (Heron)

ἄνωθεν, ov, unequal, 444 (Eucl.), ii. 50 (Archim.)

άνωτάναι, to set up, erect, ii. 78 (Archim.)

άντακολοθύλα, ἡ, reciprocity, 76 (Theol. Arith.)

άντικείσθαι, to be opposite, 114 (Nicom.); τοιαί ἀντικείμεναι, opposite branches of a hyperbola, ii. 322 (Apollon.)

άντιπάσχειν, to be reciprocally proportional, 114 (Nicom.); ἀντιπασχονθῶς, adv., reciprocally, ii. 208 (Archim.)

άντιστροφή, ἡ, conversion, converse, ii. 140 (Archim. ap. Eutoc.)

ἄξιομα, ατος, τό, axiom, postulate, ii. 42 (Archim.)

ἄξων, ovos, ὁ, axis; of a cone, ii. 286 (Apollon.); of any plane curve, ii. 288 (Apollon.); of a conic section, 282 (Eutoc.); οὖν γείσαι ἄ., conjugate axes, ii. 288 (Apollon.)

άόριστος, ov, without boundaries, undefined, πλῆθος μονάδων ἄ., ii. 522 (Dioph.)

ἄπαγωγή, ἡ, reduction of one problem or theorem to another, 252 (Procl.)

ἀπαρτίζειν, to make even; οἱ ἀπαρτιζόντες ἀριθμοί, factors, ii. 506 (Heron)

ἀπειραχῶς, in an infinite number of ways, ii. 572 (Papp.)

ἀπειρός, ov, infinite; as subst., ἀπειρόν, τό, the infinite, 424 (Aristot.); εἰς ἀπειρόν, to infinity, indefinitely, 440 (Eucl.); ἐν ἄ., ii. 580 (Papp.)

ἀπεναντίον, adv. used adjectively, opposite; αἱ ἀ. πλευραί, 444 (Eucl.)

ἀπέχειν, to be distant, 470 (Eucl.), ii. 6 (Aristarch.)

ἀπλανῆς, ἐς, motionless, fixed, ii. 2 (Archim.)

ἀπλάτης, ἐς, without breadth, 436 (Eucl.)

ἀπλοός, ἡ, ov, contr. ἀπλοῦς, ἡ, ὁ, ὁν, simple; ἀ. γραμμή, ii. 360 (Procl.)
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ἀπλῶς, simply, absolutely, 424 (Aristot.); generally, ii. 132 (Archim.)

ἀπλωσ, eow, ἡ, simplification, explanation; "A. ἐπι-

φανειᾶς σφαῖρας, Explanation of the Surface of a

Sphere, title of work by Ptolemy, ii. 408 (Suidas)

ἀπό, from; τὸ ἀπὸ τῆς δια-

μέτρου τετράγωνον, the

square on the diameter,

332 (Archim.); τὸ ἀπὸ

ΓΘ (sc. τετράγωνον), the

square on ΓΘ, 268

(Eutoc.)

ἀποδεικτικός, ἡ, ὑπο, affording

proof, demonstrative, 420

(Aristot.), 158 (Procl.)

ἀποδεικτικός, adv., theo-

retically, 260 (Eutoc.)

ἀπόδειξις, eow, ἡ, proof,

demonstration, ii. 42 (Ar-

chim.), ii. 566 (Papp.)

ἀποκαθιστάναι, to re-establish,

restore; pass., to return

to an original position, ii.

182 (Archim.)

ἀπολαμβάνειν, to cut off; ἡ

ἀπολαμβανομένη περιφέρεια,

440 (Eucl.)

ἀπορία, ἡ, difficulty, perplex-

ity, 256 (Theon Smyr.)

ἀπόστημα, atos, τὸ, distance,

interval, ii. 6 (Aristarch.)

ἀποτομή, ἡ, cutting off, sec-

tion; Λόγον ἀποτομή, Χω-

ρίον ἀποτομή, works by

Apollonius, ii. 598

(Papp.); compound ir-

rational straight line equi-

valent to binomial surd

with negative sign, apo-

tome, 456 (Eucl.)

ἀπτεων, to fasten to; mid.,

ἀπτεωθαι, to be in contact,

meet, 438 (Eucl.), ii. 106

(Archim.)

ἀρα, therefore, used for the

steps in a proof, 180

(Eucl.)

ἀρβηλος, ὁ, semicircular knife

used by leather-workers, a

geometrical figure used by

Archimedes and Pappus,

ii. 578 (Papp.)

ἀριθμεῖν, to number, reckon,

enumerate, ii. 198 (Ar-

chim.), 90 (Luc.)

ἀριθμητικός, ὁ, ὑπο, of or for

reckoning or numbers; ἡ

ἀριθμητική (sc. τέχνη),

arithmetico, 6 (Plat.), 420

(Aristot.); ἡ ἀριθμητική

μέση (sc. εὐθεία), arith-

metic mean, ii. 568

(Papp.); ἄ. μεσότητις, 110

(Iamb.)

ἀριθμός, ὁ, number, 6 (Plat.),

66 (Eucl.); πρῶτος ἂ.,

prime number, 68 (Eucl.);

πρῶτοι, δεύτεροι, τρίτοι, τέ-

ταρτοι, πέμπτοι ἂ., num-

bers of the first, second,

third, fourth, fifth order,

ii. 198-199 (Archim.); μη-

λήτης ἂ., problem about a

number of sheep, 16 (Schol.

in Plat. Charm.); φιαλίτης

ἀ., problem about a num-

ber of bowls (ibid.)
INDEX OF GREEK TERMS

άρβυμοστόν, τό, fraction whose denominator is unknown \[\frac{1}{x}\], ii. 522 (Dioph.)

άρμοζεν, to fit together, ii. 494 (Heron)

άρμονία, ἧ, musical scale, octave, music, harmony, 404 (Plat.); used to denote a square and a rectangle, 398 (Plat.)

άρμονικός, ἥ, ón, skilled in music, musical; ἥ ἄρμονική (sc. ἐπιστήμη), mathematical theory of music, harmonic; ἥ ἄρμονικη μέση, harmonic mean, 112 (Iamb.)

ἀρτιάκας, adv., an even number of times; ἀ. ἄρτιος ἄρμοσ, even-times even number, 66 (Eucl.)

ἀρτιόπλευρος, ov, having an even number of sides; πολύγωνον ἀ., ii. 88 (Archim.)

ἀρτιός, a, ov, complete, perfect; ἀ. ἄρμοσ, even number, 66 (Eucl.)

άρχη, ἡ, beginning or principle of a proof or science, 418 (Aristot.); beginning of the motion of a point describing a curve; ἄρχη ἑλικος, origin of the spiral, ii. 182

άρχικός, ἥ, ὁν, principal, fundamental; ἀ. σύμπτωμα, principal property of a curve, ii. 282 (Apollon.)

άρχικῶτατος, ov, sovereign, fundamental; ἀ. δίς, 90 (Nicom.)

άρχιτεκτονικός, ἡ, ὁν, of or for an architect; ἡ ἄρχιτεκτονική (sc. τέχνη), architecture, ii. 616 (Papp.)

άστρολογία, ἡ, astronomy, 388 (Aristox.)

άστρολογός, ὁ, astronomer, 378 (Suidas)

άστρονομία, ἡ, astronomy, 14 (Plat.)

άσύμμετρος, ov, incommensurable, irrational, 110 (Aristot.), 452 (Eucl.), ii. 214 (Archim.)

άσύμπτωτος, ov, not falling in, non-secant, asymptotic, ii. 374 (Procl.); ἀ. (sc. γραμμή), ἡ, asymptote, ii. 282 (Apollon.)

ἀσύνθετος, ov, incomposite; ἀ. γραμμή, ii. 360 (Procl.)

ἀτάκτος, ov, unordered; Περὶ ἀτ. ἀλόγων, title of work by Apollonius, ii 350 (Procl.)

ἀτελής, ἐς, incomplete; ἀ. ἀμφοτερικά, figures generated by an incomplete revolution, ii. 604 (Papp.)

ἀτόμος, ov, indivisible; ἀτομο γραμμαί, 424 (Aristot.)

ἄτομος, ov, out of place, absurd; ὅπερ ἄτομον, which is absurd, a favourite conclusion to a piece of reasoning based on a false premise, e.g. ii. 114 (Archim.)
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<tr>
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<th>English Translation</th>
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<td>αὐξάνειν</td>
<td>to increase, to multiply</td>
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<tr>
<td>αὐξηθεῖν</td>
<td>to increase, dimension</td>
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<tr>
<td>αὐξηθή</td>
<td>self-acting</td>
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<tr>
<td>τά, title of work by Heron</td>
<td>ii. 616 (Papp.)</td>
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<tr>
<td>ἀφαιρεῖν</td>
<td>to cut off, take away, subtract</td>
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<tr>
<td>ἀφή</td>
<td>point of concourse of straight lines; point of contact of circles or of a straight line and a circle, ii. 64 (Archim.)</td>
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<td>Αχιλλεύς</td>
<td>Achilles, the first of Zeno's four arguments on motion, 368 (Aristot.)</td>
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<td>Βάρος</td>
<td>weight, esp. in a lever, ii. 206 (Archim.), or system of pulleys, ii. 490 (Heron); to κέντρον τοῦ βάρους, centre of gravity, ii. 208 (Archim.)</td>
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<td>βαρουκός</td>
<td>lifting-screw invented by Archimedes, title of work by Heron, ii. 489 n. a</td>
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<td>βάσις</td>
<td>base; of a geometrical figure; of a triangle, 318 (Archim.); of a cube, 222 (Plat.); of a cylinder, ii. 42 (Archim.); of a cone, ii. 304 (Apollon.); of a segment of a sphere, ii. 40 (Archim.)</td>
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<td>Γεωδαισία</td>
<td>land dividing, mensuration, geodesy</td>
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<td>γεωμετρεῖν</td>
<td>to measure, to practise geometry; δὲ γ. τὸν θεὸν, 386 (Plat.); γεωμετροῦμεν ἑπιφάνεια, geometric surface, 292 (Eutoc.), γεωμετροῦμεν ἀπόδειξις, geometric proof, ii. 228 (Archim.)</td>
</tr>
<tr>
<td>γεωμετρησις</td>
<td>land measurer, geometry</td>
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<tr>
<td>γεωμετρικός</td>
<td>pertaining to geometry, geometrical</td>
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<tr>
<td>γεωμετρικῶς</td>
<td>geometrically</td>
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<td>γίνεθαι</td>
<td>to be brought about</td>
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<tr>
<td>γεγονέτω</td>
<td>let it be done, a formula used to open a piece of analysis; of curves, to be generated, ii. 468 (Heron); to be brought about by multiplication, i.e., the result (of the multiplication) is, ii. 480 (Heron); to γενόμενον, τὰ γενόμενα, the product, ii. 482 (Heron)</td>
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<td>γλωσσόκομον</td>
<td>chest, ii. 490 (Papp.)</td>
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<tr>
<td>γνωμονικός</td>
<td>of or concerning sun-dials</td>
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</table>
| γνωμών | carpenter's
square; pointer of a sundial, ii. 268 (Cleom.); geometrical figure known as gnomon, number added to a figured number to get the next number, 98 (Iamb.)

γραμμή, γραμμή, γράμμα, γράμμα, line, curve, 436 (Euclid.); εὐθεία γράμμα, a straight line, 438 (Euclid.); εκ τῶν γραμμῶν, rigorous proof by geometrical arguments, ii. 412 (Ptolemy)

γραμμικός, γραμμικός, γραμμικός, γραμμικός, ὑπό, linear, 348 (Appollon.)

γράφειν, to describe, 442 (Euclid.); ii. 582 (Pappus.); 298 (Eutocius.); to prove, 380 (Plato.), 260 (Eutocius.)

γράφη, γράφή, description, account, 260 (Eutocius.); writing, treatise, 260 (Eutocius.)

γωνία, γωνία, angle; επίπεδος γωνία, plane angle (presumably including angles formed by curves), 438 (Euclid.); εὐθύγραμμος γωνία, rectilinear angle (formed by straight lines), 438 (Euclid.); διάγωνια, διάγωνια, διήγερε, διήγερε, right, obtuse, acute angle, 440 (Euclid.)

Δεικνύειν, to prove; δεικτῆς, γὰρ τοῦτο, for this has been proved, ii. 220 (Archimedes.); δεικτέων διώκειν, it is required to prove that, ii. 168 (Archimedes.)

δεῖν, to be necessary, to be required; δέειν δεῖν, let it be required; ὅπερ ἐδεικτῆς, quod erat demonstrandum, which was to be proved, the customary ending to a theorem, 184 (Euclid.); ὅπερ • ὅπερ ἐδεικτῆς, ii. 610 (Pappus.)

dékagónon, τό, a regular plane figure with ten angles, decagon, ii. 196 (Archimedes.)

δήλος, δήλος, ὑπό, also ὑπό, on, manifest, clear, obvious; ὅτι μὲν οὖν οὕτα συμπίπτειν, δήλος, ii. 192 (Archimedes.)

diáγειν, to draw through, 190 (Euclid.); 290 (Eutocius.)

diáγραμμα, diagram, diagram, 428 (Aristotle.)

diáρειν, to divide, cut, ii. 286 (Apollonius.); διαφημένος, or, divided; δ. análogia, discrete proportion, 262 (Eutocius.); διελόντι, lit. to one having divided, dirimendo (or, less correctly, dividendo), indicating the transformation of the ratio a : b into a - b : b according to Eucl. v. 15, ii. 130 (Archimedes.)

diáρεις, εἰς, διάρεις, division, separation, 363 (Aristotle.); δ. λόγου, transformation of a ratio dividendo, 448 (Euclid.)

diáμενείν, to remain, to remain stationary, 258 (Eutocius.)

diámetros, or, diagonal, diametrical; as subst., δ. (sc. γράμμα), γράμμα, diagonal; of a parallelogram, ii. 218
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(Archim.); diameter of a circle, 438 (Eucl.); of a sphere, 466 (Eucl.); principal axis of a conic section in Archim., ii. 148 (Archim.); diameter of any plane curve in Apollon., ii. 286 (Apollon.); πλαγία δ., transverse diameter, ii. 286 (Apollon.); συζυγείς δ., conjugate diameters, ii. 288 (Apollon.)

diάστασις, εως, ἡ, dimension, 412 (Simpl.)
dιαστέλεων, to separate, ii. 502 (Heron)
dιάστημα, ἄτο, τά, interval; radius of a circle, i. 192 (Archim.), 442 (Eucl.); interval or distance of a conchoid, 300 (Papp.); in a proportion, the ratio between terms, τὸ τῶν μειζόνων ὀρῶν δ., 112 (Archytas ap. Porph.); dimension, 88 (Nicom.)
dιαφορά, ἡ, difference, 114 (Nicom.)
dιδόναι, to give; aor. part., δώτως, εἰσα, ἐν, given, ii. 598 (Papp.); Δεδομένα, τά, Data, title of work by Euclid, ii. 588 (Papp.); θέσει καὶ μεγέθει δεδομένη, to be given in position and magnitude, 478 (Eucl.)
dιελόντι, ὑ., διαιρεῖν
dιεκσῆς, ἐς, discontinuous; σπείρα δ., open spire, ii. 364 (Procl.)
dιορίζειν, to determine, ii. 566 (Papp.); Διωρισμένη
tομή, Determinate Section, title of work by Apollonius, ii. 598 (Papp.)

dιορισμὸς, ὁ, statement of the limits of possibility of a solution of a problem, diorismos, 150 (Procl.)
dιπλασίαζεων, to double, 258 (Eutoc.)
dιπλασιασμός, ὁ, doubling, duplication; κύβος δ., 258 (Eutoc.)
dιπλάσιος, α., ὁν, double, 302 (Papp.); δ. λόγος, duplication ratio, 446 (Eucl.)
dιπλασίων, ὁν, later form for διπλάσιος, double, 326 (Archim.)
dιπλός, ἡ, ὁν, contr. διπλός, ἡ, ὁν, twofold, double, 326 (Archim.).

δισθένης, open spire, ii. 364 (Procl.)

dίσχα, ἀδ., in two (equal) parts, 66 (Eucl.): δ. τέμνειν, to bisect, 440 (Eucl.)
dισχοτομία, ἡ, dividing in two; point of bisection, ii. 216 (Archim.); Dichotomy, first of Zeno's arguments on motion, 368 (Aristot.)

dισχοτόμος, ὁν, cut in two, halved, ii. 4 (Aristarch.)
dύναμις, εως, ἡ, power, force, ii. 488 (Heron), ii. 616 (Papp.); αἱ πέντε δ., the five mechanical powers (wheel and axle, lever, pulley, wedge, screw), ii. 492 (Heron); power in
the algebraic sense, esp. square; δύναμεν, in power, i.e., squared, 322 (Archim.); δύναμεν ὁμομετρός, commensurable in square, 450 (Eucl.); δύναμεν ἀσυμμετρός, incommensurable in square (ibid.)

δυναμοδύναμος, εἰς, ἦ, fourth power of the unknown quantity \( [x^4] \), ii. 522 (Dioph.)

δυναμοδύναμοστόν, τὸ, the fraction \( \frac{1}{x^4} \), ii. 522 (Dioph.)

δυναμοκύβοσ, ó, square multiplied by a cube, fifth power of the unknown quantity \( [x^5] \), ii. 522 (Dioph.)

δυναμοκύβοστόν, τὸ, the fraction \( \frac{1}{x^5} \), ii. 522 (Dioph.)

δύνασθαι, to be able, to be equivalent to; δύνασθαι τι, to be equivalent when squared to a number or area, ii. 96 (Archim.); ἡ δυναμένη (sc. εὐθεία), side of a square, 452 (Eucl.); αὐξῆσαι δύναμαι, 398 (Plat.); παρ' ἡν δύναται αἱ καταγόμεναι τεταγμένοι ἐπὶ τὴν ΖΗ διάμετρον, the parameter of the ordinates to the diameter ZH, ii. 308 (Apollon.)

δυναστεύειν, to be powerful; pass., to be concerned with

powers of numbers; αὐξῆσαι δύναστεύουμεν, 398 (Plat.)

δυνατός, ἢ, ὅν, possible, ii. 566 (Papp.)

δυνακενενηκοντάεδρον, τὸ, solid with ninety-two faces, ii. 196 (Archim.)

δυνακεξηκοντάεδρον, τὸ, solid with sixty-two faces, ii. 196 (Archim.)

δυνακτριακοντάεδρον, τὸ, solid with thirty-two faces, ii. 196 (Archim.)

δωδεκάεδρον, ὅν, with twelve faces; as subst., δωδεκάεδρον, τὸ, body with twelve faces, dodecahedron, 472 (Eucl.), 216 (Aét.)

Εἰσδομηκοστόμονος, ὅν, seventy-first; τὸ ἕπ., seventy-first part, 320 (Archim.)

ἐγγράφειν, to inscribe, 470 (Eucl.), ii. 46 (Archim.)

ἐγκύκλιος, ὅν, also α, ὅν, circular, ii. 618 (Papp.)

ἐἴδος, ὅν, Ἰον. ἑός, τὸ, shape or form of a figured number, 94 (Aristot.); figure giving the property of a conic section, viz., the rectangle contained by the diameter and the parameter, ii. 317 n. a, 358 (Papp.), 282 (Eutoc.); term in an equation, ii. 524 (Dioph.); species—of number, ii. 522 (Dioph.), of angles 390 (Plat.)
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εἰκοσάεδρος, οὐ, having twenty faces; εἰκοσαεδρον, τό, body with twenty faces, icosahedron, 216 (Aët.)

εἰκοσαπλάσιος, οὐ, twentyfold, ii. 6 (Aristarch.)

ἐκατοντάς, ἄδος, ἡ, the number one hundred, ii. 198 (Archim.)

ἐκβάλλειν, to produce (a straight line), 442 (Eucl.), ii. 8 (Aristarch.), 352 (Papp.)

ἐκκακοσάεδρον, τό, solid with twenty-six faces, ii. 196 (Archim.)

ἐκκείθοια, used as pass. of ἐκτιθέναι, to be set out, be taken, ii. 96 (Archim.), 298 (Papp.)

ἐκκρόνειν, to take away, eliminate, ii. 612 (Papp.)

ἐκπέπταμα, ἄτος, τό, that which is spread out, unfolded; Ἐκπεπτάματα, title of work by Democritus dealing with projection of armillary sphere on a plane, 229 n. a

ἐκπρισμα, ἄτος, τό, section sawn out of a cylinder, prismatic section, ii. 470 (Heron)

ἐκτίθεναι, to set out, ii. 568 (Papp.)

ἐκτός, adv., without, outside; as prep., ἐ. τοῦ κύκλου, 314 (Alex. Aphr.); adv. used adjectivally, ἡ ἐ. (sc. εὔθεία), external straight line, 314 (Simpl.); ἡ ἐ. γωνία τοῦ τριγώνου, the external angle of the triangle, ii. 310 (Apollon.)

ἐλάσσων, οὐ, smaller, less, 320 (Archim.); ἦτοι μείζων ἐκτός ἡ ἡ, ii. 112 (Archim.);

ἐ. ὀρθής, less than a right angle, 438 (Eucl.); ἡ ἐ. (sc. εὐθεία), minor in Euclid's classification of straight lines, 458 (Eucl.)

ἐλάχιστος, ἡ, οὐ, smallest, least, ii. 44 (Archim.)

ἐλιξ, ἐλικος, ἡ, spiral, helix, ii. 182 (Archim.); spiral on a sphere, ii. 580 (Papp.)

ἐλλειμμα, ἄτος, τό, defect, deficiency, 206 (Eucl.)

ἐλλειπεῖν, to fall short, be deficient, 394 (Plat.), 188 (Procl.)

ἐλλειψις, εῶς, ἡ, falling short, deficiency, 186 (Procl.); the conic section ellipse, so called because the square on the ordinate is equal to a rectangle whose height is equal to the abscissa applied to the parameter as base but falling short (ἐλλειπτων), ii. 316 (Apollon.), 188 (Procl.)

ἐμβαδόν, τό, area, ii. 470 (Heron)

ἐμβάλλειν, to throw in, insert, ii. 574 (Papp.); multiply, ii. 534 (Dioph.)

ἐμπλητεῖν, to fall on, to meet, to cut, 442 (Eucl.), ii. 58 (Archim.)

ἐμπλέκειν, to plait or weave in; σπείρα ἐμπεπλευμένη,
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interlaced spire, ii. 364 (Procl.)

ἐναλλάξ, adv., often used adjectively, transformation of a ratio according to the rule of Eucl. v. Def. 12, permutando, 448 (Eucl.), ii. 144 (Archim.); ἐ. γωνία, alternate angles

ἐναντίος, a, ov, opposite; κατ' ὑ., ii. 216 (Archim.)

ἐναρμόζειν, to fit in, to insert, 284 (Eutoc.)

ἐντάσις, εως, ἡ, inscription, 396 (Plat.)

ἐντελής, ἐς, perfect, complete; τρίγωνον ὡ., 90 (Procl.)

ἐντὸς, adv. used adjectively, within, inside, interior; αἱ ἡ. γωνία, 442 (Eucl.)

ἐνυπάρχειν, to exist in; εἰδὴ ἐνυπάρχοντα, τὰ, positive terms, ii. 524 (Dioph.)

ἐξαγωνικός, ἡ, ὁ, hexagonal; ἐ. ἀριθμός, 96 (Nicom.)

ἐξάγωνος, ὁ, ὁ, as subst. ἐξάγωνον, τὸ, hexagon, 470 (Eucl.)

ἐξηκοστός, ἡ, ὁ, sixtieth; in astronom., πρῶτον ἐξηκοστὸν, τὸ, first sixtieth, minute, δεῦτερον ὡ., second sixtieth, second, 50 (Theon Alex.)

ἐξής, adv., in order, successively, ii. 566 (Papp.)

ἐπαφή, ἡ, touching, tangency, contact, 314 (Simpl.).

Ἐσαφάλ, On Tangencies, title of a book by Apollonius, ii. 336 (Papp.)

ἐπιστεία, to be or come after, follow; τὸ ἐπόμενον, con-

sequence, ii. 566 (Papp.); τὰ ἐπόμενα, rearward elements, ii. 184 (Apollon.); in theory of proportion, τὰ ἐπόμενα, following terms, consequents, 448 (Eucl.)

ἐπι, prep. with acc., upon, on to, on, εὐθεία ἐπ' εὐθείαν σταθεῖσα, 438 (Eucl.)

ἐπιζευγνύσαι, to join up, ii. 608 (Papp.); αἱ ἐπιζευγ-

νυσαι εὐθεία, connecting lines, 272 (Eutoc.)

ἐπιλογίζονται, to reckon, calculate, 60 (Theon Alex.)

ἐπιλογισμός, ὁ, reckoning, calculation, ii. 412 (Ptol.)

ἐπίπεδος, ὁ, plane; ἐ. ἐπι-

φάνεια, 438 (Eucl.); ἐ. γωνία, 438 (Eucl.); ἐ. σχήμα, 438 (Eucl.); ἐ. ἀριθμός, 70 (Eucl.); ἐ. πρό-

βλημα, 348 (Papp.)

ἐπικείσως, adv., plane-wise, 88 (Nicom.)

ἐπιπλαθής, ἐς, flat, broad; σφαιροειδὲς ὡ., ii. 164 (Archim.)

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<td>μεσημβρινός, ἡ, ὅν, for μεσημβρινός, of or for noon; μ. (sc. κύκλος), ὁ, meridian, ii. 268 (Cleom.)</td>
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<td>μέσος, η, ὄν, middle; ἡ μέση (sc. εὐθεία), mean (ἀρμονική, γεωμετρική, ἀρμονική), ii. 568 (Papp.); μέση τῶν ΔΚ, ΚΓ, mean between ΔΚ, ΚΓ, 272 (Eutoc.); ἀκρός καὶ μ. λόγος, extreme and mean ratio, 472 (Eucl.), ii. 416 (Ptol.); ἡ μέση (sc. εὐθεία), medial in Euclid's classi-</td>
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<td>μέτρον, τό</td>
<td>measure, relation,</td>
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<td>μέχρι</td>
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<td>μέσον</td>
<td>τοῦ ἄξονος</td>
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<td>μήκος</td>
<td>Dor. μάκος, eis, τό, length,</td>
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<tr>
<td>μήκος</td>
<td>Dor. μάκος, eis, τό, length,</td>
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<tr>
<td>μηχανή</td>
<td>contrivance, machine, engine,</td>
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<td>μηχανικός</td>
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<td>μηχανοτότος</td>
<td>δ', maker of engines,</td>
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<td>μικρός</td>
<td>ἄ, ὁν, small, little;</td>
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<tr>
<td>μικτός</td>
<td>έ, ὁν, mixed;</td>
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<tr>
<td>μονάς</td>
<td>αδός, έ, unit, monad,</td>
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<td>μώριον</td>
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<td>μυρίας</td>
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<tr>
<td>μύριον</td>
<td>τοί, part,</td>
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<tr>
<td>μύριοι</td>
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<td>μύριαν</td>
<td>κατά, a myriad raised to the first power,</td>
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<td>κατά, a myriad raised to the first power,</td>
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<td>Nevev</td>
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<td>πεντάγωνος, ov, pentagonal, π. ἀριθμὸς, 96 (Nicom.)</td>
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<td>περιμετρός, ov, very large, well-fitting, ἡ π. (sc. γραμμή)</td>
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<td>περισσός, ἀριθμὸς, odd-times odd number, 68 (Eucl.)</td>
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<td>περιτιθέναι, to place or put around, 94 (Aristot.)</td>
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sûnôtâs, ews, ἦ, putting together in order, systematic treatise, composite volume, collection; title of work by Ptolemy, ii. 408 (Suidas)
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ταράσσειν, to disturb; τεταραγμένη ἀνάλογια, disturbed proportion, 450 (Eucl.)

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τεταγμένος, ν. τάσσειν τεταραγμένος, ov, ν. ταράσσειν
tεταρτημόριον, τό, fourth part, quadrant of a circle, ii. 582 (Papp.)

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τετράκις, adv., four times, 326 (Archim.)

τετραπλάσιος, α, ὁ, four-fold, four times as much, 332 (Archim.)

τετραπλάσιον, ov, later form of τετραπλάσιος

τετράπλευρος, ov, four-sided,
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ὑπτιός, α, οὐ, laid on one's
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φορά, ἡ, motion, 12 (Plat.)

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