SELECTIONS
ILLUSTRATING THE HISTORY OF
GREEK MATHEMATICS

WITH AN ENGLISH TRANSLATION BY
IVOR THOMAS
FORMERLY SCHOLAR OF ST. JOHN'S AND SENIOR DEMY
OF MAGDALEN COLLEGE, OXFORD

IN TWO VOLUMES
II
FROM ARISTARCHUS TO PAPPUS

LONDON
WILLIAM HEINEMANN LTD
CAMBRIDGE, MASSACHUSETTS
HARVARD UNIVERSITY PRESS
MCMLVII
First printed 1941
Reprinted 1951, 1957

Printed in Great Britain
# CONTENTS OF VOLUME II

## XVI. Aristarchus of Samos—

(a) General ........................................... 2
(b) Distances of the sun and moon ............. 4
(c) Continued fractions (?) ...................... 14

## XVII. Archimedes—

(a) General ........................................... 18
(b) Surface and volume of the cylinder and sphere ... 40
(c) Solution of a cubic equation ............... 126
(d) Conoids and spheroids—
   (i) Preface ....................................... 164
   (ii) Two lemmas .................................. 164
   (iii) Volume of a segment of a paraboloid of revolution .... 170
(e) The spiral of Archimedes—
   (i) Definitions .................................. 182
   (ii) Fundamental property ..................... 184
   (iii) A verging ................................... 186
   (iv) Property of the subtangent .......... 190
(f) Semi-regular solids ......................... 194

...
CONTENTS

(g) System of expressing large numbers .................. 198
(h) Indeterminate analysis: the Cattle Problem ............. 202
(i) Mechanics: centres of gravity—
   (i) Postulates .................................. 206
   (ii) Principle of the lever .................... 208
   (iii) Centre of gravity of a parallelogram ...... 216
(j) Mechanical method in geometry ......................... 220
(k) Hydrostatics—
   (i) Postulates .................................. 242
   (ii) Surface of fluid at rest ................... 244
   (iii) Solid immersed in a fluid ............... 248
   (iv) Stability of a paraboloid of revolution .. 252

XVIII. Eratosthenes—

(a) General .................................. 260
(b) On means .................................. 262
(c) The Platonicus ............................ 264
(d) Measurement of the earth ................. 266

XIX. Apollonius of Perga—

(a) The conic sections—
   (i) Relation to previous works ......... 276
   (ii) Scope of the work ................. 280
   (iii) Definitions ......................... 284
   (iv) Construction of the sections ...... 288
   (v) Fundamental properties ............ 304
   (vi) Transition to new diameter ...... 328
## CONTENTS

(b) Other works—

| (i) | General                                   | 336 |
| (ii) | On the Cutting-off of a Ratio             | 336 |
| (iii) | On the Cutting-off of an Area            | 338 |
| (iv) | On Determinate Section                   | 338 |
| (v)  | On Tangencies                             | 340 |
| (vi) | On Plane Loci                             | 344 |
| (vii)| On Vergings                               | 344 |
| (viii) | On the dodecahedron and the icosaahedron | 348 |
| (ix) | Principles of mathematics                 | 348 |
| (x)  | On the Cochlias                           | 350 |
| (xi) | On unordered irrationals                  | 350 |
| (xii)| Measurement of a circle                   | 352 |
| (xiii)| Continued multiplications                | 352 |
| (xiv)| On the Burning Mirror                    | 356 |

### XX. LATER DEVELOPMENTS IN GEOMETRY—

(a) Classification of curves                         | 360 |

(b) Attempts to prove the parallel postulate—

| (i) | General                                   | 366 |
| (ii) | Posidonius and Geminus                    | 370 |
| (iii) | Ptolemy                                  | 372 |
| (iv) | Proclus                                   | 382 |

(c) Isoperimetric figures                             | 386 |

(d) Division of zodiac circle into 360 parts: Hypsicles | 394 |

(e) Handbooks—

| (i) | Cleomedes                                 | 396 |
| (ii) | Theon of Smyrna                           | 400 |

vii
CONTENTS

XXI. Trigonometry—

1. Hipparchus and Menelaus  ...  406
2. Ptolemy—
   (a) General  ...  408
   (b) Table of sines—
       (i) Introduction  ...  412
       (ii) \( \sin 18^\circ \) and \( \sin 36^\circ \)  ...  414
       (iii) \( \sin^2 \theta + \cos^2 \theta = 1 \)  ...  420
       (iv) Ptolemy’s theorem  ...  422
       (v) \( \sin(\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi \)  ...  424
       (vi) \( \sin^2 \frac{1}{2} \theta = \frac{1}{2}(1 - \cos \theta) \)  ...  428
       (vii) \( \cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \)  ...  430
       (viii) Method of interpolation  ...  434
       (ix) The table  ...  442
   (c) Menelaus’s theorem—
       (i) Lemmas  ...  446
       (ii) The theorem  ...  458

XXII. Mensuration: Heron of Alexandria—

   (a) Definitions  ...  466
   (b) Measurement of areas and volumes—
       (i) Area of a triangle given the sides  ...  470
       (ii) Volume of a spire  ...  476
       (iii) Division of a circle  ...  482
       (iv) Measurement of an irregular area  ...  484
CONTENTS

(c) Mechanics . . . . .  488
(d) Optics: equality of angles of incidence and reflection . .  496
(e) Quadratic equations . . .  502
(f) Indeterminate analysis . .  504

XXIII. ALGEBRA: DIOPHANTUS—

(a) General . . . . .  512
(b) Notation . . . . .  518
(c) Determinate equations—
   (i) Pure determinate equations .  524
   (ii) Quadratic equations . .  526
   (iii) Simultaneous equations leading to a quadratic . .  536
   (iv) Cubic equation . . .  538
(d) Indeterminate equations—
   (i) Indeterminate equations of the second degree—
      (a) Single equations . . .  540
      (b) Double equations . .  542
   (ii) Indeterminate equations of higher degree . .  548
(e) Theory of numbers: sums of squares . . .  550
(f) Polygonal numbers . . .  560

XXIV. REVIVAL OF GEOMETRY: PAPPUS OF ALEXANDRIA—

(a) General . . . . .  564
(b) Problems and theorems . .  566
## CONTENTS

| (c) The theory of means | . | . | 568 |
| (d) The paradoxes of Erycinus | . | . | 570 |
| (e) The regular solids | . | . | 572 |
| (f) Extension of Pythagoras’s theorem | 574 |
| (g) Circles inscribed in the ἀρβηλος | . | . | 578 |
| (h) Spiral on a sphere | . | . | 580 |
| (i) Isoperimetric figures | . | . | 588 |
| (j) Apparent form of a circle | . | . | 592 |
| (k) The Treasury of Analysis | . | . | 596 |
| (l) Locus with respect to five or six lines | 600 |
| (m) Anticipation of Guldin’s theorem | . | 604 |
| (n) Lemmas to the treatises— | | | |
| (i) To the *Determinate Section* of Apollonius | . | . | 606 |
| (ii) To the *Porisms* of Euclid | . | . | 610 |
| (o) Mechanics | . | . | 614 |

**Index** | . | . | 622

**List of Ancient Texts cited** | . | . | 650

**Index of Greek Terms** | . | . | 655
XVI. ARISTARCHUS OF SAMOS
XVI. ARISTARCHUS OF SAMOS

(a) General

Aët. i. 15. 5; Doxographi Graeci, ed. Diels 313. 16-18

'Αρίσταρχος Σάμιος μαθηματικὸς ἀκούστης Στράτωνος φῶς εἶναι τὸ χρῶμα τοῖς ὑποκειμένοις ἐπιπίπτον.

Archim. Aren. 1, Archim. ed. Heiberg ii. 218. 7-18

'Αρίσταρχος δὲ ὁ Σάμιος ὑποθεσίων τινῶν ἐξ-ἐδωκεν γραφάς, ἐν αἷς ἐκ τῶν ὑποκειμένων συμβαίνει τὸν κόσμον πολλαπλάσιον εἴμεν τοῦ νῦν εἰρημένου. ὑποτίθεται γὰρ τὰ μὲν ἀπλανεὰ τῶν ἀστρων καὶ τὸν ἀλιον μένειν ἀκίνητον, τὰν δὲ γὰν περιφέρεσθαι περὶ τὸν ἀλιον κατὰ κύκλου περιφέρειαν, ὦς ἔστιν ἐν μέσῳ τῷ δρόμῳ κείμενος, τὰν δὲ τῶν ἀπλανέων ἀστρων σφαῖραν περὶ τὸ

---

4 Strato of Lampsacus was head of the Lyceum from 288/287 to 270/269 B.C. The next extract shows that Aristarchus formulated his heliocentric hypothesis before Archimedes wrote the Sand-Reckoner, which can be shown to have been written before 216 B.C. From Ptolemy, Syntaxis iii. 2, Aristarchus is known to have made an observation of the summer solstice in 281/280 B.C. He is ranked by Vitruvius, De Architectura i. 1. 17 among those rare men, such as Philolaus, Archytas, Apollonius, Eratosthenes,
Aristarchus of Samos, a mathematician and pupil of Strato, held that colour was light impinging on a substratum.

Aristarchus of Samos produced a book based on certain hypotheses, in which it follows from the premises that the universe is many times greater than the universe now so called. His hypotheses are that the fixed stars and the sun remain motionless, that the earth revolves in the circumference of a circle about the sun, which lies in the middle of the orbit, and that the sphere of the fixed stars, situated Archimedes and Scopinas of Syracuse, who were equally proficient in all branches of science. Vitruvius, loc. cit. ix. 8. 1, is also our authority for believing that he invented a sun-dial with a hemispherical bowl. His greatest achievement, of course, was the hypothesis that the earth moves round the sun, but as that belongs to astronomy it can be mentioned only casually here. A full and admirable discussion will be found in Heath, Aristarchus of Samos: The Ancient Copernicus, together with a critical text of Aristarchus’s only extant work.
GREEK MATHEMATICS

αὐτὸ κέντρον τῷ ἀλής κειμέναιν τῷ μεγέθει ταλικαῦταν εἶμεν, ὡςτε τὸν κύκλον, καθ’ ὅν τὰν γὰν ὑποτίθεται περιφέρεσθαι, τοιαῦταν ἔχειν ἀναλογίαν ποτὶ τὰν ἄπλανέων ἀποστασίαν, οἷαν ἔχει τὸ κέντρον τὰς σφαῖρας ποτὶ τὰν ἐπιφάνειαν.

Plut. De facie in orbe lunae 6, 922 f–923 a

Καὶ ο Λεύκιος γελᾶσας, "Μόνον," εἶπεν, "ὁ τὰν, μὴ κρίσαι ἡμῖν ἀσεβείας ἐπαγγεῖλης, ὡσπερ Ἀρισταρχὸν ἴστη ἀνάκλιθε τὸν Σάμων ἀσεβείας προσκαλεῖσθαι τοὺς "Ελλήνας, ὡς κυνοῦντα τὸν κόσμον τὴν ἐστίαν, ὅτι τὰ φαινόμενα σφῖζειν ἀνήρ ἑπειράτο, μένειν τὸν οὐρανὸν ὑποτιθέμενοι, ἔξελίπτεσθαι δὲ κατὰ λοξοῦ κύκλου τὴν γῆν, ἀμα καὶ περὶ τὸν αὐτῆς ἄξονα δινομένην."

(b) Distances of the Sun and Moon


〈Ὑποθέσεις1〉

α’. Τὴν σελήνην παρὰ τοῦ ἦλιου τὸ φῶς λαμβάνειν.

β’. Τὴν γῆν σημείου τε καὶ κέντρου λόγον ἔχειν πρὸς τὴν τῆς σελήνης σφαῖραν.

γ’. Ὡταν ἡ σελήνη διχότομος ἡμῖν φαίνεται,

1 ὑποθέσεις add. Heath.

* Aristarchus’s last hypothesis, if taken literally, would mean that the sphere of the fixed stars is infinite. All that he implies, however, is that in relation to the distance of the
ARISTARCHUS OF SAMOS

about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve has such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface.

Plutarch, *On the Face in the Moon* 6, 922 f–923 a

Lucius thereupon laughed and said: “Do not, my good fellow, bring an action against me for impiety after the manner of Cleanthes, who held that the Greeks ought to indict Aristarchus of Samos on a charge of impiety because he set in motion the hearth of the universe; for he tried to save the phenomena by supposing the heaven to remain at rest, and the earth to revolve in an inclined circle, while rotating at the same time about its own axis.”

(b) Distances of the Sun and Moon


HYPOTHESES

1. The moon receives its light from the sun.
2. The earth has the relation of a point and centre to the sphere in which the moon moves.
3. When the moon appears to us halved, the great fixed stars the diameter of the earth’s orbit may be neglected. The phrase appears to be traditional (*v.* Aristarchus’s second hypothesis, *infra*).

* Heraclides of Pontus (along with Ecphantus, a Pythagorean) had preceded Aristarchus in making the earth revolve on its own axis, but he did not give the earth a motion of translation as well.
* Lit. “sphere of the moon.”
GREEK MATHEMATICS

νεύειν εἰς τὴν ἱμετέραν ὁμιν τὸν διορίζοντα τὸ τε σκιερὸν καὶ τὸ λαμπρὸν τῆς σελήνης μέγιστον κύκλον.

δ'. Ὅταν η σελήνη διχότομος ἤμων φαίνεται, τότε αὐτὴν ἀπέχει τοῦ ἡλίου ἐλασσον τεταρτημορίου τῷ τοῦ τεταρτημορίου τριακοστῷ.

ε'. Τὸ τῆς σκιᾶς πλάτος σελήνων εἶναι δύο.

ε'. Τὴν σελήνην ὑποτείνειν ὑπὸ πεντεκαιδεκατον μέρος ζῳδίου.

Επιλογιζεται οὖν τὸ τοῦ ἡλίου ἀπόστημα ἀπὸ τῆς γῆς τοῦ τῆς σελήνης ἀποστήματος μείζον μὲν ἡ ὀκτωκαιδεκάπλασιον, ἐλασσὸν δὲ ἡ εἰκοσαπλάσιον, διὰ τῆς περὶ τὴν διχοτομίαν ὑποθέσεως τοῦ αὐτοῦ δὲ λόγου ἔχειν τὴν τοῦ ἡλίου διάμετρον πρὸς τὴν τῆς σελήνης διάμετρον. τὴν δὲ τοῦ ἡλίου διάμετρον πρὸς τὴν τῆς γῆς διάμετρον μείζονα μὲν λόγου ἔχειν ἦ ὁν τὰ θι πρὸς γ', ἐλάσσονα δὲ ἦ ὁν μὴ πρὸς 5, διὰ τοῦ εὐρεθέντος περὶ τὰ ἀποστήματα λόγου, τῆς (τε1) περὶ τὴν σκιὰν ὑποθέσεως, καὶ τοῦ τῆς σελήνην ὑπὸ πεντεκαιδεκατον μέρος ζῳδίου ὑποτείνειν.

Ibid., Prop. 7, ed. Heath 376. 1–380. 28

Τὸ ἀπόστημα δ ἀπέχει δ ἡλίος ἀπὸ τῆς γῆς τοῦ

1 τε add. Heath.

Lit. "verges towards our eye." For "verging," v. vol. i. p. 244 n. a. Aristarchus means that the observer's eye lies in the plane of the great circle in question. A great circle is a circle described on the surface of the sphere and having the same centre as the sphere; as the Greek implies, a great circle is the "greatest circle" that can be described on the sphere.
circle dividing the dark and the bright portions of the moon is in the direction of our eye.\(^a\)

4. When the moon appears to us halved, its distance from the sun is less than a quadrant by one-thirtieth of a quadrant.\(^b\)

5. The breadth of the earth’s shadow is that of two moons.\(^c\)

6. The moon subtends one-fifteenth part of a sign of the zodiac.\(^d\)

It may now be proved that the distance of the sun from the earth is greater than eighteen times, but less than twenty times, the distance of the moon—this follows from the hypothesis about the halved moon; that the diameter of the sun has the aforesaid ratio to the diameter of the moon; and that the diameter of the sun has to the diameter of the earth a ratio which is greater than 19 : 3 but less than 43 : 6—this follows from the ratio discovered about the distances, the hypothesis about the shadow, and the hypothesis that the moon subtends one-fifteenth part of a sign of the zodiac.

\(^a\) i.e., is less than 90° by 3°, and so is 87°. The true value is 89° 50'.

\(^b\) i.e., the breadth of the earth’s shadow where the moon traverses it during an eclipse. The figure is presumably based on records of eclipses. Hipparchus made the figure 2½ for the time when the moon is at its mean distance, and Ptolemy a little less than 2½ for the time when the moon is at its greatest distance.

\(^c\) i.e., the angular diameter of the moon is one-fifteenth of 30°, or 2°. The true value is about ½°, and in the Sand-Reckoner (Archim. ed. Heiberg ii. 223. 6-8) Archimedes says that Aristarchus “discovered that the sun appeared to be about 7½ th part of the circle of the Zodiac”; as he believed
that the sun and moon had the same angular diameter he
must, therefore, have found the approximately correct angular
diameter of $\frac{1}{2}^\circ$ after writing his treatise *On the Sizes and
Distances of the Sun and Moon.*
eighteen times, but less than twenty times, the distance of the moon from the earth.

For let $A$ be the centre of the sun, $B$ that of the earth; let $AB$ be joined and produced; let $\Gamma$ be the centre of the moon when halved; let a plane be drawn through $AB$ and $\Gamma$, and let the section made by it in the sphere on which the centre of the sun moves be the great circle $A\Delta E$, let $A\Gamma$, $\Gamma B$ be joined, and let $B\Gamma$ be produced to $\Delta$.

Then, because the point $\Gamma$ is the centre of the moon when halved, the angle $A\Gamma B$ will be right.
GREEK MATHEMATICS

ΑΓΒ. ἦχθω δὴ ἀπὸ τοῦ Β τῇ ΒΑ πρὸς ὀρθᾶς ἤ ΒΕ. ἔσται δὴ ἡ ΕΔ περιφέρεια τῆς ΕΔΑ περιφερείας λ'. ὑπόκειται γάρ, ὅταν ἡ σελήνη διχότομος ἡμῖν φαίνηται, ἀπέχειν ἀπὸ τοῦ ἥλιου ἐλασσον τεταρτημορίου τῷ τοῦ τεταρτημορίου λ'. ὥστε καὶ ἡ ὑπὸ τῶν ΕΒΓ γωνία ὀρθῆς ἐστὶ λ'. συμπεπληρώσθω δὴ τὸ ΑΕ παραλληλόγραμμον, καὶ ἐπεζεύχθω ἡ ΒΖ. ἔσται δὴ ἡ ὑπὸ τῶν ΖΒΕ γωνία ἡμίσεια ὀρθῆς. τετμήσθω ἡ ὑπὸ τῶν ΖΒΕ γωνία δίχα τῇ ΒΖ ἐυθείᾳ. ἡ ἀρα ὑπὸ τῶν ΖΒΕ γωνία τέταρτον μέρος ἐστὶν ὀρθῆς. ἀλλὰ καὶ ἡ ὑπὸ τῶν ΔΒΕ γωνία λ' ἐστὶ μέρος ὀρθῆς. λόγος ἀρα τῆς ὑπὸ τῶν ΖΒΕ γωνίας πρὸς τὴν ὑπὸ τῶν ΔΒΕ γωνίαν ἐστὶν τὰ ἐπὶ πρὸς τὰ δύο. οἰων γάρ ἐστὶν ὀρθῇ γωνίᾳ, τοιούτων ἐστὶν ἡ μὲν ὑπὸ τῶν ΖΒΕ τε, ἡ δὲ ὑπὸ τῶν ΔΒΕ δύο. καὶ ἐπεὶ ἡ ΕΕ πρὸς τὴν ΕΘ μείζωνα λόγον ἔχει ἣπερ ἡ ὑπὸ τῶν ΖΒΕ γωνία πρὸς τὴν ὑπὸ τῶν ΔΒΕ γωνίαν, ἡ ἀρα ΗΕ πρὸς τὴν ΕΘ μείζωνα λόγον ἔχει ἢπερ τὰ ἐπὶ πρὸς τὰ β. καὶ ἐπεὶ ἢσθ ἐστὶν ἡ ΒΕ τῇ ΕΖ, καὶ ἐστὶν ὀρθῇ ἡ πρὸς τῷ Ε, τὸ ἀρα ἀπὸ τῆς ΖΒ τοῦ ἀπὸ ΒΕ διπλάσιον ἐστὶν· ὥστε τὸ ἀπὸ ΖΒ πρὸς τὸ ἀπὸ ΒΕ, οὕτως ἐστὶ τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ ΗΕ· τὸ ἀρα ἀπὸ ΖΗ τοῦ ἀπὸ ΗΕ διπλάσιον ἐστὶ. τὰ δὲ μθ τῶν κε ἐλάσσονα ἐστὶν ἡ διπλάσια, ὥστε τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ ΗΕ μείζωνα λόγον ἔχει ἡ ὅν τὰ ἀπὸ ΖΗ πρὸς κε. καὶ ἡ ΖΗ ἀρα πρὸς τὴν ΗΕ μείζωνα λόγον

1 ἐστὶν add. Nizze.
2 ἔχει add. Wallis.
3 δν τὰ add. Wallis.

* Lit. “circumference,” as in several other places in this proposition.
10
ARISTARCHUS OF SAMOS

From B let BE be drawn at right angles to BA. Then the arc \( a \) \( E\Delta \) will be one-thirtieth of the arc \( E\Delta A \); for, by hypothesis, when the moon appears to us halved, its distance from the sun is less than a quadrant by one-thirtieth of a quadrant [Hypothesis 4]. Therefore the angle \( EBI \) is also one-thirtieth of a right angle. Let the parallelogram \( AE \) be completed, and let \( BZ \) be joined. Then the angle \( ZBE \) will be one-half of a right angle. Let the angle \( ZBE \) be bisected by the straight line \( BH \); then the angle \( HBE \) is one-fourth part of a right angle. But the angle \( \Delta BE \) is one-thirtieth part of a right angle; therefore angle \( HBE : \angle \Delta BE = 15 : 2 \); for, of those parts of which a right angle contains 60, the angle \( HBE \) contains 15 and the angle \( \Delta BE \) contains 2.

Now since

\[ HE : E\Theta > \angle HBE : \angle \Delta BE, \]

therefore \( HE : E\Theta > 15 : 2. \)

And since \( BE = EZ \), and the angle at \( E \) is right, therefore

\[ ZB^2 = 2BE^2. \]

But

\[ ZB^2 : BE^2 = ZH^2 : HE^2. \]

Therefore

\[ ZH^2 = 2HE^2. \]

Now

\[ 49 < 2 \cdot 25, \]

so that

\[ ZH^2 : HE^2 > 49 : 25. \]

Therefore

\[ ZH : HE > 7 : 5. \]

\* Aristarchus's assumption is equivalent to the theorem

\[ \frac{\tan \alpha}{\tan \beta} > \alpha > \beta \]

where \( \beta < \alpha < \frac{1}{2} \pi \). Euclid's proof in *Optics* 8 is given in vol. i. pp. 502-505.
GREEK MATHEMATICS

ἐχεῖ ἡ (ὁν¹) τὰ ξ πρὸς τὰ ἐ· καὶ συνθέντι ἡ ΖΕ ἀρα πρὸς τὴν ΕΗ μείζονα λόγον ἐχεῖ ἡ ὃν τὰ ἴβ πρὸς τὰ ἐ, τοντέστων, ἡ ὃν (τὰ*) λς πρὸς τὰ ἴε. ἐδείχθη δὲ καὶ ἡ ΗΕ πρὸς τὴν ΕΘ μείζονα λόγον ἔχουσα ἡ ὃν τὰ ἴε πρὸς τὰ δύο. δι’ ἵσου ἀρα ἡ ΖΕ πρὸς τὴν ΕΘ μείζονα λόγον ἐχεῖ ἡ ὃν τὰ λς πρὸς τὰ δύο, τοντέστων, ἡ ὃν τὰ ἴη πρὸς ἃ· ἡ ἀρα ΖΕ τῆς ΕΘ μείζων ἐστὶν ἡ ἴη. ἡ δὲ ΖΕ ἴση ἐστὶν τῇ ΒΕ· καὶ ἡ ΒΕ ἀρα τῆς ΕΘ μείζων ἐστὶν ἡ ἴη· πολλῷ ἀρα ἡ BH τῆς ΘΕ μείζων ἐστὶν ἡ ἴη. ἀλλ’ ὥς ἡ ΘΘ πρὸς τὴν ΘΕ, οὕτως ἐστὶν ἡ AB πρὸς τὴν ΒΓ, διὰ τὴν ὁμοιότητα τῶν τριγώνων· καὶ ἡ AB ἀρα τῆς ΒΓ μείζων ἐστὶν ἡ ἴη. καὶ ἐστὶν ἡ μὲν AB τὸ ἀπόστημα ὃ ἀπέχει ὁ ἓλωσ ἀπὸ τῆς γῆς, ἡ δὲ ΓΒ τὸ ἀπόστημα ὃ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς· τὸ ἀρα ἀπόστημα ὃ ἀπέχει ὁ ἓλωσ ἀπὸ τῆς γῆς τοῦ ἀποστήματος, οὐ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς, μείζων ἐστὶν ἡ ἴη.

Δέγω δὴ ὅτι καὶ ἐλάσσον ἡ ἐχθὺς γάρ διὰ τοῦ Δ τῇ EB παράλληλος ἡ ΔΚ, καὶ περὶ τὸ ΔΚΒ τρίγωνον κύκλος γεγράφθω ὁ ΔΚΒ· ἐσται δὴ αὐτοῦ διάμετρος ἡ ΔΒ, διὰ τὸ ὀρθὴν εἶναι τὴν πρὸς τῷ K γωνίαν. καὶ ἐνημοῦσθω ἡ ΒΑ ἐξαγώνον. καὶ ἐπεὶ ἡ ὑπὸ τῶν ΔΒΕ γωνία λ’ ἐστὶν ὀρθής, καὶ ἡ ὑπὸ τῶν ΒΔΚ ἀρα λ’ ἐστὶν ὀρθής· ἡ ἀρα BK περιφέρεια ξ’ ἐστὶν τοῦ ὄλου κύκλου. ἐστὶν δὲ καὶ ἡ ΒΑ ἐκτὸς μέρος τοῦ ὄλου κύκλου· ἡ ἀρα ΒΑ περιφέρεια τῆς BK περιφέρειας ἐ ἐστὶν. καὶ ἐχεῖ ἡ ΒΑ περιφέρεια πρὸς τὴν BK περιφέρειαν μείζων λόγον ὑπέρ ἡ ΒΑ

¹ ὁν add. Wallis.
² τὰ add. Wallis.
ARISTARCHUS OF SAMOS

Therefore, *componendo*, \( \frac{ZE}{EH} > \frac{12}{5} \),
that is, \( \frac{ZE}{EH} > \frac{36}{15} \).

But it was also proved that
\( \frac{HE}{EO} > \frac{15}{2} \).
Therefore, *ex aequali*,
\( \frac{ZE}{EO} > \frac{36}{2} \),
that is,
\( \frac{ZE}{EO} > \frac{18}{1} \).

Therefore \( ZE \) is greater than eighteen times \( EO \).
And \( ZE \) is equal to \( BE \). Therefore \( BE \) is also greater than eighteen times \( EO \). Therefore \( BH \) is much greater than eighteen times \( OE \).

But \( \frac{BO}{OE} = \frac{AB}{BG} \),
by similarity of triangles. Therefore \( AB \) is also greater than eighteen times \( BG \). And \( AB \) is the distance of the sun from the earth, while \( GB \) is the distance of the moon from the earth; therefore the distance of the sun from the earth is greater than eighteen times the distance of the moon from the earth.

I say now that it is less than twenty times. For through \( \Delta \) let \( \Delta K \) be drawn parallel to \( EB \), and about the triangle \( \Delta KB \) let the circle \( \Delta KB \) be drawn; its diameter will be \( \Delta B \), by reason of the angle at \( K \) being right. Let \( BA \), the side of a hexagon, be fitted into the circle. Then, since the angle \( \Delta BE \) is one-thirtieth of a right angle, therefore the angle \( B \Delta K \) is also one-thirtieth of a right angle. Therefore the arc \( BK \) is one-sixtieth of the whole circle. But \( BA \) is one-sixth part of the whole circle.

Therefore \( \text{arc } BA = 10 \cdot \text{arc } BK \).

And the arc \( BA \) has to the arc \( BK \) a ratio greater

* For the proportion *ex aequali*, v. vol. i. pp. 448-451.
GREEK MATHEMATICS

εὐθεῖα πρὸς τὴν ΒΚ εὐθείαν· ἡ ἀρα ΒΛ εὐθεῖα τῆς ΒΚ εὐθείας ἐλάσσων ἐστὶν ἡ ἑ. καὶ ἐστὶν αὐτῆς διπλῆ ἡ ΒΔ· ἡ ἀρα ΒΔ τῆς ΒΚ ἐλάσσων ἐστὶν ἡ ἑ. ὥσ τε ἡ ΒΔ πρὸς τὴν ΒΚ, ἡ ΑΒ πρὸς τὴν ΒΓ, ὥστε καὶ ἡ ΑΒ τῆς ΒΓ ἐλάσσων ἐστὶν ἡ ἑ. καὶ ἐστὶν ἡ μὲν ΑΒ τὸ ἀπόστημα δ ἀπέχει δ ἕλιος ἀπὸ τῆς γῆς, ἡ δὲ ΒΓ τὸ ἀπόστημα δ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς· τὸ ἀρα ἀπόστημα δ ἀπέχει ὁ ἕλιος ἀπὸ τῆς γῆς τοῦ ἀποστήματος, οὐ ἀπέχει ἡ σελήνη ἀπὸ τῆς γῆς, ἔλασσον ἐστὶν ἡ ἑ. ἐδείχθη δέ καὶ μείζον ἡ ἑ.

(c) CONTINUED FRACTIONS (?)

Ibid., Prop. 13, ed. Heath 396. 1-2

"Εχει δὲ καὶ τὰ ζύγια πρὸς ,διν μείζονα λόγον ἕπερ τὰ πή πρὸς μὲ.

Ibid., Prop. 15, ed. Heath 406. 23-24

"Εχει δὲ καὶ ζροε Ἑσοι πρὸς Ζρογ, μείζονα λόγον ἦ δν τὰ μγ πρὸς λζ.

1 τὴν add. Wallis.

---

a This is proved in Ptolemy’s Syntaxis i. 10, v. infra, pp. 435-439.

b If \( \frac{7921}{4050} \) is developed as a continued fraction, we obtain the approximation \( 1 + \frac{1}{1 + \frac{1}{21 + \frac{1}{2}}} \), which is \( \frac{88}{45} \). Similarly, if \( \frac{71755875}{61735500} \) or \( \frac{21261}{18292} \) is developed as a continued fraction, we
ARISTARCHUS OF SAMOS

than that which the straight line BA has to the straight line BK.a

Therefore  \( BA < 10 \cdot BK. \)

And  \( BA = 2 \cdot BA. \)

Therefore  \( BA < 20 \cdot BK. \)

But  \( BA : BK = AB : BG. \)

Therefore  \( AB < 20 \cdot BG. \)

And AB is the distance of the sun from the earth, while BG is the distance of the moon from the earth; therefore the distance of the sun from the earth is less than twenty times the distance of the moon from the earth. And it was proved to be greater than eighteen times.

(c) Continued Fractions (?)


But 7921 has to 4050 a ratio greater than that which 88 has to 45.


But 71755875 has to 61735500 a ratio greater than that which 43 has to 37.b

obtain the approximation \( 1 + \frac{1}{6} + \frac{1}{6} \) or \( \frac{43}{37} \). The latter result was first noticed in 1823 by the Comte de Fortia D'Urban (*Traité d'Aristarque de Samos*, p. 186 n. 1), who added: “Ainsi les Grecs, malgré l'imperfection de leur numération, avaient des méthodes semblables aux nôtres.” Though these relations are hardly sufficient to enable us to say that the Greeks knew how to develop continued fractions, they lend some support to the theory developed by D'Arcy W. Thompson in *Mind*, xxxviii. pp. 43-55, 1929.
XVII. ARCHIMEDES
A life of Archimedes was written by a certain Heraclides—perhaps the Heraclides who is mentioned by Archimedes himself in the preface to his book On Spirals (Archim. ed. Heiberg ii. 2. 3) as having taken his books to Dositheus. We know this from two references by Eutocius (Archim. ed. Heiberg iii. 228. 20, Apollon. ed. Heiberg ii. 163. 3, where, however, the name is given as 'Ἡράκλειος'), but it has not survived. The surviving writings of Archimedes, together with the commentaries of Eutocius of Ascalon (φ. A.D. 520), have been edited by J. L. Heiberg in three volumes of the Teubner series (references in this volume are to the 2nd ed., Leipzig, 1910–1915). They have been put into mathematical notation by T. L. Heath, The Works of Archimedes (Cam-
Archimedes the wise, the famous maker of engines, was a Syracusan by race, and worked at geometry till old age, surviving five-and-seventy-years; he reduced to his service many mechanical powers, and with his triple-pulley device, using only his left hand, he drew a vessel of fifty thousand medimni burden. Once, when Marcellus, the Roman general, was assaulting Syracuse by land and sea, first by his engines he drew up some merchant-vessels, lifted them up against the wall of Syracuse, and sent them in a heap again to the bottom, crews and all. When Marcellus had withdrawn his ships a little distance, the old man gave all the Syracusans power to lift bridge, 1897), supplemented by The Method of Archimedes (Cambridge, 1922), and have been translated into French by Paul Ver Eecke, Les Œuvres complètes d'Archimède (Brussels, 1921).

The lines which follow are an example of the "political" (πολιτικὸς, popular) verse which prevailed in Byzantine times. The name is given to verse composed by accent instead of quantity, with an accent on the last syllable but one, especially an iambic verse of fifteen syllables. The twelfth-century Byzantine pedant, John Tzetzes, preserved in his Book of Histories a great treasure of literary, historical, theological and scientific detail, but it needs to be used with caution. The work is often called the Chiliades from its arbitrary division by its first editor (N. Gerbel, 1546) into books of 1000 lines each—it actually contains 12,674 lines.

As he perished in the sack of Syracuse in 212 B.C., he was therefore born about 287 B.C.
Unfortunately, the earliest authority for this story is Lucian, *Hipp.* 2: τὸν δὲ (sc. Ἀρχιμήδην) τὰς τῶν πολεμῶν τριήμερες καταφλέξαντα τῇ τέχνῃ. It is also found in Galen, *Περὶ κρασ.* iii. 2, and Zonaras xiv. 3 relates it on the authority of Dion Cassius, but makes Proclus the hero of it.

Further evidence is given by Tzetzes, *Chil.* xii. 995 and Eutocius (Archim. ed. Heiberg iii. 132. 5-6) that Archimedes wrote in the Doric dialect, but the extant text of his best-known works, *On the Sphere and Cylinder* and the *Measurement of a Circle*, retains only one genuine trace of its original Doric—the form τῆν. Partial losses have occurred in other books, the *Sand-Reckoner* having suffered least. The subject is fully treated by Heiberg, *Quaestiones Archimedeae*, pp. 69-94, and in a preface to the second volume of his edition of Archimedes he indicates the words which he has restored to their Doric form despite the manuscripts; his text is adopted in this selection.

The loss of the original Doric is not the only defect in the

---

1. πέμποντας Cary, πέμποντα codd.
stones large enough to load a waggon and, hurling them one after the other, to sink the ships. When Marcellus withdrew them a bow-shot, the old man constructed a kind of hexagonal mirror, and at an interval proportionate to the size of the mirror he set similar small mirrors with four edges, moved by links and by a form of hinge, and made it the centre of the sun's beams—its noon-tide beam, whether in summer or in mid-winter. Afterwards, when the beams were reflected in the mirror, a fearful kindling of fire was raised in the ships, and at the distance of a bow-shot he turned them into ashes. In this way did the old man prevail over Marcellus with his weapons. In his Doric \(^b\) dialect, and in its Syracusan variant, he declared: "If I have somewhere to stand, I will move the whole earth with my charistion." \(^c\) The hand of an interpolator—often not particularly skilful—can be repeatedly detected, and there are many loose expressions which Archimedes would not have used, and occasional omissions of an essential step in his argument. Sometimes the original text can be inferred from the commentaries written by Eutocius, but these extend only to the books On the Sphere and Cylinder, the Measurement of a Circle, and On Plane Equilibriums. A partial loss of Doric forms had already occurred by the time of Eutocius, and it is believed that the works most widely read were completely recast a little later in the school of Isidorus of Miletus to make them more easily intelligible to pupils.

\(^a\) The instrument is otherwise mentioned by Simplicius (in Aristot. Phys., ed. Diels 1110, 2-5) and it is implied that it was used for weighing: ταύτη δὲ τῇ ἀναλογίᾳ τοῦ κινοῦντος καὶ τοῦ κινουμένου καὶ τοῦ διαστήματος τὸ σταθμιστικὸν ῥῆμαν τὸν καλούμενον χαριστώνα συντήσας ὁ 'Αρχιμήδης ὡς μέχρι παντὸς τῆς ἀναλογίας προχωρούσης ἐκόμπασεν ἐκείνο τὸ "πά βώ καὶ κωδὸ τὰν γάν." As Tzetzes in another place (Chil. iii. 61: ὁ γὰρ ἀνασπῶν μηχανὴ τῇ τριπαστῳ βοῶν "όπα βώ καὶ σιλεύων τὴν χδόνα ") writes of a triple-pulley device in the same connexion, it may be presumed to have been of this nature.

VOL. II B 21
GREEK MATHEMATICS

Οὕτως, κατὰ Διόδωρου, τῆς Συρακούσης ταύτης Προδότου πρὸς τὸν Μάρκελλον ἀθρόως γενομένης. Εἶτε, κατὰ τὸν Δίωνα, ἩῬωμαίους πορθηθείσης, Ἀρτέμιδι τῶν πολιτῶν τότε παννυχιζόντων, Τοιουτοτρόπως τέθυηκεν ὕπο τινὸς ἩῬωμαίου. Ἡν κεκυφώς, διάγραμμα μηχανικόν τι γράφων. Τὶς δὲ ἩῬωμαίος ἐπιστὰς εἶλκεν αἴχμαλωτίζων. Ὡς καὶ τοῦ διαγράμματος ὅλος ὑπάρχων τότε, Τὶς ὁ καθέλκων οὐκ εἶδώς, ἐλεγε πρὸς ἐκεῖνον: ὁ Ἀπόστηθι, ὁ ἀνθρώπε, τοῦ διαγράμματος μου. Ἡ δὲ εἶλκε τοῦτον συστραφεῖς καὶ γνώς ἩῬωμαίον εἶναι, Ἐβόα, τὶ μηχανημα τὶς τῶν ἔμων μοι δότων. Ὡς δὲ ἩῬωμαίος πτωθεῖς εὐθὺς ἐκεῖνον κτεῖνει, Ἀνδρα σαθρὸν καὶ γέροντα, δαμοῦν τοῖς ἔργοις.

Plut. Marcellus xiv. 7–xvii. 7

Καὶ μέντοι καὶ Ἀρχιμήδης, Ἱέρωνι τῷ βασιλεί συγγενῆς ὅν καὶ φιλος, ἐγραφεὶν ὡς τῇ δοθείᾳ δυνάμει τὸ δοθὲν βάρος κινῆσαι δυνατὸν ἐστι· καὶ νεανιευςάμενος, ὡς φασί, ρώμῃ τῆς ἀποδείξεως εἰπεν ὡς, εἰ γὴν εἰχεν ἐτέραν, ἐκύησεν ἃν ταῦτην μεταβὰς εἰς ἐκεῖνην. θαυμάσαντος δὲ τοῦ Ἱέρωνος, καὶ δεικνύσαντός εἰς ἔργον ἐξαγαγεὶν τὸ προβλῆμα καὶ δειξαὶ τὶ τῶν μεγάλων κινουμένων ὑπὸ σμικρὰς δυνάμεως, ὀλκάδα τριάμενον τῶν βασιλικῶν πόνῳ μεγάλῳ καὶ χερὶ πολλῇ νεωληθεῖσαι, ἐμβαλὼν ἀνθρώπους τε πολλοὺς καὶ τὸν συνήθη φόρτον, αὐτὸς ἀπωθεῖ καθήμενος, οὐ μετὰ σπουδῆς, ἀλλὰ

b The account of Dion Cassius has not survived.
c Zonaras ix. 5 adds that when he heard the enemy were
ARCHIMEDES

Whether, as Diodorus asserts, Syracuse was betrayed and the citizens went in a body to Marcellus, or, as Dion tells, it was plundered by the Romans, while the citizens were keeping a night festival to Artemis, he died in this fashion at the hands of one of the Romans. He was stooping down, drawing some diagram in mechanics, when a Roman came up and began to drag him away to take him prisoner. But he, being wholly intent at the time on the diagram, and not perceiving who was tugging at him, said to the man: "Stand away, fellow, from my diagram." As the man continued pulling, he turned round and, realizing that he was a Roman, he cried, "Somebody give me one of my engines." But the Roman, scared, straightway slew him, a feeble old man but wonderful in his works.

Plutarch, Marcellus xiv. 7–xvii. 7

Archimedes, who was a kinsman and friend of King Hiero, wrote to him that with a given force it was possible to move any given weight; and emboldened, as it is said, by the strength of the proof, he averred that, if there were another world and he could go to it, he would move this one. Hiero was amazed and besought him to give a practical demonstration of the problem and show some great object moved by a small force; he thereupon chose a three-masted merchantman among the king's ships which had been hauled ashore with great labour by a large band of men, and after putting on board many men and the usual cargo, sitting some distance away and without any special effort, he pulled gently with his hand at coming "πάρο κατάλαν" ἕφη "καὶ μὴ παρὰ γραμμάν"—"Let them come at my head," he said, "but not at my line."
GREEK MATHEMATICS

ἡρέμα τῇ χειρὶ σεῖων ἀρχήν τινα πολυσπάστου προσηγάγετο λείως καὶ ἀπαίστως καὶ ὠσπερ διὰ θαλάττης ἐπιθέουσαν. ἐκπλαγεὶς οὖν ὁ βασιλεὺς καὶ συννοήσας τῆς τέχνης τῆς δύναμιν, ἐπείσε τὸν Ἀρχιμήδην ὅπως αὐτῷ τὰ μὲν ἄμυνομένω, τὰ δ' ἐπιχειροῦντι μηχανῆμα κατασκευάσῃ πρὸς πάσαν ἰδέαν πολυρκίας, οἷς αὐτὸς μὲν οὐκ ἐχρήσατο, τοῦ βίου τὸ πλείστον ἀπόλεμον καὶ πανηγυρικὸν βιώσας, τότε δ' ὑπῆρχε τοῖς Συρακουσίοις εἰς δέον ἡ παρασκευή καὶ μετὰ τῆς παρασκευῆς ὁ δημιουργὸς.

Ὤς οὖν προσέβαλον οἱ Ῥωμαῖοι διχόθεν, ἕκπληξε ἣν τῶν Συρακουσίων καὶ συγγά διὰ δέος, μηδὲν ἂν ἀνθέξειν πρὸς βίαν καὶ δύναμιν οἰομένων τοσαύτην. σχάσαντος δὲ τὰς μηχανὰς τοῦ Ἀρχιμήδους ἀμα τοῖς μὲν πεζοῖς ἀπίντα τοξεύματα τε παντοδαπὰ καὶ λίθων ὑπέρογκα μεγέθη, βοίζω καὶ τάχει καταφερομένων ἀπίστῳ, καὶ μηδενὸς ὅλως τὸ βρίθος στέγοντος ἀθρόους ἀνατρεπόντων τοὺς ὑποπίπτοντας καὶ τὰς τάξεις συγχεόντων, ταῖς δὲ ναυσὶν ἀπὸ τῶν πειχῶν ἁφυῖ ὑπεραιωροῦμεναι κεραίαι τὰς μὲν ὑπὸ βρίθους στηρίζοντος ἀνωθεν ἀθοῦσαι κατέδυνοι εἰς βυθὸν, τὰς δὲ χεροὶ σιδηραῖς ἡ στόμασιν εἰκασμένοις γεράνων ἀνασπώσαι πρῶραθεν ὀρθὰς ἐπὶ πρὺμναν ἐβάπτιζον,

* πολυσπάστος. Galen, in Hipp. De Artic. iv. 47 uses the same word. Tzetzes (loc. cit.) speaks of a triple-pulley device (τῇ τρισπάστῳ μηχανῇ) in the same connexion, and Oribasius, Coll. med. xlix. 22 mentions the τρισπάστος as an invention of Archimedes; he says that it was so called because it had three ropes, but Vitruvius says it was thus named because it had three wheels. Athenaeus v. 207 a-b says that a helix was used. Heath, The Works of Archimedes, 24.
the end of a compound pulley and drew the vessel smoothly and evenly towards himself as though she were running along the surface of the water. Astonished at this, and understanding the power of his art, the king persuaded Archimedes to construct for him engines to be used in every type of siege warfare, some defensive and some offensive; he had not himself used these engines because he spent the greater part of his life remote from war and amid the rites of peace, but now his apparatus proved of great advantage to the Syracusans, and with the apparatus its inventor.

Accordingly, when the Romans attacked them from two elements, the Syracusans were struck dumb with fear, thinking that nothing would avail against such violence and power. But Archimedes began to work his engines and hurled against the land forces all sorts of missiles and huge masses of stones, which came down with incredible noise and speed; nothing at all could ward off their weight, but they knocked down in heaps those who stood in the way and threw the ranks into disorder. Furthermore, beams were suddenly thrown over the ships from the walls, and some of the ships were sent to the bottom by means of weights fixed to the beams and plunging down from above; others were drawn up by iron claws, or crane-like beaks, attached to the prow and were

p. xx, suggests that the vessel, once started, was kept in motion by the system of pulleys, but the first impulse was given by a machine similar to the κοχλίας described by Pappus viii. ed. Hultsch 1066, 1108 ff., in which a cog-wheel with oblique teeth moves on a cylindrical helix turned by a handle.

Similar stories of Archimedes’ part in the defence are told by Polybius viii. 5. 3-5 and Livy xxiv. 34.
GREEK MATHEMATICS

ή δὲ ἀντιτόνων ἐνδον ἐπιστρεφόμεναι καὶ περιαγό-
μεναι τοῖς ὑπὸ τὸ τεῖχος πεφυκόσι κρημνοῖς καὶ
σκοπέλοις προσήρασσον, ἀμα φθόρῳ τολλῷ τῶν
ἐπιβατῶν συντριβομένων. πολλάκις δὲ μετέωρος
ἐξαρθείσα ναύς ἀπὸ τῆς θαλάσσης δεῦρο κάκεισε
περιδυνομένη καὶ κρεμαμένη θέαμα φρυκώδες ἦν,
μέχρι οὗ τῶν ἀνδρῶν ἀπορριφέντων καὶ διασφεν-
δονηθέντων κενὴ προσπέσοι τοῖς τείχεσιν ἦν περι-
ολίσθοι τῆς λαβῆς ἀνείσης. ἦν δὲ ὁ Μάρκελλος
ἀπὸ τοῦ ζεύγματος ἐπῆγε μιχαγῆν, σαμβύκη μὲν
ἐκαλεῖτο δὲ ὀμοιότητά τινα σχήματος πρὸς τὸ
μονοικὸν ὄργανον, ἑτὶ δὲ ἀπώθεν αὐτῆς προσφειρο-
μένης πρὸς τὸ τεῖχος ἐξῆλατο λίθος δεκατάλαντος
ὀλικήν, εἶτα ἑτερος ἐπὶ τοῦτῳ καὶ τρίτος, ὅν οἱ
μεν αὐτὴν ἐμπεσόντες μεγάλων κτύπῳ καὶ κλύδων
tῆς μιχαγῆς τῆν τε βάσιν συνηλόσαν καὶ τὸ
γόμφωμα διέστεσαν καὶ διέσπασαν τοῦ ζεύγματος,
ὥστε τὸν Μάρκελλον ἀπορομένον αὐτῶν τε ταῖς
ναυῶν ἀποπλείν κατὰ τάχος καὶ τοῖς πεζοῖς ἀνα-
χώρησιν παρεγγυῆσαι.

Βουλευμόμενοι δὲ ἔδοξεν αὐτοῖς ἔτι νυκτὸς, ἂν
δύνωται, προσμίζει τοῖς τείχεσιν τοὺς γαρ τόνως,
οἷς χρήσθαι τὸν Ἀρχιμήδην, ρύμην ἔχοντας
ὑπερπετείς ποιήσεσθαι τὰς τῶν βελῶν ἀφέσεις,
ἐγνώθεν δὲ καὶ τελέως ἀπράκτους εἶναι διάστημα
τῆς πληγῆς οὐκ ἔχουσας. ὃ δὲ ἦν, ὡς ἐοικεν, ἐπὶ
tαύτα πάλαι παρεσκευασμένος ὄργανον τε συμ-
μέτρουσ πρὸς πάν διάστημα κινήσεις καὶ βέλη
βραχέα, καὶ διὰ τὸ τεῖχος\(^2\) οὐ μεγάλων, πολλῶν

\(^1\) αὐτὴ Coraës, αὐτῆς codd.

\(^2\) τὸ τεῖχος add. Sintenis ex Polyb.
plunged down on their sterns, or were twisted round and turned about by means of ropes within the city, and dashed against the cliffs set by Nature under the wall and against the rocks, with great destruction of the crews, who were crushed to pieces. Often there was the fearful sight of a ship lifted out of the sea into mid-air and whirled about as it hung there, until the men had been thrown out and shot in all directions, when it would fall empty upon the walls or slip from the grip that had held it. As for the engine which Marcellus was bringing up from the platform of ships, and which was called *sambuca* from some resemblance in its shape to the musical instrument, a while it was still some distance away as it was being carried to the wall a stone ten talents in weight was discharged at it, and after this a second and a third; some of these, falling upon it with a great crash and sending up a wave, crushed the base of the engine, shook the framework and dislodged it from the barrier, so that Marcellus in perplexity sailed away in his ships and passed the word to his land forces to retire.

In a council of war it was decided to approach the walls, if they could, while it was still night; for they thought that the ropes used by Archimedes, since they gave a powerful impetus, would send the missiles over their heads and would fail in their object at close quarters since there was no space for the cast. But Archimedes, it seems, had long ago prepared for such a contingency engines adapted to all distances and missiles of short range, and through openings in the

---

*The *sambuca* was a triangular musical instrument with four strings. Polybius (viii. 6) states that Marcellus had eight quinqueremes in pairs locked together, and on each pair a "sambuca" had been erected; it served as a pent-house for raising soldiers on to the battlements."
GREEK MATHEMATICS

de kai svnexhωn tptmάtwn ὑντων', oi skorpioi
braxutonoi mēn, ēγγυθεν de plēξai pαrεstηκεσαν
ārαtοι tois pοleμίοis.

'Ωs oūn prōsēμξαν οἰόμενοι λανθάνειν, αὕθις αὕ
bēlexi pοlλoīs ἐντυγχάνοντες kai pληγαίς, pετρῶν
mēn ēk kēfalῆs ἐπ' αὐτούs fερομένων ὁσπερ πρός
kάθετον, tου de teīchous toxeūmata pαntαχόθεν
ἄναπέμποντος, ἀνεχώρουν ὁπίσω. kάνταθα pάλιν
αὐτῶν εἰς μήκος ἐκτεταγμένων, bελῶν ἐκθεόντων
kai kαtαλαμβάνοντων ἀπιόντας ἐγίνετο pολὺs mēn
aὐτῶν φθόρος, pολὺs de tῶn νεῶn συγκρουμός,
oūdēn ἀντιδράσαι tοὺs pολεμίουs δυναμένων. tὰ
γαρ πλεῖστα tῶν ὄργανων ὑπὸ τὸ teίchοs ἐσκευο-
ποίητο τῷ Ἀρχιμήδει, kai theοmαχοῦσιν ἐξικεσαν
oἱ Ῥωμαῖοι, μυρίων αὐτοῖς kακῶν ἕξ ἀφανοῦ
εἰπεξομένων.

Oū μὴν ἀλλ' ὁ Mάρκελλος ἀπέφυγε τε καὶ tοὺs
sūn ἑαυτῶi σκώπτων tεχνίταs kai μηχανοτοιοῦ
ἐλεγεν: "oū pαυσόμεθα pρὸs τὸn γεωμετρικῶn
toūtov Bριάρεων pολεμοῦντεs, ὃs tαῖς mὲn ναυαῖn
ἡμῶν κυαθίζει εκ tῆs βαλάσηs, tῆn de σαμβύκηn
rαπίζων μετ' αἰσχύνηs ἐκβέβληκε, τοὺs de μυθικοὺs
ἐκατόγχειραs ὑπεραίρει τοσαύτα βάλλων ἀμα βέλη
cαθ' ἡμῶν;" tῷ γαρ ὀντὶ pάντεs οἱ lουποὶ Sυρα-
kούσιοι σῶμα tῆs Ἀρχιμῆδουs παρασκευῆs ἦσαν,
η de κυνοῦσα pάντα kai στρέφουσα ψυχῆ mία, tῶn
mēn ἄλλων ὀπλῶν ἀτρέμα kειμένων, mόνοι de tοὺs
ἐκείνου tότε tῆs pόλεωs χρωμένηs kai πρὸs
ἀμυναν kai πρὸs ἀσφάλειαν. tέλοs de tοὺs
Ῥωμαίουs σὺν pεριφόβουs γεγονόταs ὀρῶn ὁ
Mάρκελλος ὡστ', eί kαλύδιοn ἡ εύλου ὑπὲρ τοῦ

1 ὑντων add. Sintenis ex Polyb.
wall, small in size but many and continuous, short-ranged engines called scorpions could be trained on objects close at hand without being seen by the enemy.

When, therefore, the Romans approached the walls, thinking to escape notice, once again they were met by the impact of many missiles; stones fell down on them almost perpendicularly, the wall shot out arrows at them from all points, and they withdrew to the rear. Here again, when they were drawn up some distance away, missiles flew forth and caught them as they were retiring, and caused much destruction among them; many of the ships, also, were dashed together and they could not retaliate upon the enemy. For Archimedes had made the greater part of his engines under the wall, and the Romans seemed to be fighting against the gods, inasmuch as countless evils were poured upon them from an unseen source.

Nevertheless Marcellus escaped, and, twitting his artificers and craftsmen, he said: “Shall we not cease fighting against this geometrical Briareus, who uses our ships like cups to ladle water from the sea, who has whipped our sambuca and driven it off in disgrace, and who outdoes all the hundred-handed monsters of fable in hurling so many missiles against us all at once?” For in reality all the other Syracusans were only a body for Archimedes’ apparatus, and his the one soul moving and turning everything: all other weapons lay idle, and the city then used his alone, both for offence and for defence. In the end the Romans became so filled with fear that, if they saw a little piece of rope or of wood projecting over

---

2 ταῖς μὲν ναυσὶν . . . βαπίζων an anonymous correction from Polybius, τὰς μὲν ναῦς ἡμῶν καθίζων πρὸς τὴν θάλασσαν παῖζων codd.
GREEK MATHEMATICS

teíchos mikrón ὁφθεὶν προτεινόμενον, τοῦτο ἐκεῖνο, μηχανήν τινα κινεῖν ἐπ’ αὐτοῦς Ἄρχιμήδη βοῶντας ἀποτρέπεσθαι καὶ φεύγειν, ἀπέσχετο μάχης ἀπάσης καὶ προσβολῆς, τὸ λοιπὸν ἐπὶ τῶν χρόνων τῆς πολιορκίας θέμενος.

Τηλικοῦτον μέντοι φρόνημα καὶ βάθος ψυχῆς καὶ τοσοῦτον ἐκέκτητο θεωρημάτων πλοῦτον Ἄρχιμήδης ὡστε, ἐφ’ οἷς ὀνομα καὶ δόξαν οὐκ ἀνθρωπίνης, ἀλλὰ δαμνονίου τινὸς ἐσχε συνέσεως, μηθὲν ἔθελήσαι σύγγραμμα περὶ τούτων ἀπολπεῖν, ἀλλὰ τὴν περὶ τὰ μηχανικὰ πραγματείαν καὶ πᾶσαν ὅλως τέχνην χρείας ἐφαπτομένην ἁγενῆ καὶ βάραυσον ἡγησάμενος, εἰς ἐκεῖνα καταθέσθαι μόνα τὴν αὐτοῦ φιλοτιμίαν οἷς τὸ καλὸν καὶ περιττὸν ἀμιγῆς τοῦ ἀναγκαῖον πρόσεστιν, ἀσύγκριτα μὲν ὅντα τοῖς ἄλλοις, ἔρων δὲ παρέχοντα πρὸς τὴν ὕλην τῇ ἀποδείξει, τῆς μὲν τὸ μέγεθος καὶ τὸ κάλλος, τῆς δὲ τὴν ἀκριβείαν καὶ τὴν δύναμιν ὑπερφυῆ παρεχομένης. οὐ γὰρ ἔστιν ἐν γεωμετρίᾳ χαλεπωτέρας καὶ βαρυτέρας ὑποθέσεις ἐν ἀπλούστεροις λαβεῖν καὶ καθαρωτέρους στοιχείους γραφομένας καὶ τοῦθ’ οἱ μὲν εὐφυῖα τοῦ ἀνδρὸς προσάπτοσιν, οἱ δὲ υπερβολῆ τινὶ πόνῳ νομίζουσιν ἀπόνως πεποιημένως καὶ ῥαδίως ἔκαστον ἐοίκος γεγονέναι. ξητῶν μὲν γὰρ οὐκ ἂν τὶς εὑροὶ δι’ αὐτοῦ τὴν ἀπόδειξιν, ἀμα δὲ τῇ μαθῆσει παρισταται δόξα τοῦ κἂν αὐτοῦ εὐρεῖν. οὔτω λείαν ὁδὸν ἄγει καὶ ταχείαν ἐπὶ τὸ δεικνύμενον. οὕκουν οὔδὲ ἀπιστήσαι τοῖς περὶ αὐτοῦ λεγομένως ἐστίν, ἐς ὑπ’ οἰκείας ὅτι τινος καὶ συνοικίου βελγόμενος ἀεὶ σειρῆνος ἐλέλυστο καὶ σῖτον καὶ θεραπείας σώματος ἐξέλειπε, βία δὲ πολλάκις ἐλκόμενοι ἐπ’ ἀλείμμα καὶ
the wall, they cried, "There it is, Archimedes is training some engine upon us," and fled; seeing this Marcellus abandoned all fighting and assault, and for the future relied on a long siege.

Yet Archimedes possessed so lofty a spirit, so profound a soul, and such a wealth of scientific inquiry, that although he had acquired through his inventions a name and reputation for divine rather than human intelligence, he would not deign to leave behind a single writing on such subjects. Regarding the business of mechanics and every utilitarian art as ignoble and vulgar, he gave his zealous devotion only to those subjects whose elegance and subtlety are untrammelled by the necessities of life; these subjects, he held, cannot be compared with any others; in them the subject-matter vies with the demonstration, the former possessing strength and beauty, the latter precision and surpassing power; for it is not possible to find in geometry more difficult and weighty questions treated in simpler and purer terms. Some attribute this to the natural endowments of the man, others think it was the result of exceeding labour that everything done by him appeared to have been done without labour and with ease. For although by his own efforts no one could discover the proof, yet as soon as he learns it, he takes credit that he could have discovered it: so smooth and rapid is the path by which he leads to the conclusion. For these reasons there is no need to disbelieve the stories told about him—how, continually bewitched by some familiar siren dwelling with him, he forgot his food and neglected the care of his body; and how, when he was dragged by main force, as often happened, to the

1 αγει Bryan, αγευ codd.
GREEK MATHEMATICS

λοντρόν, ἐν ταῖς ἐσχάραις ἔγραφε σχῆματα τῶν
gewmetrikῶν, καὶ τοῦ σώματος ἀληλιμένου διήγε
τῷ δακτύλῳ γραμμάς, ὑπὸ ἡδονῆς μεγάλης κάτοχος
ῶν καὶ μουσώληπος ἀληθῶς. πολλῶν δὲ καὶ
καλῶν εὐρετῆς γεγονός λέγεται τῶν φίλων δεη-
θήναι καὶ τῶν συγγενῶν ὅπως αὐτοῖς μετὰ τὴν
teleutην ἐπιστήσωσι τῷ τάφῳ τὸν περιλαμβάνοντα
tὴν σφαῖραν ἐντὸς κῦλινδρον, ἐπιγράφαντες τὸν
λόγον τῆς ὑπεροχῆς τοῦ περιέχοντος στερεοῦ πρὸς
tὸ περιεχόμενον.

Ibid. xix. 4-6

Μάλιστα δὲ τὸ 'Ἀρχιμήδους πάθος ἤνιασε Μάρ-
κελλον. ἔτυχε μὲν γὰρ αὐτὸς τι καθ’ ἐαυτὸν
ἀνασκοπῶν ἐπὶ διαγράμματος καὶ τῇ θεωρίᾳ
dedwkon ἁμα τὴν τε δίανοιαν καὶ τὴν πρόσωπῳ
οὐ προῆσθετο τὴν καταδρομήν τῶν Ἄρμαῖων
όυδὲ τὴν ἀλώσων τῆς πόλεως, ἄφνω δὲ ἐπιστάντος
αὐτῷ στρατιώτου καὶ κελεύοντος ἀκολουθεῖν πρὸς
Μάρκελλον οὐκ ἐβούλετο πρὶν ἢ τελέσαι τὸ πρό-
βλημα καὶ καταστήσαι πρὸς τὴν ἀπόδειξιν. δὲ
ὄργυσθεὶς καὶ σπασάμενος τὸ ἔφος ἀνείλεν αὐτῶν.
ἐτεροὶ μὲν οὖν λέγουσιν ἐπιστήμην μὲν εὐθὺς ὡς
ἀποκτενοῦντα ἐξήρη τὸν Ἀρμαῖον, ἐκεῖνον δὲ
ἰδόντα δεισθαί καὶ ἀντιβολεῖν ἀναμείναι βραχὺν
χρόνον, ὡς μὴ καταλήπτη τὸ ζητοῦμενον ἀτελές
καὶ ἀθεώρητον, τὸν δὲ οὐ φροντίσαντα διαχρῆ-
σαθαί. καὶ τρίτος ἔστὶ λόγος, ὃς κομίζοντι
πρὸς Μάρκελλον αὐτῷ τῶν μαθηματικῶν ὄργανων
σκιόθηρα καὶ σφαίρας καὶ γωνιῶς, αὐτὸ ἐναρμόττει

* Cicero, when quaestor in Sicily, found this tomb over-
ARCHIMEDES

place for bathing and anointing, he would draw geometrical figures in the hearths, and draw lines with his finger in the oil with which his body was anointed, being overcome by great pleasure and in truth inspired of the Muses. And though he made many elegant discoveries, he is said to have besought his friends and kinsmen to place on his grave after his death a cylinder enclosing a sphere, with an inscription giving the proportion by which the including solid exceeds the included.*

Ibid. xix. 4-6

But what specially grieved Marcellus was the death of Archimedes. For it chanced that he was alone, examining a diagram closely; and having fixed both his mind and his eyes on the object of his inquiry, he perceived neither the inroad of the Romans nor the taking of the city. Suddenly a soldier came up to him and bade him follow to Marcellus, but he would not go until he had finished the problem and worked it out to the demonstration. Thereupon the soldier became enraged, drew his sword and dispatched him. Others, however, say that the Roman came upon him with drawn sword intending to kill him at once, and that Archimedes, on seeing him, besought and entreated him to wait a little while so that he might not leave the question unfinished and only partly investigated; but the soldier did not understand and slew him. There is also a third story, that as he was carrying to Marcellus some of his mathematical instruments, such as sundials, spheres and grown with vegetation, but still bearing the cylinder with the sphere, and he restored it (Tusc. Disp. v. 64-66). The theorem proving the proportion is given infra, pp. 124-127.
GREEK MATHEMATICS

to tov ἥλιου μέγεθος πρὸς τὴν ὀψιν, στρατιωταὶ περιτυχόντες καὶ χρυσιῶν ἐν τῷ τεύχει δόξαντες φέρειν ἀπέκτειναν. ὥστε μέντοι Μάρκελλος ἢλγησε καὶ τὸν αὐτόχειρα τοῦ ἀνδρὸς ἀπεστράφη καθάπερ ἐναγη, τοὺς δὲ οἰκείους ἀνευρών ἐτύμησεν, ὁμολογεῖτα.

Papp. Coll. viii. 11. 19, ed. Hultsch 1060. 1-4

Τῆς αὐτῆς δὲ ἐστὶν θεωρίας τὸ δοθὲν βάρος τῇ δοθείσῃ δυνάμει κινῆσαι· τοῦτο γὰρ Ἀρχιμήδους μὲν εὔρημα [λέγεται] μηχανικόν, ἐφ’ ὡς λέγεται εἰρηκεύων· "δός μοί (φησι) ποῦ στῶ καὶ κινῶ τὴν γῆν."

Diod. Sic. i. 34. 2

Ποταμόχωστος γὰρ οὕσα καὶ κατάρρυτος πολλοὺς καὶ πανταδαποῦς ἐκφέρει καρποὺς, τοῦ μὲν ποταμοῦ διὰ τὴν κατ’ ἐτος ἀνάβασιν νεαρὰν ἐλυν ἄει καταχέοντος, τῶν δ’ ἀνθρώπων ῥαδίως ἀπασαν ἀρδευόντων διὰ τινὸς μηχανῆς, ἤν ἐπενόησε μὲν Ἀρχιμήδης ὁ Συρακόσιος, ὁνομάζεται δὲ ἀπὸ τοῦ σχῆματος κοχλίας.

Ibid. v. 37. 3

Τὸ πάντων παραδοξότατον, ἀπαρύτοιο τὰς ρύσεις τῶν ὑδάτων τοῖς Αἰγυπτιακοῖς λεγομένοις κοχλίαις, οὕς Ἀρχιμήδης ὁ Συρακόσιος ἐδρευν, διὸς παρέβαλεν εἰς Αἰγυπτιον.

1 λέγεται om. Hultsch.

a Diodorus is writing of the island in the delta of the Nile.

b It may be inferred that he studied with the successors of Euclid at Alexandria, and it was there perhaps that he made the acquaintance of Conon of Samos, to whom, as 34

34
angles adjusted to the apparent size of the sun, some soldiers fell in with him and, under the impression that he carried treasure in the box, killed him. What is, however, agreed is that Marcellus was distressed, and turned away from the slayer as from a polluted person, and sought out the relatives of Archimedes to do them honour.

Pappus, Collection viii. 11. 19, ed. Hultsch 1060. 1-4

To the same type of inquiry belongs the problem: To move a given weight by a given force. This is one of Archimedes' discoveries in mechanics, whereupon he is said to have exclaimed: "Give me somewhere to stand and I will move the earth."

Diodorus Siculus i. 34. 2

As it is made of silt watered by the river after being deposited, it\(^a\) bears an abundance of fruits of all kinds; for in the annual rising the river continually pours over it fresh alluvium, and men easily irrigate the whole of it by means of a certain instrument conceived by Archimedes of Syracuse, and which gets its name because it has the form of a spiral or screw.

Ibid. v. 37. 3

Most remarkable of all, they draw off streams of water by the so-called Egyptian screws, which Archimedes of Syracuse invented when he went by ship to Egypt.\(^b\)

the preface to his books On the Sphere and Cylinder shows, he used to communicate his discoveries before publication, and Eratosthenes of Cyrene, to whom he sent the Method and probably the Cattle Problem.
GREEK MATHEMATICS

Vitr. De Arch. ix., Praef. 9-12

Archimedes vero cum multa miranda inventa et varia fuerint, ex omnibus etiam infinita sollertia id quod exponam videtur esse expressum. Nimium Hiero Syracusis auctus regia potestate, rebus bene gestis cum auream coronam votivam diis immortalibus in quodam fano constituisset ponendam, manupretio locavit faciendum et aurum ad sacoma adpendit redemptori. Is ad tempus opus manu factum subtiliter regi adprobavit et ad sacoma pondus coronae visus est praestitisse. Posteaquam indicium est factum dempto auro tantundem argenti in id coronarium opus admixtum esse, indignatus Hiero se contemptum esse neque inveniens qua ratione id furtum deprehenderet, rogavit Archimeden uti insumeret sibi de eo cogitationem. Tunc is cum haberet eius rei curam, casu venit in balineum ibique cum in solium descenderet, animadvertit quantum corporis sui in eo insideret tantum aquae extra solium effluere. Idque cum eius rei rationem explicationis ostendisset, non est moratus sed exsiluit gaudio motus de solio et nudus vadens domum versus significabat clara voce invenisse quod quaereret. Nam currens identidem graece clamabat εὑρηκα εὑρηκα.

Tum vero ex eo inventionis ingressu duas fecisse dicitur massas aequo pondere quo etiam fuerat corona, unam ex auro et alteram ex argento. Cum ita fecisset, vas amplum ad summa labra implevit

* "I have found, I have found."
Archimedes made many wonderful discoveries of different kinds, but of all these that which I shall now explain seems to exhibit a boundless ingenuity. When Hiero was greatly exalted in the royal power at Syracuse, in return for the success of his policy he determined to set up in a certain shrine a golden crown as a votive offering to the immortal gods. He let out the work for a stipulated payment, and weighed out the exact amount of gold for the contractor. At the appointed time the contractor brought his work skillfully executed for the king's approval, and he seemed to have fulfilled exactly the requirement about the weight of the crown. Later information was given that gold had been removed and an equal weight of silver added in the making of the crown. Hiero was indignant at this disrespect for himself, and, being unable to discover any means by which he might unmask the fraud, he asked Archimedes to give it his attention. While Archimedes was turning the problem over, he chanced to come to the place of bathing, and there, as he was sitting down in the tub, he noticed that the amount of water which flowed over the tub was equal to the amount by which his body was immersed. This indicated to him a means of solving the problem, and he did not delay, but in his joy leapt out of the tub and, rushing naked towards his home, he cried out with a loud voice that he had found what he sought. For as he ran he repeatedly shouted in Greek, heureka, heureka. Then, following up his discovery, he is said to have made two masses of the same weight as the crown, the one of gold and the other of silver. When he had so done, he filled a large vessel right up to the brim
GREEK MATHEMATICS

aqua, in quo demisit argenteam massam. Cuius quanta magnitudo in vase depressa est, tantum aquae effluxit. Ita exempta massa quanto minus factum fuerat refudit sextario mensus, ut eodem modo quo prius fuerat ad labra aequaretur. Ita ex eo invenit quantum pondus argenti ad certam aquae mensuram responderet.

Cum id expertus esset, tum auream massam similiter pleno vase demisit et ea exempta eadem ratione mensura addita invenit deesse aquae non tantum sed minus, quanto minus magno corpore eodem pondere auri massa esset quam argenti. Postea vero repleto vase in eadem aqua ipsa corona demissa invenit plus aquae defluxisse in coronam quam in auream eodem pondere massam, et ita ex eo quod defuerit plus aquae in corona quam in massa, ratiocinatus deprehendit argenti in auro mixtionem et manifestum furtum redemptoris.

* The method maybe thus expressed analytically.
Let \( w \) be the weight of the crown, and let it be made up of a weight \( w_1 \) of gold and a weight \( w_2 \) of silver, so that \( w = w_1 + w_2 \).

Let the crown displace a volume \( v \) of water.
Let the weight \( w \) of gold displace a volume \( v_1 \) of water; then a weight \( w_1 \) of gold displaces a volume \( \frac{w_1}{w} \cdot v_1 \) of water.
Let the weight \( w \) of silver displace a volume \( v_2 \) of water;
with water, into which he dropped the silver mass. The amount by which it was immersed in the vessel was the amount of water which overflowed. Taking out the mass, he poured back the amount by which the water had been depleted, measuring it with a pint pot, so that as before the water was made level with the brim. In this way he found what weight of silver answered to a certain measure of water.

When he had made this test, in like manner he dropped the golden mass into the full vessel. Taking it out again, for the same reason he added a measured quantity of water, and found that the deficiency of water was not the same, but less; and the amount by which it was less corresponded with the excess of a mass of silver, having the same weight, over a mass of gold. After filling the vessel again, he then dropped the crown itself into the water, and found that more water overflowed in the case of the crown than in the case of the golden mass of identical weight; and so, from the fact that more water was needed to make up the deficiency in the case of the crown than in the case of the mass, he calculated and detected the mixture of silver with the gold and the contractor’s fraud stood revealed.\(^a\)

then a weight \(w_2\) of silver displaces a volume \(\frac{w_2}{w} \cdot v_2\) of water.

It follows that

\[
v = \frac{w_1}{w} \cdot v_1 + \frac{w_2}{w} \cdot v_2
\]

\[
= \frac{w_1v_1 + w_2v_2}{w_1 + w_2},
\]

so that

\[
\frac{w_1}{w_2} = \frac{v_2 - v}{v - v_1}.
\]


39
GREEK MATHEMATICS

(b) Surface and Volume of the Cylinder and Sphere

Archim. De Sphaera et Cyl. i., Archim. ed. Heiberg i. 2–132. 3

'Αρχιμήδης Δοσιθέω χαίρειν

Πρότερον μὲν ἀπέσταλκά σοι τῶν υφ' ἠμῶν τεθεωρημένων γράφας μετὰ ἀποδείξεως, ὅτι πᾶν τμήμα τὸ περιεχόμενον ὑπὸ τε εὐθεῖας καὶ ὀρθογωνίου κώνου τομῆς ἑπτάρτων ἦστι τριγώνου τοῦ βάσιν τῆν αὐτὴν ἔχοντος τῷ τμήματι καὶ ὑψος ἵσον· ὑστερον δὲ ἠμῶν ὑποπεσόντων θεωρημάτων ἀξίων λόγου1 πεπραγματεύμεθα περὶ τὰς ἀποδείξεις αὐτῶν. ἦστιν δὲ τάδε· πρῶτον μὲν, ὅτι πάσης σφαίρας ἡ ἐπιφάνεια τετραπλασία ἦστιν τοῦ μεγίστον κύκλου τῶν ἐν αὐτῇ ἐπειτα δὲ, ὅτι παντὸς τμῆματος σφαίρας τῇ ἐπιφάνεια ἴσος ἦστι κύκλος, οὔ ἐκ τοῦ κέντρου ἴση ἦστι τῇ εὐθείᾳ τῇ ἀπὸ τῆς κορυφῆς τοῦ τμῆματος ἀγομένη ἐπὶ τὴν περιφέρειαν τοῦ κύκλου, ὃς ἦστι βάσις τοῦ τμῆματος·

1 ἀξίων λόγου cod., ἀνελέγκτων coni. Heath.

* The chief results of this book are described in the prefatory letter to Dositheus. In this selection as much as possible is given of what is essential to finding the proportions between the surface and volume of the sphere and the surface and volume of the enclosing cylinder, which Archimedes regarded as his crowning achievement (supra, p. 32). In the case of the surface, the whole series of propositions is reproduced so that the reader may follow in detail the majestic chain of reasoning by which Archimedes, starting from seemingly remote premises, reaches the desired conclusion; in the case of the volume only the final proposition (34) can be given, for reasons of space, but the reader will be able to prove the omitted theorems for himself. Pari passu with 40
ARCHIMEDES

(b) Surface and Volume of the Cylinder and Sphere

Archimedes. On the Sphere and Cylinder i., Archim. ed. Heiberg i. 2-132. 3

Archimedes to Dositheus greeting

On a previous occasion I sent you, together with the proof, so much of my investigations as I had set down in writing, namely, that any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle having the same base as the segment and equal height. Subsequently certain theorems deserving notice occurred to me, and I have worked out the proofs. They are these: first, that the surface of any sphere is four times the greatest of the circles in it; then, that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of the circle which is the base of the segment; and, this demonstration, Archimedes finds the surface and volume of any segment of a sphere. The method in each case is to inscribe in the sphere or segment of a sphere, and to circumscribe about it, figures composed of cones and frusta of cones. The sphere or segment of a sphere is intermediate in surface and volume between the inscribed and circumscribed figures, and in the limit, when the number of sides in the inscribed and circumscribed figures is indefinitely increased, it would become identical with them. It will readily be appreciated that Archimedes' method is fundamentally the same as integration, and on p. 116 n. b this will be shown trigonometrically.

b This is proved in Props. 17 and 24 of the Quadrature of the Parabola, sent to Dositheus of Pelusium with a prefatory letter, v. pp. 228-243, infra.

c De Sphaera et Cyl. i. 30. "The greatest of the circles," here and elsewhere, is equivalent to "a great circle."

d Ibid. i. 42, 43.

41
GREEK MATHEMATICS

τροπόν τοῦτος, ὅτι πάσης σφαίρας ὁ κύλινδρος ὁ βάσις μὲν ἔχων ᾗς τῷ μεγίστῳ κύκλῳ τῶν ἐν τῇ σφαίρᾳ, ὕψος δὲ ᾗς τῇ διαμέτρῳ τῆς σφαίρας αὐτὸς τε ἡμισφαῖρος ἐστὶν τῆς σφαίρας, καὶ ἡ ἐπιφάνεια αὐτοῦ τῆς ἐπιφανείας τῆς σφαίρας. ταῦτα δὲ τὰ συμπτώματα τῇ φύσει προσήκοψεν περὶ τὰ εἰρημένα σχήματα, ἠγνοεῖτο δὲ ὑπὸ τῶν πρὸ ἡμῶν περὶ γεωμετρίαν ἀνεστραμμένων οὐδενὸς αὐτῶν ἐπιφανεικτόσ, ὅτι τούτων τῶν σχημάτων ἐστὶν συμμετρία. . . ἐξέσται δὲ περὶ τούτων ἐπισκέψασθαι τοῖς δυνησμένοισ. ὥθειλε μὲν οὖν Κόνωνος ἐτὶ ζώντος ἐκδίδασθαι ταῦτα τῇν γὰρ ὑπολαμβάνομεν που μάλιστα ἂν δύνασθαι κατανοῆσαι ταῦτα καὶ τὴν ἀρμόζουσαν ὑπὲρ αὐτῶν ἀπόφασιν ποιήσασθαι· δοκιμάζοντες δὲ καλῶς ἔχειν μεταδίδοναι τοῖς οἰκείοις τῶν μαθημάτων ἀποστέλλομεν σοι τὰς ἀποδείξεις ἀναγράψαντες, ὕπερ δὲν ἐξέσται τοῖς περὶ τὰ μαθήματα ἀναστρεφομένοις ἐπισκέψασθαι. ἐρωμένως.

Γράφονται πρῶτον τὰ τε ἀξιώματα καὶ τὰ λαμβανόμενα εἰς τὰς ἀποδείξεις αὐτῶν.

'Αξιώματα

α'. Εἰσὶ τινες ἐν ἐπιπέδῳ καμπύλαι γραμμαὶ πεπερασμέναι, αἱ τῶν τὰ πέρατα ἐπιζευγνυσοῦν αὐτῶν εὑθείων ἦτοι ὅλαι ἐπὶ τὰ αὐτὰ εἰσὶν ἡ οὐδὲν ἔχουσιν ἐπὶ τὰ ἑτέρα.

β'. Ἐπὶ τὰ αὐτὰ δὴ κοίλην καλῶ τὴν τοιαύτην γραμμῆν, ἐν ἡ ἐὰν δύο σημείων λαμβανομένων

---

* De Sphaera et Cyl. i. 34 coroll. The surface of the cylinder here includes the bases.

42
further, that, in the case of any sphere, the cylinder having its base equal to the greatest of the circles in the sphere, and height equal to the diameter of the sphere, is one-and-a-half times the sphere, and its surface is one-and-a-half times the surface of the sphere. Now these properties were inherent in the nature of the figures mentioned, but they were unknown to all who studied geometry before me, nor did any of them suspect such a relationship in these figures. But now it will be possible for those who have the capacity to examine these discoveries of mine. They ought to have been published while Conon was still alive, for I opine that he more than others would have been able to grasp them and pronounce a fitting verdict upon them; but, holding it well to communicate them to students of mathematics, I send you the proofs that I have written out, which proofs will now be open to those who are conversant with mathematics. Farewell.

In the first place, the axioms and the assumptions used for the proofs of these theorems are here set out.

**AXIOMS** c

1. There are in a plane certain finite bent lines which either lie wholly on the same side of the straight lines joining their extremities or have no part on the other side.

2. I call *concave in the same direction* a line such that, if any two points whatsoever are taken on it, either

---

*b* In the omitted passage which follows, Archimedes compares his discoveries with those of Eudoxus; it has already been given, vol. i. pp. 408-411.

c These so-called axioms are more in the nature of definitions.
GREEK MATHEMATICS

ὅπως οὖν αἱ μεταξὺ τῶν σημείων εὐθεῖαι ἦτοι πᾶσαι ἐπὶ τὰ αὐτὰ πίπτουσιν τῆς γραμμῆς, ἢ τινὲς μὲν ἐπὶ τὰ αὐτὰ, τινὲς δὲ κατ’ αὐτὴς, ἐπὶ τὰ ἑτερα δὲ μηδεμία.

γ’. Ὠμοίως δὴ καὶ ἐπιφάνειαι τινὲς εἰσὶν πε- περασμέναι, αὐταὶ μὲν οὐκ ἐν ἐπίπεδῳ, τὰ δὲ πέρατα ἔχουσαι ἐν ἐπίπεδῳ, αἱ τοῦ ἐπιπέδου, ἐν ὧ τὰ πέρατα ἔχουσιν, ἦτοι οὖλα ἐπὶ τὰ αὐτὰ ἐσονται ἢ οὐδὲν ἔχουσιν ἐπὶ τὰ ἑτερα.

δ’. Ἐπὶ τὰ αὐτὰ δὴ κοίλας καλὸς τὰς τοιαύτας ἐπιφανείας, ἐν αἰσ ἄν δύο σημείων λαμβανομένων αἱ μεταξὺ τῶν σημείων εὐθεῖαι ἦτοι πᾶσαι ἐπὶ τὰ αὐτὰ πίπτουσιν τῆς ἐπιφανείας, ἢ τινὲς μὲν ἐπὶ τὰ αὐτὰ, τινὲς δὲ κατ’ αὐτής, ἐπὶ τὰ ἑτερα δὲ μηδεμία.

ε’. Τομέα δὲ στερεῶν καλῶν, ἐπειδὴν σφαῖραν κῶνος τέμνη κορυφὴν ἔχων πρὸς τῷ κέντρῳ τῆς σφαίρας, τὸ ἐμπεριεχόμενον σχῆμα ὑπὸ τε τῆς ἐπιφανείας τοῦ κώνου καὶ τῆς ἐπιφανείας τῆς σφαίρας ἐν τοῦ κώνου.

ζ’. Ρόμβου δὲ καλῶ στερεῶν, ἐπειδὴν δύο κῶνοι τὴν αὐτὴν βάσιν ἔχουσιν τὰς κορυφὰς ἔχουσιν ἐφ’ ἐκάτερα τοῦ ἐπιπέδου τῆς βάσεως, ὅπως οἱ ἄξονες αὐτῶν ἐπ’ εὐθείας ὅσοι κείμενοι, τὸ ἐξ ἀμφότερον τοῦ κώνου συγκείμενον στερεόν σχῆμα.

Λαμβανόμενα

Λαμβάνω δὲ ταῦτα.

α’. Τῶν τὰ αὐτὰ πέρατα ἔχουσῶν γραμμῶν ἐλαχίστην εἶναι τὴν εὐθείαν.
ARCHIMEDES

all the straight lines joining the points fall on the same side of the line, or some fall on one and the same side while others fall along the line itself, but none fall on the other side.

3. Similarly also there are certain finite surfaces, not in a plane themselves but having their extremities in a plane, and such that they will either lie wholly on the same side of the plane containing their extremities or will have no part on the other side.

4. I call concave in the same direction surfaces such that, if any two points on them are taken, either the straight lines between the points all fall upon the same side of the surface, or some fall on one and the same side while others fall along the surface itself, but none falls on the other side.

5. When a cone cuts a sphere, and has its vertex at the centre of the sphere, I call the figure comprehended by the surface of the cone and the surface of the sphere within the cone a solid sector.

6. When two cones having the same base have their vertices on opposite sides of the plane of the base in such a way that their axes lie in a straight line, I call the solid figure formed by the two cones a solid rhombus.

POSTULATES

I make these postulates:

1. Of all lines which have the same extremities the straight line is the least.a

a Proclus (in Eucl., ed. Friedlein 110. 10-14) saw in this statement a connexion with Euclid’s definition of a straight line as lying evenly with the points on itself: ὁ δ’ αὐτ’ Ἀρχιμήδης τὴν εὐθεῖαν ὀφθαλμόν γραμμὴν ἐλαχίστην τῶν τὰ αὐτὰ πέρατα ἔχουσών. διότι γάρ, ὡς ὁ Ἐὐκλείδης λόγος φησίν, ἐξ ἰσού κεῖται τοῖς ἐφ’ ἐαυτῆς σημεῖοι, διὰ τούτο ἐλαχίστη ἠστίν τῶν τὰ αὐτὰ πέρατα ἔχοσών.
This famous "Axiom of Archimedes" is, in fact, generally used by him in the alternative form in which it is proved.
2. Of other lines lying in a plane and having the same extremities, [any two] such are unequal when both are concave in the same direction and one is either wholly included between the other and the straight line having the same extremities with it, or is partly included by and partly common with the other; and the included line is the lesser.

3. Similarly, of surfaces which have the same extremities, if those extremities be in a plane, the plane is the least.

4. Of other surfaces having the same extremities, if the extremities be in a plane, [any two] such are unequal when both are concave in the same direction, and one surface is either wholly included between the other and the plane having the same extremities with it, or is partly included by and partly common with the other; and the included surface is the lesser.

5. Further, of unequal lines and unequal surfaces and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those comparable with one another.*

With these premises, if a polygon be inscribed in a circle, it is clear that the perimeter of the inscribed polygon is less than the circumference of the circle; for each of the sides of the polygon is less than the arc of the circle cut off by it.

in Euclid x. 1, for which v. vol. i, pp. 452-455. The axiom can be shown to be equivalent to Dedekind's principle, that a section of the rational points in which they are divided into two classes is made by a single point. Applied to straight lines, it is equivalent to saying that there is a complete correspondence between the aggregate of real numbers and the aggregate of points in a straight line; v. E. W. Hobson, The Theory of Functions of a Real Variable, 2nd ed., vol. i, p. 55.
GREEK MATHEMATICS

α'

'Εὰν περὶ κύκλου πολύγωνον περιγραφῇ, ἡ τοῦ περιγραφέντος πολυγώνου περίμετρος μεῖζων ἐστὶν τῆς περιμέτρου τοῦ κύκλου.

Περὶ γὰρ κύκλου πολύγωνον περιγεγράφθω τὸ ύποκείμενον. λέγω, ὅτι ἡ περίμετρος τοῦ πολυγώνου μεῖζων ἐστὶν τῆς περιμέτρου τοῦ κύκλου.

Επεί γὰρ συναμφότερος ἡ ΒΑΛ μεῖζων ἐστὶ τῆς ΒΛ περιφερείας διὰ τὸ τὰ αὐτὰ πέρατα ἔχουσαν περιλαμβάνειν τὴν περιφερείαν, ὡμοίως δὲ καὶ συναμφότερος μὲν ἡ ΔΓ, ΓΒ τῆς ΔΒ, συναμφότερος δὲ ἡ ΛΚ, ΚΘ τῆς ΛΘ, συναμφότερος δὲ ἡ ΖΗΘ τῆς ΖΘ, ἔτι δὲ συναμφότερος ἡ ΔΕ, ΕΖ τῆς ΔΖ, ὥλη ἀρα ἡ περίμετρος τοῦ πολυγώνου μεῖζων ἐστὶ τῆς περιφερείας τοῦ κύκλου.

* It is here indicated, as in Prop. 3, that Archimedes added a figure to his own demonstration.
ARCHIMEDES

Prop. 1

If a polygon be circumscribed about a circle, the perimeter of the circumscribed polygon is greater than the circumference of the circle.

For let the polygon be circumscribed about the circle as below. I say that the perimeter of the polygon is greater than the circumference of the circle.

For since $BA + AA > \text{arc } BA$, owing to the fact that they have the same extremities as the arc and include it, and similarly

$\Delta \Gamma + \Gamma B > [\text{arc }] \Delta B,$

$\Delta K + K\Theta > [\text{arc }] \Delta \Theta,$

$ZH + H\Theta > [\text{arc }] Z\Theta,$

and further $\Delta E + EZ > [\text{arc }] \Delta Z,$

therefore the whole perimeter of the polygon is greater than the circumference of the circle.
Δύο μεγεθῶν ἀνίσων δοθέντων δυνατὸν ἔστω εὐρείων δύο εὐθείας ἀνίσους, ὅστε τὴν μεῖζον εὐθείαν πρὸς τὴν ἐλάσσονα λόγον ἔχειν ἐλάσσονα ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλάσσον.

"Εστω δύο μεγέθη ἀνίσα τὰ ΑΒ, Δ, καὶ ἔστω μεῖζον τὸ ΑΒ. λέγω, ὅτι δυνατὸν ἔστι δύο εὐθείας ἀνίσους εὐρείων τὸ εἰρημένον ἐπίταγμα ποιοῦσας.

Κείσθω διὰ τὸ β' τοῦ α' τῶν Εὐκλείδου τῷ Δ ὑσον τὸ ΒΓ, καὶ κείσθω τις εὐθεία γραμμὴ ἢ ΖΗ τὸ δὴ ΓΑ ἐαυτῷ ἐπισυντηθὲς ὑπερέχει τοῦ Δ. πεπολλαπλασιάζων οὖν, καὶ ἔστω τὸ ΑΘ, καὶ ὀδαπλασίον ἔστη τὸ ΑΘ τοῦ ΑΓ, τοσαυταπλάσιος ἔστω ἢ ΖΗ τῆς ΗΕ. ἔστιν ἀρα, ὡς τὸ ΘΑ πρὸς ΑΓ, οὔτως ἢ ΖΗ πρὸς ΗΕ. καὶ ἀνάπαλιν ἔστω, ὡς ἢ ΗΘ πρὸς ΗΖ, οὔτως τὸ ΑΓ πρὸς ΑΘ. καὶ ἐπει μεῖζον ἔστιν τὸ ΑΘ τοῦ Δ, τούτεστι τοῦ ΓΒ, τὸ ἀρα ΓΑ πρὸς τὸ ΑΘ λόγον ἐλάσσονα ἔχει ἦπερ το ΓΑ πρὸς ΓΒ. ἀλλ' ὡς τὸ ΓΑ πρὸς ΑΘ, οὔτως ἢ ΗΘ πρὸς ΗΖ. ἢ ΗΘ ἀρα πρὸς ΗΖ ἐλάσσονα λόγον ἔχει ἦπερ τὸ ΓΑ πρὸς ΓΒ. καὶ συνθέντι ἢ ΕΖ [ἀρα]¹ πρὸς ΖΗ ἐλάσσονα λόγον ἔχει ἦπερ τὸ ΑΒ πρὸς ΒΓ [διὰ λήμμα].² ὢσον δὲ τὸ ΒΓ τῷ Δ. ἢ ΕΖ ἀρα πρὸς ΖΗ ἐλάσσονα λόγον ἔχει ἦπερ τὸ ΑΒ πρὸς τὸ Δ.

50
Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less.

Let $AB, \Delta$ be two unequal magnitudes, and let $AB$ be the greater. I say that it is possible to find two unequal straight lines satisfying the aforesaid requirement.

By the second proposition in the first book of Euclid let $BG$ be placed equal to $\Delta$, and let $ZH$ be any straight line; then $GA$, if added to itself, will exceed $\Delta$. [Post. 5.] Let it be multiplied, therefore, and let the result be $A\Theta$, and as $A\Theta$ is to $AG$, so let $ZH$ be to $HE$; therefore

$$\Theta A : AG = ZH : HE \quad [cf. \text{ Eucl. v. 15}]$$

and conversely, $EH : HZ = AG : A\Theta$. [Eucl. v. 7, coroll.]

And since $A\Theta > \Delta$

$$\Gamma A : A\Theta < \Gamma A : \Gamma B. \quad [\text{Eucl. v. 8}]$$
But $$\Gamma A : A\Theta = EH : HZ;$$

therefore $$EH : HZ < \Gamma A : \Gamma B;$$

componendo,

$$EZ : ZH < AB : BG. \quad ^a$$

Now $BG = \Delta$;

therefore $$EZ : ZH < AB : \Delta.$$

---

^a This and related propositions are proved by Eutocius [Archim. ed. Heiberg iii. 16. 11-18. 22] and by Pappus, Coll. ed. Hultsch 684. 20 ff. It may be simply proved thus. If $a : b < c : d$, it is required to prove that $a + b : b < c + d : d$. Let $e$ be taken so that $a : b = e : d$. Then $e : d < c : d$. Therefore $e < c$, and $e + d : d < c + d : d$. But $e : d = a + b : b$ (ex hypothesi, componendo). Therefore $a + b : b < c + d : d$. 

\footnote{\text{äpa om. Heiberg.}} \footnote{\text{διὰ λῆμμα om. Heiberg.}}
Εὑρημέναι εἰσὶν ἄρα δύο εὐθείαι ἀνίσοι ποιοῦσαι τὸ εἰρημένον ἐπίταγμα [τουτέστων τὴν μεῖζον πρὸς τὴν ἐλάσσονα λόγον ἐχεῖν ἐλάσσονα ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλασσόν].

γ'

Δύο μεγεθῶν ἀνίσων δοθέντων καὶ κύκλου δυνατὸν ἔστιν εἰς τὸν κύκλον πολύγωνον ἐγγράψαι καὶ ἄλλο περιγράψαι, ὡς ἡ τοῦ περιγραφομένου πολυγώνου πλευρὰ πρὸς τὴν τοῦ ἐγγραφομένου πολυγώνου πλευρὰν ἐλάσσονα λόγον ἐχεῖ ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλαττον.

Ἐστὶ διὰ τὰ δοθέντα δύο μεγεθῆ τὰ Α, Β, ὡς δὲ δοθεῖς κύκλος ὁ ὑποκείμενος. λέγω οὖν, ὅτι δυνατὸν ἔστι ποιεῖν τὸ ἐπίταγμα.

Εὑρήσθωσαν γὰρ δύο εὐθείαι αἱ Θ, ΚΛ, ὡς μεῖζων ἔστω η Ἡ, ὥστε τὴν Θ πρὸς τὴν ΚΛ.
ARCHIMEDES

Accordingly there have been discovered two unequal straight lines fulfilling the aforesaid requirement.

Prop. 3

Given two unequal magnitudes and a circle, it is possible to inscribe a [regular] polygon in the circle and to circumscribe another, in such a manner that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that which the greater magnitude has to the less.

Let \( A, B \) be the two given magnitudes, and let the given circle be that set out below. I say then that it is possible to do what is required.

For let there be found two straight lines \( \Theta, KA \), of which \( \Theta \) is the greater, such that \( \Theta \) has to \( KA \) a ratio

\[ \text{vol. ii c 53} \]
GREEK MATHEMATICS

ἐλάσσονα λόγον ἐχειν ἡ τὸ μείζον μέγεθος πρὸς τὸ ἐλασστὸν, καὶ ἡχθω ἀπὸ τοῦ Λ τῇ ΛΚ πρὸς ὀρθάς ἡ ΛΜ, καὶ ἀπὸ τοῦ Κ τῇ Θ ἡ γωνία κατήχθω ἡ ΚΜ [δυνατὸν γὰρ τοῦτο],1 καὶ ἡχθωσαν τοῦ κύκλου δύο διαμετροὶ πρὸς ὀρθάς ἀλλήλως αἱ ΓΕ, ΔΖ. τέμνοντες οὐν τὴν ὑπὸ τῶν ΔΗΓ γωνίαν δίχα καὶ τὴν ἡμίσειαν αὐτῆς δίχα καὶ αἰεὶ τοῦτο ποιοῦσαν λείψομεν τινα γωνίαν ἐλάσσονα ἡ διπλασία τῆς ὑπὸ ΛΚΜ. λειτεθήσον καὶ ἐστω ἡ ὑπὸ ΝΗΓ, καὶ ἐπεζεύξω ἡ ΝΓ. ἡ ἀρα ΝΓ πολυγώνου ἐστὶ πλευρά ἰσοπλεύρου [ἐπείπερ ἡ ὑπὸ ΝΗΓ γωνία μετρεί τὴν ὑπὸ ΔΗΓ ὀρθὴν οὖσαν, καὶ ἡ ΝΓ ἀρα περιφέρεια μετρεί τὴν ΓΔ τέταρτον οὖσαν κύκλου. ὧστε καὶ τὸν κύκλον μετρεῖ. πολυγώνου ἀρα ἐστὶ πλευρά ἰσοπλεύρου. φανερῶν γάρ ἐστὶ τοῦτο].2 καὶ τετμήσωθη ἡ ὑπὸ ΓΗΝ γωνία δίχα τῇ ΗΞ εὐθεία, καὶ ἀπὸ τοῦ Ξ ἐφαπτέσθω τοῦ κύκλου ἡ ΟΞΠ, καὶ ἐκβεβληθωσαν αἱ ΗΝΠ, ΗΓΟ. ὧστε καὶ ἡ ΠΟ πολυγώνου ἐστὶ πλευρά τοῦ περιγραφόμενου περὶ τὸν κύκλον καὶ ἰσοπλεύρου φανερῶν, ὅτι καὶ ὁμοίων τῷ ἐγγραφομένῳ, οὗ πλευρά ἡ ΝΓ].3 ἐπεὶ δὲ ἐλάσσων ἐστὶν ἡ διπλασία ἡ ὑπὸ ΝΗΓ τῆς ὑπὸ ΛΚΜ, διπλασία δὲ τῆς ὑπὸ ΤΗΓ, ἐλάσσων ἀρα ἡ ὑπὸ ΤΗΓ τῆς ὑπὸ ΛΚΜ. καὶ εἰσιν ὁρθαὶ αἱ πρὸς τοὺς Λ, Τ. ἡ ἀρα ΜΚ πρὸς ΛΚ μείζωνα λόγον ἐχει ἦπερ ἡ ΓΗ πρὸς ΗΤ. ἢ ὡς ἡ ΓΗ τῇ ΗΞ. ὧστε ἡ ΗΞ πρὸς ΗΤ ἐλάσσονα λόγον ἐχει, τούτεστιν ἡ ΠΟ πρὸς ΝΓ, ἦπερ ἡ ΜΚ πρὸς ΚΛ. ἐτι δὲ ἡ ΜΚ πρὸς ΚΛ ἐλάσσονα λόγον ἐχει ἦπερ τὸ Α πρὸς τὸ Β. καὶ ἐστιν ἡ μὲν ΠΟ πλευρά.
less than that which the greater magnitude has to the less [Prop. 2], and from $\Lambda$ let $\Delta \overline{M}$ be drawn at right angles to $\Delta K$, and from $K$ let $K\overline{M}$ be drawn equal to $\Theta$, and let there be drawn two diameters of the circle, $\Gamma E$, $\Delta Z$, at right angles one to another. If we bisect the angle $\Delta H \Gamma$ and then bisect the half and so on continually we shall leave a certain angle less than double the angle $\Lambda K \overline{M}$. Let it be left and let it be the angle $N H \Gamma$, and let $N \Gamma$ be joined; then $N \Gamma$ is the side of an equilateral polygon. Let the angle $\Gamma H N$ be bisected by the straight line $H \Xi$, and through $\Xi$ let the tangent $O \Xi \Pi$ be drawn, and let $H \Pi \Pi$, $H \Gamma O$ be produced; then $\Pi O$ is a side of an equilateral polygon circumscribed about the circle. Since the angle $N H \Gamma$ is less than double the angle $\Lambda K \overline{M}$ and is double the angle $T H \Gamma$, therefore the angle $T H \Gamma$ is less than the angle $\Lambda K \overline{M}$. And the angles at $\Lambda, T$ are right; therefore

$$MK : \Lambda K > \Gamma H : HT.$$ 

But

$$\Gamma H = H \Xi.$$ 

Therefore

$$H \Xi : HT < MK : KA,$$

that is,

$$\Pi O : \Pi \Gamma < MK : KA.$$ 

Further,

$$MK : KA < A : B.$$ 

[Therefore

$$\Pi O : \Pi \Gamma < A : B.$$]

\footnote{This is proved by Eutocius and is equivalent to the assertion that if $a < \beta \leq \frac{\pi}{2}$, cosec $\beta >$ cosec $a$.}

\footnote{For $H \Xi : HT = \Pi O : \Pi \Gamma$, since $H \Xi : HT = O \Xi : \Gamma \Gamma = 2 O \Xi : 2 \Gamma \Gamma = \Pi O : \Pi \Gamma$.}

\footnote{For by hypothesis $\Theta : KA < A : B$, and $\Theta = MK$.}
GREEK MATHEMATICS

τοῦ περιγραφομένου πολυγώνου, ἢ δὲ ΓΝ τοῦ ἐγγραφομένου, ὁπερ προέκειτο εὑρεῖν.

ε'

Κύκλου δοθέντος καὶ δύο μεγεθῶν ἀνίσων περιγράφαι περὶ τὸν κύκλου πολύγωνον καὶ ἄλλο ἐγγράφαι, ὅστε τὸ περιγραφὲν πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἐχεῖν ἢ τὸ μεῖζον μέγεθος πρὸς τὸ ἐλασσον.

Ἐκκείσθω κύκλος ὁ Α καὶ δύο μεγέθη ἀνίσα

![Diagram of a circle with lines indicating chords and radii](image)

τὰ Ε, Ζ καὶ μεῖζον τὸ Ε. δεῖ οὖν πολύγωνον ἐγγράφαι εἰς τὸν κύκλου καὶ ἄλλο περιγράφαι, ἵνα γενηται τὸ ἐπιταχθέν.

Λαμβάνω γὰρ δύο εὐθείας ἀνίσους τὰς Γ, Δ, ὅν μεῖζων ἐστὶν ἢ Γ, ὅστε τὴν Γ πρὸς τὴν Δ

56
And ΠΟ is a side of the circumscribed polygon, ΠΝ of the inscribed; which was to be found.

Prop. 5

Given a circle and two unequal magnitudes, to circumscribe a polygon about the circle and to inscribe another, so that the circumscribed polygon has to the inscribed polygon a ratio less than the greater magnitude has to the less.

Let there be set out the circle A and the two unequal magnitudes E, Z, and let E be the greater; it is therefore required to inscribe a polygon in the circle and to circumscribe another, so that what is required may be done.

For I take two unequal straight lines Γ', Δ, of which let Γ be the greater, so that Γ has to Δ a ratio
GREEK MATHEMATICS

\[ \text{έλάσσονα λόγον ἔχειν ἢ τὴν Ε πρὸς τὴν Ζ. καὶ τῶν Γ, Δ μέσης ἀνάλογον ληφθείσης τῆς Η μείζων ἢ καὶ ἡ Γ τῆς Η. περιγεγράφθω δὴ περὶ κύκλων πολύγων καὶ ἄλλο ἐγγεγράφθω, ὡστε τὴν τοῦ περιγραφέντος πολύγων πλευρὰν πρὸς τὴν τοῦ ἐγγραφέντος ἐλάσσονα λόγον ἔχειν ἢ τὴν Γ πρὸς τὴν Η [καθὼς ἐμάθομεν]\textsuperscript{1}. διὰ τούτο δὴ καὶ ὁ διπλάσιος λόγος τοῦ διπλασίου ἐλάσσων ἐστὶ. καὶ τοῦ μὲν τῆς πλευρᾶς πρὸς τὴν πλευρὰν διπλάσιός ἐστι ο τοῦ πολύγων πρὸς τὸν πολύγων [ὁμοία γάρ],\textsuperscript{2} τῆς δὲ Γ πρὸς τὴν Η ὁ τῆς Γ πρὸς τὴν Δ. καὶ τὸ περιγραφὲν ἢ ἀρα πολύγων πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἔχει ἢπερ ἢ Γ πρὸς τὴν Δ. πολλῷ ἢ ἀρα τὸ περιγραφὲν πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἔχει ἢπερ τὸ Ε πρὸς τὸ Ζ.

η'

'Εὰν περὶ κώνον ἰσοσκελῆ πυραμίδις περιγραφῆ, ἢ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς βάσεως ἢς ἐστιν τριγώνω βάσιν μὲν ἔχοντι τὴν ἢτην τῇ περιμέτρῳ τῆς βάσεως, ύπος δὲ τὴν πλευρὰν τοῦ κώνου. . . .

θ'

'Εὰν κώνου τῶν ἰσοσκελοῦς εἰς τὸν κύκλον, ὡς ἐστὶ βάσις τοῦ κώνου, εὐθείᾳ γραμμή ἐμπέσῃ, ἀπὸ δὲ τῶν περάτων αὐτῆς εὐθείαι γραμμαὶ ἀκριβῶς ἐπὶ τὴν κορυφὴν τοῦ κώνου, τὸ περιλῃφθὲν τρίγωνον ὑπὸ τε τῆς ἐμπεσοῦσης καὶ τῶν ἐπὶ-ξευθειῶν ἐπὶ τὴν κορυφὴν ἐλάσσον ἐσται τῆς

58
ARCHIMEDES

less than that which \(E\) has to \(Z\) [Prop. 2]; if a mean proportional \(H\) be taken between \(\Gamma, \Delta\), then \(\Gamma\) will be greater than \(H\) [Eucl. vi. 13]. Let a polygon be circumscribed about the circle and another inscribed, so that the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that which \(\Gamma\) has to \(H\) [Prop. 3]; it follows that the duplicate ratio is less than the duplicate ratio. Now the duplicate ratio of the sides is the ratio of the polygons [Eucl. vi. 20], and the duplicate ratio of \(\Gamma\) to \(H\) is the ratio of \(\Gamma\) to \(\Delta\) [Eucl. v. Def. 9]; therefore the circumscribed polygon has to the inscribed polygon a ratio less than that which \(\Gamma\) has to \(\Delta\); by much more therefore the circumscribed polygon has to the inscribed polygon a ratio less than that which \(E\) has to \(Z\).

Prop. 8

*If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid without the base is equal to a triangle having its base equal to the perimeter of the base [of the pyramid] and its height equal to the side of the cone.*

Prop. 9

*If in an isosceles cone a straight line [chord] fall in the circle which is the base of the cone, and from its extremities straight lines be drawn to the vertex of the cone, the triangle formed by the chord and the lines joining it to

* The "side of the cone" is a generator. The proof is obvious.

---

1 καθώς ἐμάθομεν om. Heiberg.
2 ὅμοια γὰρ om. Heiberg.
ἐπιφανείας τοῦ κώνου τῆς μεταξὺ τῶν ἐπὶ τὴν κορυφὴν ἐπιζευγθεῖσῶν.

Ἔστω κώνου ἴσοπερίμετρος βάσις ὁ ΑΒΓ κύκλος, κορυφή δὲ τὸ Δ, καὶ διήκει τὸς εἰς αὐτὸν εὐθεία ἡ ΑΓ, καὶ ἀπὸ τῆς κορυφῆς ἐπὶ τὰ Α, Γ ἐπεζευ-χθοσαν αἱ ΑΔ, ΔΓ· λέγω, ὅτι τὸ ΑΔΓ τριγώνον ἐλασσὸν ἐστὶν τῆς ἐπιφανείας τῆς κωνικῆς τῆς μεταξὺ τῶν ΑΔΓ.

Τετμῆσθω ἡ ΑΒΓ περιφέρεια δίχα κατὰ τὸ Β, καὶ ἐπεζευχθοσαν αἱ ΑΒ, ΓΒ, ΔΒ· ἔσται δὴ τὰ ΑΒΔ, ΒΓΔ τρίγωνα μεῖζονα τοῦ ΑΔΓ τριγώνου. ὅ δὲ ὑπερέχει τὰ εἰρημένα τρίγωνα τοῦ ΑΔΓ τριγώνου, ἐστω τὸ τὸ Θ. τὸ δὴ Θ ἦτοι τῶν ΑΒ, ΒΓ τμημάτων ἐλασσὸν ἐστὶν ἡ οὐ.
ARCHIMEDES

the vertex will be less than the surface of the cone between the lines drawn to the vertex.

Let the circle $AB\Gamma$ be the base of an isosceles cone, let $\Delta$ be its vertex, let the straight line $A\Gamma$ be drawn in it, and let $A\Delta, \Delta\Gamma$ be drawn from the vertex to $A, \Gamma$; I say that the triangle $A\Delta\Gamma$ is less than the surface of the cone between $A\Delta, \Delta\Gamma$.

Let the arc $AB\Gamma$ be bisected at $B$, and let $AB, \Gamma B, \Delta B$ be joined; then the triangles $AB\Delta, \Gamma B\Delta$ will be greater than the triangle $A\Delta\Gamma$. Let $\Theta$ be the excess by which the aforesaid triangles exceed the triangle $A\Delta\Gamma$. Now $\Theta$ is either less than the sum of the segments $AB, \Gamma B$ or not less.

\[^a\text{ For if } h \text{ be the length of a generator of the isosceles cone, } \triangle AB\Delta = \frac{1}{2}h \cdot AB, \triangle \Gamma B\Delta = \frac{1}{2}h \cdot \Gamma B, \triangle A\Delta\Gamma = \frac{1}{2}h \cdot \Delta \Gamma, \text{ and } AB + \Gamma B > \Delta \Gamma.\]

\[^1\text{ ἔσται . . τρίγωνον: ex Eutocio videtur Archimedes scripsisse: μεῖζονα ἄρα ἔστι τὰ } AB\Delta, B\Delta\Gamma \tauρίγωνα τοῦ } A\Delta\Gamma \tauρίγωνον.\]
"Εστώ μὴ ἔλασσον πρότερον. ἔπει ὁν δύο ἐστὶν ἐπιφάνεια τὰ τῶν ΑΔΒ μετὰ τοῦ ΑΕΒ τμῆματος καὶ ἡ τοῦ ΑΔΒ τριγώνου τὸ αὐτὸ πέρας ἔχουσαι τὴν περίμετρον τοῦ τριγώνου τοῦ ΑΔΒ, μείζων ἐστι τῆς περιλαμβάνουσα τῆς περιλαμβανομένης· μείζων ἀρα ἐστὶν η ἱκετή ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΒ μετὰ τοῦ ΑΕΒ τμῆματος τοῦ ΑΒΔ τριγώνου. ὅμως δὲ καὶ ἡ μεταξὺ τῶν ΒΔΓ μετὰ τοῦ ΓΖΒ τμῆματος μείζων ἐστὶν τοῦ ΒΔΓ τριγώνου· ὅλη ἀρα ἡ κωνικὴ ἐπιφάνεια μετὰ τοῦ Θ χωρίου μείζων ἐστὶ τῶν εἰρημένων τριγώνων. τὰ δὲ εἰρημένα τρίγωνα ἵσα ἐστὶν τὸ τε ΑΔΓ τριγώνω καὶ τὸ Θ χωρίω· κοινὸν ἀφηρήσθω τὸ Θ χωρίον· λοιπὴ ἀρα ἡ κωνικὴ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μείζων ἐστὶν τοῦ ΑΔΓ τριγώνου.

"Εστώ δὴ τὸ τὰ Θ ἔλασσον τῶν ΑΒ, ΒΓ τμημάτων. τέμνοντες δὴ τὰς ΑΒ, ΒΓ περιφερείας δίχα καὶ τὰς ἥμισειας αὐτῶν δίχα λείψομεν τμῆματα ἔλασσον δύνα ὁ τοῦ Θ χωρίου. λειψάθω τὰ ἐπὶ τῶν ΑΕ, ΕΒ, ΒΖ, ΖΓ εὐθείων, καὶ ἐπεζεύγθωσαν αἱ ΔΕ, ΔΖ. πάλιν τῶν κατὰ τὰ αὐτὰ ἡ μὲν ἐπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν ΑΕΔ μετὰ τοῦ ἐπὶ τῆς ΑΕ τμῆματος μείζων ἐστὶν τοῦ ΑΕΔ τριγώνου, ἡ δὲ μεταξὺ τῶν ΕΒΔ μετὰ τοῦ ἐπὶ τῆς ΕΒ τμῆματος μείζων ἐστὶν τοῦ ΕΒΔ τριγώνου· ἡ ἀρα ἐπιφάνεια ἡ μεταξὺ τῶν ΑΕΔ μετὰ τῶν ἐπὶ τῶν ΑΕ, ΕΒ τμημάτων μείζων ἐστὶν τῶν ΑΕΔ, ΕΒΔ τριγώνων. ἔπει δὲ τὰ ΑΕΔ, ΔΕΒ τρίγωνα μείζονα ἐστὶν τοῦ ΑΒΔ τριγώνου, καθὼς δέδεικται, πολλῷ ἀρα ἡ ἐπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν ΑΔΒ μετὰ τῶν ἐπὶ τῶν ΑΕ, 62
ARCHIMEDES

Firstly, let it be not less. Then since there are two surfaces, the surface of the cone between $A\Delta$, $\Delta B$ together with the segment $AEB$ and the triangle $A\Delta B$, having the same extremity, that is, the perimeter of the triangle $A\Delta B$, the surface which includes the other is greater than the included surface [Post. 3]; therefore the surface of the cone between the straight lines $A\Delta$, $\Delta B$ together with the segment $AEB$ is greater than the triangle $A\overline{B}\Delta$. Similarly the [surface of the cone] between $B\Delta$, $\Delta \Gamma$ together with the segment $\Gamma ZB$ is greater than the triangle $B\Delta \Gamma$ ; therefore the whole surface of the cone together with the area $\Theta$ is greater than the aforesaid triangles. Now the aforesaid triangles are equal to the triangle $A\Delta \Gamma$ and the area $\Theta$. Let the common area $\Theta$ be taken away; therefore the remainder, the surface of the cone between $A\Delta$, $\Delta \Gamma$ is greater than the triangle $A\Delta \Gamma$.

Now let $\Theta$ be less than the segments $AB$, $B\Gamma$. Bisecting the arcs $AB$, $B\Gamma$ and then bisecting their halves, we shall leave segments less than the area $\Theta$ [Eucl. xii. 2]. Let the segments so left be those on the straight lines $AE$, $EB$, $BZ$, $Z\Gamma$, and let $\Delta E$, $\Delta Z$ be joined. Then once more by the same reasoning the surface of the cone between $A\Delta$, $\Delta E$ together with the segment $AE$ is greater than the triangle $A\Delta E$, while that between $E\Delta$, $\Delta B$ together with the segment $EB$ is greater than the triangle $E\Delta B$; therefore the surface between $A\Delta$, $\Delta B$ together with the segments $AE$, $EB$ is greater than the triangles $A\Delta E$, $EB\Delta$. Now since the triangles $AE\Delta$, $\Delta EB$ are greater than the triangle $AB\Delta$, as was proved, by much more therefore the surface of the cone between $A\Delta$, $\Delta B$ together with the segments $AE$, $EB$ is
ΕΒ τμημάτων μειζον ἐστὶ τοῦ ΛΔΒ τριγώνου. διὰ τὰ αὐτὰ δὴ καὶ ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΒΔΓ μετὰ τῶν ἐπὶ τῶν ΒΖ, ΖΓ τμημάτων μειζον ἐστὶν τοῦ ΒΔΓ τριγώνου· ὅλη ἄρα ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μετὰ τῶν εἰρημένων τμημάτων μειζον ἐστὶ τῶν ΑΒΔ, ΔΒΓ τριγώνων. ταύτα δὲ ἐστιν ἵσα τῷ ΑΔΓ τριγώνῳ καὶ τῷ Θ χωρίῳ· ὅν τὰ εἰρημένα τμήματα ἐλάσσονα τοῦ Θ χωρίου· λοιπὴ ἄρα ἡ ἐπιφάνεια ἡ μεταξὺ τῶν ΑΔΓ μειζον ἐστὶν τοῦ ΑΔΓ τριγώνου.

―

Ἐὰν ἐπιμαθοῦσαί ἀκθῶσιν τοῦ κύκλου, ὅς ἐστι βάσις τοῦ κώνου, ἐν τῷ αὐτῷ ἐπιπέδῳ οὔσαι τῷ κύκλῳ καὶ συμπίπτουσαι ἀλλήλαις, ἀπὸ δὲ τῶν ἀφῶν καὶ τῆς συμπτώσεως ἐπὶ τὴν κορυφὴν τοῦ κώνου εὐθειαὶ ἀκθῶσιν, τὰ περιεχόμενα τρίγωνα ὑπὸ τῶν ἐπιμαθοῦσῶν καὶ τῶν ἐπὶ τὴν κορυφὴν τοῦ κώνου ἐπιζευγθεισῶν εὐθειῶν μειζονά ἐστιν τῆς τοῦ κώνου ἐπιφανείας τῆς ἀπολαμβανομένης ὑπ’ αὐτῶν. . . .

β'

. . . Τούτων δὲ δεδειγμένων φανερῶν [ἐπὶ μὲν τῶν προειρημένων], 1 ὃτι, ἐὰν εἰς κώνον ἵσοσκελῇ πυραμίδις ἐγγραφῇ, ἡ ἐπιφάνεια τῆς πυραμίδος χωρίς τῆς βάσεως ἐλάσσον ἐστὶ τῆς κωνικῆς ἐπιφανείας [ἐκαστὸν γὰρ τῶν περιεχόμενων τῆς πυραμίδα τριγώνων ἐλάσσον ἐστιν τῆς κωνικῆς ἐπιφανείας τῆς μεταξὺ τῶν τοῦ τριγώνου πλευρῶν· ὥστε καὶ ὅλη ἡ ἐπιφάνεια τῆς πυραμίδος χωρὶς τῆς 64
ARCHIMEDES

greater than the triangle $\Delta \Delta B$. By the same reasoning the surface between $B\Delta, \Delta \Gamma$ together with the segments $BZ, Z\Gamma$ is greater than the triangle $B\Delta \Gamma$; therefore the whole surface between $A\Delta, \Delta \Gamma$ together with the aforesaid segments is greater than the triangles $AB\Delta, AB\Gamma$. Now these are equal to the triangle $A\Delta \Gamma$ and the area $\Theta$; and the aforesaid segments are less than the area $\Theta$; therefore the remainder, the surface between $A\Delta, \Delta \Gamma$ is greater than the triangle $A\Delta \Gamma$.

Prop. 10

If tangents be drawn to the circle which is the base of an [isosceles] cone, being in the same plane as the circle and meeting one another, and from the points of contact and the point of meeting straight lines be drawn to the vertex of the cone, the triangles formed by the tangents and the lines drawn to the vertex of the cone are together greater than the portion of the surface of the cone included by them. . . .

Prop. 12

. . . From what has been proved it is clear that, if a pyramid is inscribed in an isosceles cone, the surface of the pyramid without the base is less than the surface of the cone [Prop. 9], and that, if a pyramid

* The proof is on lines similar to the preceding proposition.

---

1 ἐπὶ . . . προειρημένων om. Heiberg.
GREEK MATHEMATICS

βάσεως ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κώνου χωρίς τῆς βάσεως], καὶ ὅτι, ἐὰν περὶ κώνου ἵσοςκελῆ πυραμίδα περιγραφῇ, ἡ ἐπιφάνεια τῆς πυραμίδος χωρίς τῆς βάσεως μεῖζων ἐστὶν τῆς ἐπιφανείας τοῦ κώνου χωρίς τῆς βάσεως [κατὰ τὸ συνεχὲς ἐκείνῳ].

Φανερὸν δὲ ἐκ τῶν ἀποδεδειγμένων, ὅτι τε, ἐὰν εἰς κυλίνδρον ὀρθὸν πρόσμα ἐγγραφῇ, ἡ ἐπιφάνεια τοῦ πρύμνατος ἢ ἐκ τῶν παραλληλογράμμων συγκεμένη ἐλάσσων ἐστὶ τῆς ἐπιφανείας τοῦ κυλίνδρου χωρίς τῆς βάσεως [ἐλασσον γὰρ ἐκαστὸν παραλληλόγραμμον τοῦ πρύμνατος ἐστὶ τῆς καθ’ αὐτὸ τοῦ κυλίνδρου ἐπιφανείας], καὶ ὅτι, ἐὰν περὶ κυλίνδρον ὀρθὸν πρόσμα περιγραφῇ, ἡ ἐπιφάνεια τοῦ πρύμνατος ἢ ἐκ τῶν παραλληλογράμμων συγκεμένη μεῖζων ἐστὶ τῆς ἐπιφανείας τοῦ κυλίνδρου χωρίς τῆς βάσεως.

ι"π

Πάντως κυλίνδρου ὀρθοῦ ἡ ἐπιφάνεια χωρίς τῆς βάσεως ἵση ἐστὶ κύκλῳ, οὗ ἡ ἐκ τοῦ κέντρου μέσον λόγον ἔχει τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς διαμέτρου τῆς βάσεως τοῦ κυλίνδρου.

"Εστὼ κυλίνδρου τυνὸς ὀρθοὶ βάσεις οὐ Α κύκλος, καὶ ἐστὼ τῇ μὲν διαμέτρῳ τοῦ Α κύκλου ἴση τῇ ΓΔ, τῇ δὲ πλευρᾷ τοῦ κυλίνδρου ἡ EZ, ἔχετω δὲ μέσον λόγον τῶν ΔΓ, EZ ἡ H, καὶ κεῖσθω κύκλος, οὗ ἡ ἐκ τοῦ κέντρου ἴση ἐστὶ τῇ H, ὁ Β· δεικτέων, ὅτι ὁ B κύκλος ἱσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κυλίνδρου χωρίς τῆς βάσεως.

Εἰ γὰρ μὴ ἐστὶν ἱσος, ἡτοι μεῖζων ἐστὶ ἡ
is circumscribed about an isosceles cone, the surface of the pyramid without the base is greater than the surface of the cone without the base [Prop. 10].

From what has been demonstrated it is also clear that, if a right prism be inscribed in a cylinder, the surface of the prism composed of the parallelograms is less than the surface of the cylinder excluding the bases [Prop. 11], and if a right prism be circumscribed about a cylinder, the surface of the prism composed of the parallelograms is greater than the surface of the cylinder excluding the bases.

Prop. 13

The surface of any right cylinder excluding the bases is equal to a circle whose radius is a mean proportional between the side of the cylinder and the diameter of the base of the cylinder.

Let the circle A be the base of a right cylinder, let $\Gamma \Delta$ be equal to the diameter of the circle A, let $EZ$ be equal to the side of the cylinder, let $H$ be a mean proportional between $\Delta \Gamma$, $EZ$, and let there be set out a circle, B, whose radius is equal to $H$; it is required to prove that the circle B is equal to the surface of the cylinder excluding the bases.

For if it is not equal, it is either greater or less.

Here, and in other places in this and the next proposition, Archimedes must have written $\chi\nu\rho\iota\varsigma\;\tau\omega\nu\;\beta\alpha\sigma\varepsilon\omega\varsigma$, not $\chi\nu\rho\iota\varsigma\;\tau\iota\varsigma\;\beta\alpha\sigma\varepsilon\omega\varsigma$.

\begin{footnotesize}
\begin{itemize}
\item 1 $\epsilon\kappa\alpha\sigma\tau\omicron\ldots\;\beta\alpha\sigma\varepsilon\omega\varsigma$. Heiberg suspects that this demonstration is interpolated. Why give a proof of what is $\phi\alpha\nu\epsilon\rho\omicron\nu$?
\item 2 $k\alpha\tau\alpha\ldots\;\epsilon\kappa\epsilon\iota\nu\omicron$ om. Heiberg.
\item 3 $\epsilon\lambda\alpha\sigma\sigma\omicron\ldots\;\epsilon\pi\iota\phi\alpha\nu\epsilon\lambda\varsigma$. Heiberg suspects that this proof is interpolated.
\end{itemize}
\end{footnotesize}
GREEK MATHEMATICS

ελάσσων. ἔστω πρότερον, εἰ δυνατόν, ελάσσων.

dύο δὴ μεγεθῶν οὐντων ἀνίσων τῆς τε ἐπιφάνειας
tοῦ κυλίνδρου καὶ τοῦ B κύκλου δυνατὸν ἔστω εἰς
tὸν B κύκλον ἱσόπλευρον πολύγωνον ἐγγράψαι
καὶ ἄλλο περιγράψαι, ὡστε τὸ περιγραφὲν πρὸς
τὸ ἐγγραφὲν ἐλάσσων λόγον ἔχειν τοῦ, ὅν ἔχει
ἡ ἐπιφάνεια τοῦ κυλίνδρου πρὸς τὸν B κύκλον.

νοεῖσθω δὴ περιγεγραμμένον καὶ ἐγγεγραμμένον,
καὶ περὶ τὸν A κύκλον περιγεγράφθω εὐθύγραμμον
ὀμοιον τῷ περὶ τὸν B περιγεγραμμένω, καὶ
ἀναγεγράφθω ἀπὸ τοῦ εὐθυγράμμου πρίσμα: ἔσται
δὴ περὶ τὸν κύλινδρον περιγεγραμμένον. ἔστω
δὲ καὶ τῇ περιμέτρῳ τοῦ εὐθυγράμμου τοῦ περὶ

- One ms. has the marginal note, "equalis altitudinis
cylindro," on which Heiberg comments: "nee hoc
omiserat Archimedes." Heiberg notes several places in
which the text is clearly not that written by Archimedes.
68
Let it first be, if possible, less. Now there are two unequal magnitudes, the surface of the cylinder and the circle B, and it is possible to inscribe in the circle B an equilateral polygon, and to circumscribe another, so that the circumscribed has to the inscribed a ratio less than that which the surface of the cylinder has to the circle B [Prop. 5]. Let the circumscribed and inscribed polygons be imagined, and about the circle A let there be circumscribed a rectilineal figure similar to that circumscribed about B, and on the rectilineal figure let a prism be erected; it will be circumscribed about the cylinder. Let $K\Delta$ be equal
τὸν Α κύκλον ἴσῃ ἡ KD καὶ τῇ KD ἴσῃ ἡ ΛΖ, τῆς δὲ ΓΔ ἡμίσεια ἐστὶν ἡ ΓΤ. ἔσται δὴ τὸ KDΤ τρίγωνον ἴσον τῷ περιγεγραμμένῳ εὐθύγραμμῷ περὶ τὸν Α κύκλον [ἐπειδὴ βάσιν μὲν ἔχει τῇ περιμέτρῳ ἴσην, ύψος δὲ ἴσον τῇ ἐκ τοῦ κέντρου τοῦ Α κύκλου],¹ τὸ δὲ ΕΔ παραλληλόγραμμον τῇ ἐπιφανείᾳ τοῦ πρίσματος τοῦ περὶ τὸν κύλινδρον περιγεγραμμένου [ἐπειδὴ περιέχεται ὑπὸ τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς ἴσης τῇ περιμέτρῳ τῆς βάσεως τοῦ πρίσματος].² κεῖσθω δὴ τῇ EZ ἴσῃ ἡ EP. ἴσον ἀρα ἐστὶν τὸ ZΠΔ τρίγωνον τῷ ΕΔ παραλληλόγραμμῳ, ὡστε καὶ τῇ ἐπιφανείᾳ τοῦ πρίσματος. καὶ ἐπεὶ ὁμοία ἐστὶν τὰ εὐθύγραμμα τὰ περὶ τοὺς Α, Β κύκλους περιγεγραμμένα, τὸν αὐτὸν ἔχει λόγον [τὰ εὐθύγραμμα],³ ὀντέρ αἴ ἐκ τῶν κέντρων δυνάμει· ἔχει ἀρα τὸ ΚΤΔ τρίγωνον πρὸς τὸ περὶ τὸν Β κύκλον εὐθύγραμμον λόγον, ὅτι η ΤΔ πρὸς Η δυνάμει [αἴ γαρ ΤΔ, Η ἴσαι εἰσὶν ταῖς ἐκ τῶν κέντρων]. ἄλλ' ὃν ἔχει λόγον ἡ ΤΔ πρὸς Η δυνάμει, τούτον ἔχει τὸν λόγον ἡ ΤΔ πρὸς ΡΖ μήκει [ἡ γαρ Η τῶν ΤΔ, ΡΖ μέση ἐστὶ ἀνάλογον διὰ τὸ καὶ τῶν ΓΔ, EZ: πώς δὲ τούτο; ἐπεὶ γαρ ἴσῃ ἐστὶν ἡ μὲν ΔΤ τῇ ΤΓ, ἡ δὲ PE τῇ EZ, διπλασίᾳ ἀρα ἐστὶν ἡ ΓΔ τῆς ΤΔ, καὶ ἡ ΡΖ τῆς PE· ἐστὶν ἀρα, ώς ἡ ΔΓ πρὸς ΔΤ, οὕτως ἡ ΡΖ πρὸς ΖΕ. τὸ ἀρα ὑπὸ τῶν ΓΔ, EZ ἴσον ἐστὶν τῷ ὑπὸ τῶν ΤΔ, ΡΖ. τῷ δὲ ὑπὸ τῶν ΓΔ, EZ ἴσον ἐστὶν τὸ ἀπὸ Η· καὶ τῷ ὑπὸ τῶν ΤΔ, ΡΖ ἀρα ἴσον ἐστὶ τὸ ἀπὸ τῆς Η. ἐστὶν ἀρα,

¹ ἐπειδὴ . . . κύκλου om. Heiberg.
² ἐπειδὴ . . . πρίσματος om. Heiberg.
³ τὰ εὐθύγραμμα om. Torellius.
ARCHIMEDES

to the perimeter of the rectilineal figure about the circle A, let $\Delta Z$ be equal to $K\Delta$, and let $\Gamma T$ be half of $\Gamma \Delta$; then the triangle $K\Delta T$ will be equal to the rectilineal figure circumscribed about the circle A, while the parallelogram $EA$ will be equal to the surface of the prism circumscribed about the cylinder.

Let $EP$ be set out equal to $EZ$; then the triangle $ZP\Delta$ is equal to the parallelogram $EA$ [Eucl. i. 41], and so to the surface of the prism. And since the rectilineal figures circumscribed about the circles $A$, $B$ are similar, they will stand in the same ratio as the squares on the radii; therefore the triangle $KT\Delta$ will have to the rectilineal figure circumscribed about the circle $B$ the ratio $T\Delta^2 : H^2$.

But $T\Delta^2 : H^2 = T\Delta : PZ$.

---

* Because the base $K\Delta$ is equal to the perimeter of the polygon, and the altitude $\Delta T$ is equal to the radius of the circle $A$, i.e., to the perpendiculars drawn from the centre of $A$ to the sides of the polygon.

* Because the base $\Delta Z$ is made equal to $\Delta K$ and so is equal to the perimeter of the polygon forming the base of the prism, while the altitude $EZ$ is equal to the side of the cylinder and therefore to the height of the prism.

* Eutocius supplies a proof based on Eucl. xii. 1, which proves a similar theorem for inscribed figures.

* For, by hypothesis, $H^2 = \Delta \Gamma \cdot EZ$
  
  
  
  
  $= 2T\Delta \cdot \frac{1}{2}PZ$
  
  
  
  $= T\Delta \cdot PZ$

Heiberg would delete the demonstration in the text on the ground of excessive verbosity, as Nizze had already perceived to be necessary.
GREEK MATHEMATICS

ως η ΤΔ προς Η, ούτως η Η προς ΡΖ· ἐστὶν ἄρα, ως η ΤΔ προς ΡΖ, το ἀπὸ τῆς ΤΔ προς τὸ ἀπὸ τῆς Η· ἐὰν γὰρ τρεῖς εὐθεῖαι ἀνάλογον ᾤσιν, ἐστὶν, ως η πρώτη πρὸς τὴν τρίτην, τὸ ἀπὸ τῆς δευτέρας εἴδος πρὸς τὸ ἀπὸ τῆς ὁμοιοῦν καὶ ὁμοίως ἀναγεγραμμένον]. ὃν δὲ λόγον ἔχει η ΤΔ πρὸς ΡΖ μήκει, τούτων ἔχει τὸ ΚΤΔ τρίγωνον πρὸς τὸ ΡΔΖ [ἐπειδήπερ ἵσαι εἰσίν αἱ ΚΔ, ΛΖ]. τὸν αὐτὸν ἄρα λόγον ἔχει τὸ ΚΤΔ τρίγωνον πρὸς τὸ εὐθύγραμμον τὸ περὶ τὸν Β κύκλον περιγεγραμμένον, ὅπερ τὸ ΤΚΔ τρίγωνον πρὸς τὸ ΡΔΔ τρίγωνον. ἴσον ἄρα ἐστὶν τὸ ΖΔΡ τρίγωνον τῷ περὶ τὸν Β κύκλον περιγεγραμμένῳ εὐθύγραμμῷ· ὥστε καὶ ἡ ἐπιφάνεια τοῦ πρώσατος τοῦ περὶ τὸν Α κυλινδροῦν περιγεγραμμένου τῷ εὐθύγραμμῳ τῷ περὶ τὸν Β κύκλον ἴσῃ ἐστὶν, καὶ ἐπεὶ ἐλάσσονα λόγον ἔχει τὸ εὐθύγραμμον τὸ περὶ τὸν Β κύκλον πρὸς τὸ ἐγγεγραμμένον ἐν τῷ κύκλῳ του, ὃν ἔχει ἡ ἐπιφάνεια τοῦ Α κυλινδροῦ πρὸς τὸν Β κύκλον, ἐλάσσονα λόγον ἔχει καὶ ἡ ἐπιφάνεια τοῦ πρώσατος τοῦ περὶ τὸν κυλινδρον περιγεγραμμένον πρὸς τὸ εὐθύγραμμον τὸ ἐν τῷ κύκλῳ τῷ Β ἐγγεγραμμένον ἑπερ ἡ ἐπιφάνεια τοῦ κυλινδροῦ πρὸς τὸν Β κύκλον καὶ ἐναλλάξ· ὅπερ ἀδύνατον [ἡ μὲν γὰρ ἐπιφάνεια τοῦ πρώσματος τοῦ περιγεγραμμένου περὶ τὸν κυλινδρον μεῖζων ὁδὸν δέδεικται τῆς ἐπιφάνειας τοῦ κυλινδροῦ, τὸ δὲ ἐγγεγραμμένον εὐθύγραμμον ἐν τῷ Β κύκλῳ ἐλάσσον ἐστὶν τοῦ Β κύκλου]. οὐκ ἄρα ἐστὶν ὁ Β κύκλος ἐλάσσων τῆς ἐπιφανείας τοῦ κυλινδροῦ.

1 ἡ γὰρ . . . ὁμοίως ἀναγεγραμμένον om. Heiberg.
2 ἐπειδήπερ . . . ΚΔ, ΛΖ om. Heiberg.

72
ARCHIMEDES

And $T\Delta : PZ = \triangle K\Delta : \triangle P\Delta Z$.\(^a\)

Therefore the ratio which the triangle $K\Delta$ has to the rectilineal figure circumscribed about the circle $B$ is the same as the ratio of the triangle $TK\Delta$ to the triangle $PZ\Delta$. Therefore the triangle $TK\Delta$ is equal to the rectilineal figure circumscribed about the circle $B$ [Eucl. v. 9]; and so the surface of the prism circumscribed about the cylinder $A$ is equal to the rectilineal figure about $B$. And since the rectilineal figure about the circle $B$ has to the inscribed figure in the circle a ratio less than that which the surface of the cylinder $A$ has to the circle $B$ [ex hypothesi], the surface of the prism circumscribed about the cylinder will have to the rectilineal figure inscribed in the circle $B$ a ratio less than that which the surface of the cylinder has to the circle $B$; and, permutando, [the prism will have to the cylinder a ratio less than that which the rectilineal figure inscribed in the circle $B$ has to the circle $B$] \(^b\); which is absurd.\(^c\) Therefore the circle $B$ is not less than the surface of the cylinder.

\(^a\) By Eucl. vi. 1, since $\Delta Z = K\Delta$.

\(^b\) From Eutocius's comment it appears that Archimedes wrote, in place of καὶ ἕναλλάς ὑπερ ἀδύνατον in our text: ἕναλλάς ἀρα ἑλάσσονα λόγον ἔχει τὸ πρίσμα πρὸς τὸν κύλινδρον ὑπὸ τὸ ἐγγεγραμμένον εἰς τὸν Β κύκλον πολύγωνον πρὸς τὸν Β κύκλον ὑπερ ἀτοπον. This is what I translate.

\(^c\) For the surface of the prism is greater than the surface of the cylinder [Prop. 12], but the inscribed figure is less than the circle $B$; the explanation in our text to this effect is shown to be an interpolation by the fact that Eutocius supplies a proof in his own words.

\(^{73}\) η μέν . . . τοῦ B κύκλου om. Heiberg ex Eutocio.
GREEK MATHEMATICS

"Εστω δή, εἰ δυνατόν, μείζων. πάλιν δὴ νοείσθω εἰς τὸν Β κύκλον εὐθύγραμμον ἐγγεγραμμένον καὶ ἄλλο περιγεγραμμένον, ὥστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἐλάσσονα λόγον ἔχειν ἣ τὸν Β κύκλον πρὸς τὴν ἐπιφάνειαν τοῦ κυλίνδρου, καὶ ἐγγεγράφθω εἰς τὸν Α κύκλον πολύγωνον ὁμοιοῦ τῷ εἰς τὸν Β κύκλον ἐγγεγραμμένῳ, καὶ πρίσμα ἀναγεγράφθω ἀπὸ τοῦ ἐν τῷ κύκλῳ ἐγγεγραμμένῳ πολυγώνου· καὶ πάλιν ἢ ΚΔ ἢς ἐστω τῇ περιμέτρῳ τοῦ εὐθύγραμμου τοῦ ἐν τῷ Α κύκλῳ ἐγγεγραμμένου, καὶ ἢ ΖΔ ἢς αὐτῇ ἐστω· ἐσται δὴ τὸ μὲν ΚΔ τρίγωνον μεῖζον τοῦ εὐθύγραμμου τοῦ ἐν τῷ Α κύκλῳ ἐγγεγραμμένου [διότι βάσιν μὲν ἔχει τὴν περιμέτρον αὐτοῦ, ὡς ὅ δὲ μεῖζον τῆς ἀπὸ τοῦ κέντρου ἐπὶ μίαν πλευρὰν τοῦ πολυγώνου ἀγομένης καθέτου], τὸ δὲ ΕΔ παραλληλόγραμμον ἢσον τῇ ἐπιφάνειᾳ τοῦ πρίσματος τῇ ἐκ τῶν παραλληλόγραμμων συγκεκμένην [διότι περιέχεται ύπὸ τῆς πλευρᾶς τοῦ κυλίνδρου καὶ τῆς ἢς τῆς περιμέτρῳ τοῦ εὐθύγραμμου, ὃ ἐστὶ βάσις τοῦ πρίσματος]. Ὡστε καὶ τὸ ΡΖ τρίγωνον ἢσον ἐστὶ τῇ ἐπιφάνειᾳ τοῦ πρίσματος. καὶ ἐπεὶ ὢμοιὰ ἐστὶ τὰ εὐθύγραμμα τὰ ἐν τοῖς Α, Β κύκλοις ἐγγεγραμμένα, τὸν αὐτὸν ἔχει λόγον πρὸς ἄλληλα, ὅν αἱ ἐκ τῶν κέντρων αὐτῶν δυνάμει. ἔχει δὲ καὶ τὰ ΚΔ, ΖΡΑ τρίγωνα πρὸς ἄλληλα λόγον, ὅν αἱ ἐκ τῶν κέντρων τῶν κύκλων δυνάμει· τὸν αὐτὸν ἀρα λόγον ἔχει

1 διότι . . . καθέτου om. Heiberg.

For the base ΚΔ is equal to the perimeter of the polygon and the altitude ΔΤ, which is equal to the radius of the
ARCHIMEDES

Now let it be, if possible, greater. Again, let there be imagined a rectilineal figure inscribed in the circle B, and another circumscribed, so that the circumscribed figure has to the inscribed a ratio less than that which the circle B has to the surface of the cylinder [Prop. 5], and let there be inscribed in the circle A a polygon similar to the figure inscribed in the circle B, and let a prism be erected on the polygon inscribed in the circle [A]; and again let KΔ be equal to the perimeter of the rectilineal figure inscribed in the circle A, and let ZΔ be equal to it. Then the triangle KΤΔ will be greater than the rectilineal figure inscribed in the circle A, and the parallelogram ΕΑ will be equal to the surface of the prism composed of the parallelograms; and so the triangle PΔZ is equal to the surface of the prism. And since the rectilineal figures inscribed in the circles A, B are similar, they have the same ratio one to the other as the squares of their radii [Eucl. xii. 1]. But the triangles KΤΔ, ZPΔ have one to the other the same ratio as the squares of the radii; therefore the rectilineal figure inscribed in circle A, is greater than the perpendiculars drawn from the centre of the circle to the sides of the polygon; but Heiberg regards the explanation to this effect in the text as an interpolation.

b Because the base ZΔ is made equal to KΔ, and so is equal to the perimeter of the polygon forming the base of the prism, while the altitude EZ is equal to the side of the cylinder and therefore to the height of the prism.

c For triangle KΤΔ: triangle ZPΔ = TΔ : ZP = TΔ² : H²

[cf. p. 71 n. d.]

But TΔ is equal to the radius of the circle A, and H to the radius of the circle B.
Παντὸς κόνων ἴσοσκελεᾶς χωρὶς τῆς βάσεως ἡ ἐπιφάνεια ὡσθ ἐστὶ κύκλω, οὗ ἡ ἐκ τοῦ κέντρου μέσον λόγον ἔχει τῆς πλευρᾶς τοῦ κόνου καὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ὅσ ἐστὶν βάσις τοῦ κώνου.

'Εστὼ κόνως ἴσοσκελῆς, οὗ βάσις ὁ Α κύκλος, ἡ δὲ ἐκ τοῦ κέντρου ἐστὼ η Γ, τῇ δὲ πλευρᾷ τοῦ

1 ἐπεὶ . . . πρίσματος om. Heiberg.

---

1 For since the figure circumscribed about the circle B has to the inscribed figure a ratio less than that which the circle B has to the surface of the cylinder [ex hypothesi], and the circle B is less than the circumscribed figure, therefore the 76
ARCHIMEDES

the circle A has to the rectilineal figure inscribed in the circle B the same ratio as the triangle KTΔ has to the triangle ΔZP. But the rectilineal figure inscribed in the circle A is less than the triangle KTΔ; therefore the rectilineal figure inscribed in the circle B is less than the triangle ZPΔ; and so it is less than the surface of the prism inscribed in the cylinder; which is impossible. Therefore the circle B is not greater than the surface of the cylinder. But it was proved not to be less. Therefore it is equal.

Prop. 14

The surface of any cone without the base is equal to a circle, whose radius is a mean proportional between the side of the cone and the radius of the circle which is the base of the cone.

Let there be an isosceles cone, whose base is the circle A, and let its radius be Γ, and let Δ be equal inscribed figure is greater than the surface of the cylinder, and a fortiori is greater than the surface of the prism [Prop. 12]. An explanation on these lines is found in our text, but as the corresponding proof in the first half of the proposition was unknown to Eutocius, this also must be presumed an interpolation.
GREEK MATHEMATICS

κόνων ἔστω ἵση ἡ Δ, τῶν δὲ Γ, Δ μέση ἀνάλογον ἡ Ε, ὁ δὲ Β κύκλος ἐχέτω τὴν ἐκ τοῦ κέντρου τῆς Ε ἱσην· λέγω, ὅτι ὁ Β κύκλος ἔστιν ἴσος τῆς ἐπιφάνειας τοῦ κόνων χωρίς τῆς βάσεως.

Εἰ γὰρ μὴ ἔστω ἴσος, ἦτοι μείζων ἔστιν ἡ ἑλάσσων. ἔστω πρότερον ἑλάσσων. ἔστι δὴ δύο μεγέθη ἄνωσα ἡ τε ἐπιφάνεια τοῦ κόνων καὶ ὁ Β κύκλος, καὶ μείζων ἡ ἐπιφάνεια τοῦ κόνων· δυνατὸν ἄρα εἰς τὸν Β κύκλον πολύγωνον ἰσόπλευρον ἐγγράψαι καὶ άλλο περιγράψαι ὀμοιον τῷ ἐγγεγραμμένῳ, ὥστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένου ἑλάσσονα λόγον ἔχειν τοῦ, ὅπερ ἔχει ἡ ἐπιφάνεια τοῦ κόνων πρὸς τὸν Β κύκλον. νοεῦσθω δὴ καὶ περὶ τὸν Α κύκλον πολύγωνον περιγεγραμμένον ὀμοιον τῷ περὶ τὸν Β κύκλον περιγεγραμμένῳ, καὶ ἀπὸ τοῦ περὶ τοῦ Α κύκλου περιγεγραμμένου πολυγώνου πυραμίδι ἀνεστάτω ἀναγεγραμμένη τὴν αὐτὴν κορυφὴν ἑξουσα τῷ κώνῳ. ἐπεὶ οὖν ὀμοιά ἔστιν τά πολύγωνα τὰ περὶ

78
to the side of the cone, and let $E$ be a mean proportional between $T$, $\Delta$, and let the circle $B$ have its radius equal to $E$; I say that the circle $B$ is equal to the surface of the cone without the base.

For if it is not equal, it is either greater or less. First let it be less. Then there are two unequal magnitudes, the surface of the cone and the circle $B$, and the surface of the cone is the greater; it is therefore possible to inscribe an equilateral polygon in the circle $B$ and to circumscribe another similar to the inscribed polygon, so that the circumscribed polygon has to the inscribed polygon a ratio less than that which the surface of the cone has to the circle $B$ [Prop. 5]. Let this be imagined, and about the circle $A$ let a polygon be circumscribed similar to the polygon circumscribed about the circle $B$, and on the polygon circumscribed about the circle $A$ let a pyramid be raised having the same vertex as the cone. Now since the polygons circumscribed about
GREEK MATHEMATICS

tous A, B kúklous perigegramména, tôn autón étéi lógon proós alla, ón aì ék tôn kéntrou dynámìei proós állhlas, toutéstw ón étéi h Γ prós E dúnammi, toutéstw h Γ prós Δ mìkei. ón dé lógon étéi h Γ prós Δ mìkei, touton étéi to perigegramménon polýgownon peri tôn A kúklon prós thn épifánëias tìs pyramídos tìs perigegramménhs peri tôn kównon [h mèn gar Γ ish éstì th apò tôn kéntrou kathetw èpi mìa plenuràn toù polýgównou, h dé Δ th plenurà toù kównou, kównon dé úphos h perìmetro tròu polýgównou prós tì hmiost tôn épifaneiwn]. tôn autón ari lógon étéi to eubhúgrammon to peri tôn A kúklon prós to eubhúgrammon to peri tôn B kúklon kai autò to eubhúgrammon prós thn épifánëias tìs pyramídos tìs perigegramménhs peri tôn kównon. óste ìsa éstìn h épifánëia tìs pyramídos tòv eubhúgrammou to peri tôn B kúklon perigegramménw. épeì oûn elásson or lógon étéi to eubhúgrammon to peri tôn B kúklon perigegramménon prós to énggegramménon úper h épifánëia toù kównou prós tôn B kúklon, elásson or lógon étéi h épifánëia tòvpyramídos tìs peri tôn kłówon perigegramménhs prós to eubhúgrammon to én tòv B kúklw énggegramménon úper h épifánëia toù kównou prós tôn B kúklon. óper áðunaton [h mèn gar épifánëia tìs pyramídos meòwv ouvà deódektai tìs épifaneias tôn kównou, to dé énggegramménon eubhúgrammon én tòv B kúklw elásson éstai tôn B kúklon]. ouk ari ó B kúklw elásson éstai tòv épifaneias tôn kównou.
ARCHIMEDES

the circles A, B are similar, they have the same ratio one toward the other as the square of the radii have one toward the other, that is \( \Gamma^2 : E^2 \), or \( \Gamma : \Delta \) [Eucl. vi. 20, coroll. 2]. But \( \Gamma : \Delta \) is the same ratio as that of the polygon circumscribed about the circle A to the surface of the pyramid circumscribed about the cone \(^a\); therefore the rectilineal figure about the circle A has to the rectilineal figure about the circle B the same ratio as this rectilineal figure [about A] has to the surface of the pyramid circumscribed about the cone; therefore the surface of the pyramid is equal to the rectilineal figure circumscribed about the circle B. Since the rectilineal figure circumscribed about the circle B has towards the inscribed [rectilineal figure] a ratio less than that which the surface of the cone has to the circle B, therefore the surface of the pyramid circumscribed about the cone will have to the rectilineal figure inscribed in the circle B a ratio less than that which the surface of the cone has to the circle B; which is impossible.\(^b\) Therefore the circle B will not be less than the surface of the cone.

\(^a\) For the circumscribed polygon is equal to a triangle, whose base is equal to the perimeter of the polygon and whose height is equal to \( \Gamma \), while the surface of the pyramid is equal to a triangle having the same base and height \( \Delta \) [Prop. 8]. There is an explanation to this effect in the Greek, but so obscurely worded that Heiberg attributes it to an interpolator.

\(^b\) For the surface of the pyramid is greater than the surface of the cone [Prop. 12], while the inscribed polygon is less than the circle B.

---

1 ἦ μὲν ... ἐπιφανεῖον om. Heiberg.
2 ἦ μὲν ... τοῦ ἐπίκυκλου om. Heiberg.
Δέγω δὴ, οτι οὐδὲ μεῖζων. εἰ γὰρ δυνατὸν ἔστιν, ἔστω μεῖζων. πάλιν δὴ νοεῖσθω εἰς τὸν Β κύκλον πολύγωνον ἐγγεγραμμένον καὶ ἄλλο περιγεγραμμένον, ὥστε τὸ περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἔλασσον λόγον ἔχειν τοῦ, δὴ ἔχει οὐ κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κόνων, καὶ εἰς τὸν Α κύκλον νοεῖσθω ἐγγεγραμμένον πολύγωνον ὰμοιον τῇ εἰς τὸν Β κύκλον ἐγγεγραμμένω, καὶ ἀναγεγράφθω ἀπ' αὐτοῦ πυραμίς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κόνω. ἐπεὶ οὖν ὀμοιά ἐστὶ τὰ ἐν τοῖς Α, Β κύκλοις ἐγγεγραμμένα, τὸν αὐτὸν ἔξει λόγον πρὸς ἀλλήλα, δὴ τῶν κέντρων δυνάμει πρὸς ἀλλήλας· τὸν αὐτὸν ᾗρα λόγον ἔχει τὸ πολύγωνον πρὸς τὸ πολύγωνον καὶ ἢ Γ πρὸς τὴν Δ μήκει. ἢ δὲ Γ πρὸς τὴν Δ μεῖζονα λόγον ἔχει ἢ τὸ πολύγωνον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς ἐγγεγραμμένης εἰς τὸν κόνων [ἃ γὰρ ἐκ τοῦ κέντρου τοῦ Α κύκλου πρὸς τὴν πλευρὰν τοῦ κόνων μεῖζονα λόγον ἔχει ἢπερ ἢ ἀπὸ τοῦ κέντρου ἀγομένη κάθετος ἐπὶ μίαν πλευρὰν τοῦ πολυγώνου πρὸς τὴν ἐπὶ τὴν πλευρὰν τοῦ πολυγώνου κάθετον ἀγομένην ἀπὸ τῆς κορυφῆς τοῦ κόνων].

1  ἢ γὰρ . . . τοῦ κόνων om. Heiberg.

* Eutocius supplies a proof. ΖΘΚ is the polygon inscribed in the circle A (of centre A), AH is drawn perpendicular to
I say now that neither will it be greater. For if it is possible, let it be greater. Then again let there be imagined a polygon inscribed in the circle B and another circumscribed, so that the circumscribed has to the inscribed a ratio less than that which the circle B has to the surface of the cone [Prop. 5], and in the circle A let there be imagined an inscribed polygon similar to that inscribed in the circle B, and on it let there be drawn a pyramid having the same vertex as the cone. Since the polygons inscribed in the circles A, B are similar, therefore they will have one toward the other the same ratio as the squares of the radii have one toward the other; therefore the one polygon has to the other polygon the same ratio as \( \Gamma \) to \( \Delta \) [Eucl. vi. 20, coroll. 2]. But \( \Gamma \) has to \( \Delta \) a ratio greater than that which the polygon inscribed in the circle A has to the surface of the pyramid inscribed in the cone; therefore the polygon in-

\[ K\Theta \text{ and meets the circle in } M, A \text{ is the vertex of the isosceles cone (so that } \Delta H \text{ is perpendicular to } K\Theta), \text{ and } HN \text{ is drawn parallel to } MA \text{ to meet } \Delta A \text{ in } N. \text{ Then the area of the polygon inscribed in the circle} = \frac{1}{2} \text{ perimeter of polygon } \cdot \Delta H, \text{ and the area of the pyramid inscribed in the cone} = \frac{1}{2} \text{ perimeter of polygon } \cdot \Delta H, \text{ so that the area of the polygon has to the area of the pyramid the ratio } \Delta H : \Delta H. \text{ Now, by similar triangles, } AM : MA = AH : HN, \text{ and } AH : HN > AH : HA, \text{ for } H\Lambda > HN. \text{ Therefore } AM : MA > AH : HA; \text{ that is, } \Gamma : \Delta \text{ exceeds the ratio of the polygon to the surface of the pyramid.} \]
GREEK MATHEMATICS

ζόνα ἀρα λόγον ἔχει τὸ πολύγωνον τὸ ἐν τῷ Α κύκλῳ ἐγγεγραμμένον πρὸς τὸ πολύγωνον τὸ ἐν τῷ Β ἐγγεγραμμένον ἢ αὐτὸ τὸ πολύγωνον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος: μείζων ἄρα ἐστὶν ἢ ἐπιφάνεια τῆς πυραμίδος τοῦ ἐν τῷ Β πολυγώνου ἐγγεγραμμένον. ἔλάσσονα δὲ λόγον ἔχει τὸ πολύγωνον τὸ περὶ τὸν Β κύκλων περιγεγραμμένον πρὸς τὸ ἐγγεγραμμένον ἢ ὁ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου· πολλῷ ἄρα τὸ πολύγωνον τὸ περὶ τὸν Β κύκλων περιγεγραμμένον πρὸς τὴν ἐπιφάνειαν τῆς πυραμίδος τῆς ἐν τῷ κώνῳ ἐγγεγραμμένης ἔλάσσονα λόγον ἔχει ἢ ὁ Β κύκλος πρὸς τὴν ἐπιφάνειαν τοῦ κώνου· ὅπερ ἀδύνατον [τὸ μὲν γὰρ περιγεγραμμένον πολύγωνον μείζων ἐστὶν τοῦ Β κύκλου, ἢ δὲ ἐπιφάνεια τῆς πυραμίδος τῆς ἐν τῷ κώνῳ ἔλασσων ἐστὶ τῆς ἐπιφάνειας τοῦ κώνου]. ὅνικ ἄρα οὐδὲ μείζων ἐστὶν ὁ κύκλος τῆς ἐπιφάνειας τοῦ κώνου. ἐδείχθη δὲ, ὅτι οὐδὲ ἔλάσσων· ἰσος ἀρα.

15

'Εὰν κῶνος ἱσοσκελὴς ἐπιπέδῳ τμηθὴ παραλληλῆς τῇ βάσει, τῇ μεταξὺ τῶν παραλληλῶν ἐπιπέδων ἐπιφανεία τοῦ κώνου ἰσος ἐστὶ κύκλος, οὔ ἢ ἐκ τοῦ κέντρου μέσον λόγον ἔχει τῆς τε πλευρᾶς τοῦ κώνου τῆς μεταξὺ τῶν παραλληλῶν ἐπιπέδων καὶ τῆς ἰσης ἀμφοτέρας ταῖς ἐκ τῶν κέντρων τῶν κύκλων τῶν ἐν τοῖς παραλληλοῖς ἐπιπέδοις.

'Εστω κῶνος, οὔ τὸ διὰ τοῦ ἄξονος τρίγωνον ἰσον τῷ ΑΒΓ, καὶ τετμήθω παραλληλῆς ἐπιπέδῳ τῇ βάσει, καὶ ποιεῖτω τομὴν τὴν ΔΕ, ἀξων δὲ τοῦ κώνου ἐστὶν ὁ ΒΗ κύκλος δὲ τῆς ἐκκείσθω, οὔ ἢ
scribed in the circle A has to the polygon inscribed in the circle B a ratio greater than that which the same polygon [inscribed in the circle A] has to the surface of the pyramid; therefore the surface of the pyramid is greater than the polygon inscribed in B. Now the polygon circumscribed about the circle B has to the inscribed polygon a ratio less than that which the circle B has to the surface of the cone; by much more therefore the polygon circumscribed about the circle B has to the surface of the pyramid inscribed in the cone a ratio less than that which the circle B has to the surface of the cone; which is impossible.\(^1\) Therefore the circle is not greater than the surface of the cone. And it was proved not to be less; therefore it is equal.

Prop. 16

*If an isosceles cone be cut by a plane parallel to the base, the portion of the surface of the cone between the parallel planes is equal to a circle whose radius is a mean proportional between the portion of the side of the cone between the parallel planes and a straight line equal to the sum of the radii of the circles in the parallel planes.*

Let there be a cone, in which the triangle through the axis is equal to \(\Delta B\Gamma\), and let it be cut by a plane parallel to the base, and let [the cutting plane] make the section \(\Delta E\), and let \(BH\) be the axis of the cone,

\(^1\) For the circumscribed polygon is greater than the circle B, but the surface of the inscribed pyramid is less than the surface of the cone [Prop. 12]; the explanation to this effect in the text is attributed by Heiberg to an interpolator.
λέγω, ὅτι ὁ Θ κύκλος ἵσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κόνου τῇ μεταξὺ τῶν ΔΕ, ΑΓ.

Ἐκκεῖσθων γὰρ κύκλου οἱ Λ, Κ, καὶ τοῦ μὲν Κ κύκλου ἡ ἐκ τοῦ κέντρου δυνάσθω τὸ ὑπὸ ΒΔΖ, τοῦ δὲ Λ ἡ ἐκ τοῦ κέντρου δυνάσθω τὸ ὑπὸ ΒΑΗ. ὁ μὲν άρα Λ κύκλος ἵσος ἐστὶν τῇ ἐπιφανείᾳ τοῦ ΑΒΓ κόνου, ὁ δὲ Κ κύκλος ἵσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ ΔΕΒ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΒΑ, ΑΗ ἵσον ἐστὶ τῷ τῷ ὑπὸ τῶν ΒΔ, ΔΖ καὶ τῷ ὑπὸ τῆς ΑΔ καὶ συναμφοτέρου τῆς ΔΖ, ΑΗ διὰ τὸ παράλληλον εἶναι τῇ ΔΖ τῇ ΑΗ, ἀλλὰ τὸ μὲν ὑπὸ ΑΒ, ΑΗ δύναται ἡ ἐκ τοῦ κέντρου τοῦ Α κύκλου, τὸ δὲ ὑπὸ ΒΔ, ΔΖ δύναται ἡ ἐκ τοῦ κέντρου τοῦ Κ κύκλου, τὸ δὲ ὑπὸ τῆς ΔΑ καὶ συναμφοτέρου τῆς ΔΖ, ΑΗ δύναται ἡ ἐκ τοῦ κέντρου τοῦ Θ, τὸ ἀρα ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Λ κύκλου ἵσον ἐστὶ τοῖς ἀπὸ τῶν ἐκ τῶν κέντρων τῶν Κ, Θ κύκλων· ὡστε καὶ ὁ Λ κύκλος ἵσος ἐστὶ τοῖς Κ, Θ κύκλοις.
ARCHIMEDES

and let there be set out a circle whose radius is a mean proportional between $\Delta \Delta$ and the sum of $\Delta Z$, $HA$, and let $\Theta$ be the circle; I say that the circle $\Theta$ is equal to the portion of the surface of the cone between $\Delta E$, $AG$.

For let the circles $\Lambda$, $K$ be set out, and let the square of the radius of $K$ be equal to the rectangle contained by $B\Delta$, $\Delta Z$, and let the square of the radius of $\Lambda$ be equal to the rectangle contained by $BA$, $AH$; therefore the circle $\Lambda$ is equal to the surface of the cone $ABG$, while the circle $K$ is equal to the surface of the cone $\Delta EB$ [Prop. 14]. And since

$$BA \cdot AH = B\Delta \cdot \Delta Z + A\Delta \cdot (\Delta Z + AH)$$

because $\Delta Z$ is parallel to $AH$, while the square of the radius of $\Lambda$ is equal to $AB \cdot AH$, the square of the radius of $K$ is equal to $B\Delta \cdot \Delta Z$, and the square of the radius of $\Theta$ is equal to $\Delta A \cdot (\Delta Z + AH)$, therefore the square on the radius of the circle $\Lambda$ is equal to the sum of the squares on the radii of the circles $K$, $\Theta$; so that the circle $\Lambda$ is equal to the sum of the circles

* The proof is given by Eutocius as follows:

$$BA : AH = B\Delta : \Delta Z$$

$$BA \cdot \Delta Z = B\Delta \cdot AH.$$  \[Eucl. \, vi. \, 16\]

But $$BA \cdot \Delta Z = B\Delta \cdot \Delta Z + A\Delta \cdot \Delta Z.$$  \[Eucl. \, ii. \, 1\]

$$B\Delta \cdot AH = B\Delta \cdot \Delta Z + A\Delta \cdot \Delta Z.$$  \[\therefore\]

Let $\Delta A \cdot AH$ be added to both sides.

Then $$B\Delta \cdot AH + \Delta A + AH,$$

i.e. $$BA \cdot AH = B\Delta \cdot \Delta Z + A\Delta \cdot \Delta Z + A\Delta \cdot AH.$$
ἀλλ’ ὁ μὲν Λ ἵσος ἐστὶ τῇ ἑπιφάνειᾳ τοῦ βαξ
κώνου, ὁ δὲ Κ τῇ ἑπιφάνειᾳ τοῦ ΔΒΕ κώνου·
λοιπῇ ἃρα ἡ ἑπιφάνεια τοῦ κώνου ἡ μεταξὺ τῶν
παραλλήλων ἐπιπέδων τῶν ΔΕ, ΑΓ ἵση ἐστὶ τῷ
Θ κύκλῳ.

κα’

'Εὰν εἰς κύκλον πολύγωνον ἐγγραφῇ ἄρτιο-
πλευρόν τε καὶ ἱσόπλευρον, καὶ διαχώσων εὐθεῖαι
ἐπιζευγνύουσαι τὰς πλευρὰς τοῦ πολυγώνου, ὥστε
αὐτὰς παραλλήλους εἶναι μιᾷ ὀποίᾳ τῶν ὑπὸ
dύο πλευρὰς τοῦ πολυγώνου ὑποτευνοῦσῶν, αἱ ἐπι-
ζευγνύουσαι πᾶσαι πρὸς τὴν τοῦ κύκλου διάμετρον
tοῦτον ἔχουσι τὸν λόγον, ὅπερ ἔχει ἡ ὑποτεύνουσα τὰς
μιὰς ἑλάσσονας τῶν ἡμίσεων πρὸς τὴν πλευρὰν τοῦ
πολυγώνου.

'Εστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ πολύγωνον
ἐγγεγράφθω τὸ ΛΕΖΒΗΘΡΜΝΔΛΚ, καὶ ἐπι-
ζευγχωσαν αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ· δήλον δὴ,
οτί παραλληλοί εἰσιν τῇ ὑπὸ δύο πλευρὰς τοῦ
πολυγώνου ὑποτευνούσῃ λέγων οὖν, ὅτι αἱ εἴρημέναι
πᾶσαι πρὸς τὴν τοῦ κύκλου διάμετρον τὴν ΑΓ τοῦ
αὐτοῦ λόγον ἔχουσι τῷ τῆς ΓΕ πρὸς ΕΑ.

'Επεζευγχωσαν γὰρ αἱ ΖΚ, ΛΒ, ΗΔ, ΘΝ·
παραλληλοὶ ἃρα ἡ μὲν ΖΚ τῇ ΕΑ, ἡ δὲ ΒΑ τῇ
ΖΚ, καὶ ἐτι ἡ μὲν ΔΗ τῇ ΒΑ, ἡ δὲ ΘΝ τῇ ΔΗ,
καὶ ἡ ΓΜ τῇ ΘΝ [καὶ ἐπεὶ δύο παραλληλοὶ εἰσιν
αἱ ΕΑ, ΚΖ, καὶ δύο διηγοῦμεν εἰσιν αἱ ΕΚ, ΑΟ]¹·
ἔστιν ἃρα, ὡς ἡ ΕΞ πρὸς ΕΑ, ὁ ΚΞ πρὸς ΞΟ.
ὡς δ’ ἡ ΚΞ πρὸς ΞΟ, ἡ ΖΠ πρὸς ΠΟ, ὡς δὲ

¹ καὶ ἐπεὶ . . . ΕΚ, ΑΟ om. Heiberg.
ARCHIMEDES

K, Θ. But Λ is equal to the surface of the cone BAΓ, while K is equal to the surface of the cone ΔBE; therefore the remainder, the portion of the surface of the cone between the parallel planes ΔE, ΑΓ, is equal to the circle Θ.

Prop. 21

If a regular polygon with an even number of sides be inscribed in a circle, and straight lines be drawn joining the angles a of the polygon, in such a manner as to be parallel to any one whatsoever of the lines subtended by two sides of the polygon, the sum of these connecting lines bears to the diameter of the circle the same ratio as the straight line subtended by half the sides less one bears to the side of the polygon.

Let ΑΒΓΔ be a circle, and in it let the polygon ΑΕΖΒΗΘΓΜΝΔΛΚ be inscribed, and let ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ be joined; then it is clear that they are parallel to a straight line subtended by two sides of the polygon b; I say therefore that the sum of the aforementioned straight lines bears to ΑΓ, the diameter of the circle, the same ratio as ΓΕ bears to ΕΑ.

For let ΖΚ, ΔΒ, ΗΔ, ΘΝ be joined; then ΖΚ is parallel to ΕΑ, ΒΔ to ΖΚ, also ΔΗ to ΒΔ, ΘΝ to ΔΗ and ΓΜ to ΘΝ c; therefore

EΞ : ΞΑ = ΚΞ : ΞΟ.  [Eucl. vi. 4]

But

ΚΞ : ΞΟ = ΖΠ : ΠΟ,

“Sides” according to the text, but Heiberg thinks Archimedes probably wrote γωνίας where we have πλευρὰς.

For, because the arcs KA, EZ are equal, \( \angle EKZ = \angle KZA \) [Eucl. iii. 27]; therefore EK is parallel to AZ; and so on.

For, as the arcs AK, EZ are equal, \( \angle AEK = \angle EKZ \), and therefore AE is parallel to ZK; and so on.
GREEK MATHEMATICS

η ΖΠ πρὸς ΠΟ, η ΛΠ πρὸς ΠΡ, ὡς δὲ η ΛΠ πρὸς ΠΡ, οὔτως η ΒΣ πρὸς ΣΡ, καὶ ἐτι, ὡς ἡ μὲν ΒΣ πρὸς ΣΡ, η ΔΣ πρὸς ΣΤ, ὡς δὲ η ΔΣ πρὸς ΣΤ, η ΥΗ πρὸς ΥΤ, καὶ ἐτι, ὡς ἡ μὲν ΥΗ πρὸς ΥΤ, η ΝΥ πρὸς ΥΦ, ὡς δὲ η ΝΥ πρὸς ΥΦ, η ΘΧ πρὸς ΧΦ, καὶ ἐτι, ὡς μὲν η ΘΧ πρὸς ΧΦ, η ΜΧ πρὸς ΧΓ [καὶ πάντα ἀρά πρὸς πάντα ἐστίν, ὡς εἰς τῶν λόγων πρὸς ἑνα]. ὡς ἀρὰ η ΕΞ πρὸς ΕΑ, οὔτως αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ πρὸς τὴν ΑΓ διάμετρον. ὡς δὲ η ΕΞ πρὸς ΕΑ, οὔτως η ΓΕ πρὸς ΕΑ. ἐσται ἀρα καὶ, ὡς ἡ ΓΕ πρὸς ΕΑ, οὔτω πᾶσαι αἱ ΕΚ, ΖΛ, ΒΔ, ΗΝ, ΘΜ πρὸς τὴν ΑΓ διάμετρον.

1 καὶ ... ἑνα om. Heiberg.
ARCHIMEDES

while \( Z\Pi : \Pi \theta = \Lambda \Pi : \Pi \rho, \) [ibid.
and \( \Lambda \Pi : \Pi \rho = \nu \Sigma : \Sigma \rho. \) [ibid.
Again, \( \nu \Sigma : \Sigma \rho = \Delta \Sigma : \Sigma \tau, \) [ibid.
while \( \Delta \Sigma : \Sigma \tau = \nu \gamma : \gamma \tau. \) [ibid.
Again, \( \nu \gamma : \gamma \tau = \nu \gamma \gamma : \gamma \phi, \) [ibid.
while \( \nu \gamma \gamma : \gamma \phi = \Theta \chi : \chi \phi. \) [ibid.
Again, \( \Theta \chi : \chi \phi = \mu \chi : \chi \gamma, \) [ibid.
therefore \( E \xi : \xi \alpha = E \kappa + Z \lambda + B \delta + H \eta + \Theta \mu : A \gamma. \) [Eucl. v. 12
But \( E \xi : \xi \alpha = \xi \gamma : \xi \rho; \) [Eucl. vi. 4
therefore \( \xi \gamma : \xi \rho = E \kappa + Z \lambda + B \delta + H \eta + \Theta \mu : A \gamma. \) [Eucl. vi. 4

* By adding all the antecedents and consequents, for
\( \xi \gamma : \xi \rho = E \kappa + K \xi + Z \Pi + \Lambda \Pi + B \Sigma + \Delta \Sigma + \nu \gamma + \nu \gamma + \Theta \chi + \mu \chi + \chi \gamma \)
\( = E \kappa + Z \lambda + B \delta + H \eta + \Theta \mu : A \gamma. \)

* If the polygon has \( 4n \) sides, then
\( \angle E \kappa \gamma = \frac{\pi}{2n} \) and \( E \kappa : A \gamma = \sin \frac{\pi}{2n} \),
\( \angle Z \gamma \lambda = \frac{2\pi}{2n} \) and \( Z \lambda : A \gamma = \sin \frac{2\pi}{2n} \),
\( \angle \Theta \gamma \mu = (2n - 1) \frac{\pi}{2n} \) and \( \Theta \mu : A \gamma = \sin (2n - 1) \frac{\pi}{2n} \).

Further, \( \angle A \gamma \xi = \frac{\pi}{4n} \) and \( \xi \gamma : \xi \rho = \cot \frac{\pi}{4n} \).
Therefore the proposition shows that
\( \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin (2n - 1) \frac{\pi}{2n} = \cot \frac{\pi}{4n}. \)
κυ'

"Εστω ἐν σφαίρᾳ μέγιστος κύκλος ὁ ΑΒΓΔ, καὶ ἐγγεγράφθω εἰς αὐτὸν πολύγωνον ἵσοπλευρον, τὸ

\[ \text{Diagram of a circle with points labeled A, B, C, D, Z, H, N, M, Δ.} \]

δὲ πλῆθος τῶν πλευρῶν αὐτοῦ μετρεῖσθω ὑπὸ τετράδος, αἱ δὲ ΑΓ, ΔΒ διάμετροι ἐστωσαν. ἕαν δὴ μενοῦσις τῆς ΑΓ διαμέτρου περιενεχθῇ ὁ ΑΒΓΔ κύκλος ἐχὼν τὸ πολύγωνον, δήλου, ὅτι ἡ μὲν περιφέρεια αὐτοῦ κατὰ τῆς ἐπιφανείας τῆς σφαίρας ἐνεχθήσεται, αἱ δὲ τοῦ πολυγώνου γωνίαι χωρὶς τῶν πρὸς τοῖς Α, Γ σημείοις κατὰ κύκλων περιφερεῖῶν ἐνεχθήσονται ἐν τῇ ἐπιφανείᾳ τῆς σφαίρας γεγραμμένων ὀρθῶν πρὸς τὸν ΑΒΓΔ κύκλον. διάμετροι δὲ αὐτῶν ἔσονται αἱ ἐπιζευγνύουσαι τὰς γωνίας τοῦ πολυγώνου παρὰ τὴν ΒΔ οὔσαι. αἱ δὲ τοῦ πολυγώνου πλευραὶ κατὰ τινῶν κώνων ἐνεχθήσονται, αἱ μὲν ΑΖ, ΑΝ κατ᾽ ἐπιφανείας κώνου, οὐ βάσις μὲν ὁ κύκλος ὁ περὶ διάμετρον τῆς ΖΝ, κορυφὴ δὲ τὸ Α σημεῖον, αἱ δὲ
Let $AB\Gamma\Delta$ be the greatest circle in a sphere, and let there be inscribed in it an equilateral polygon, the number of whose sides is divisible by four, and let $A\Gamma, \Delta B$ be diameters. If the diameter $A\Gamma$ remain stationary and the circle $AB\Gamma\Delta$ containing the polygon be rotated, it is clear that the circumference of the circle will traverse the surface of the sphere, while the angles of the polygon, except those at the points $A, \Gamma$, will traverse the circumferences of circles described on the surface of the sphere at right angles to the circle $AB\Gamma\Delta$; their diameters will be the [straight lines] joining the angles of the polygon, being parallel to $B\Delta$. Now the sides of the polygon will traverse certain cones; $AZ, AN$ will traverse the surface of a cone whose base is the circle about the diameter $ZN$ and whose vertex is the point $A$; $ZH,$
GREEK MATHEMATICS

ZH, MN kατά των κωνικής ἐπιφανείας οἰσθή-
σονται, ἣς βάσις μὲν ὁ κύκλος ὁ περὶ διάμετρον τὴν
ΜΗ, κορυφή δὲ τὸ σημεῖον, καθ’ ὁ συμβάλλουσιν ἕκ-
βαλλόμεναι αἱ ZH, MN ἄλληλαις τε καὶ τῇ ΑΓ, αἱ δὲ BH, MD πλευραῖ κατὰ κωνικῆς ἐπιφανείας
οἰσθήσονται, ἣς βάσις μὲν ἐστὶν ὁ κύκλος ὁ περὶ
diάμετρον τὴν BD ὀρθὸς πρὸς τὸν ΑΒΓΔ κύκλον,
κορυφή δὲ τὸ σημεῖον, καθ’ ὁ συμβάλλουσιν ἕκ-
βαλλόμεναι αἱ BH, DM ἄλληλαις τε καὶ τῇ ΤΑ-
διμοῖς δὲ καὶ αἱ ἐν τῷ ἑτέρῳ ἡμικυκλῶ πλευραὶ
catὰ κωνικῶν ἐπιφανειών οἰσθήσονται πάλιν
ὀμοίων τἀυτὰς. ἔσται δὴ τι σχήμα ἐγγεγραμ-
μένον ἐν τῇ σφαίρᾳ ὑπὸ κωνικῶν ἐπιφανειῶν
περιεχόμενον τῶν προερημένων, ὥς ἡ ἐπιφάνεια
ἐλάσσων ἔσται τῆς ἐπιφανείας τῆς σφαίρας.

Διαπερίβλησις γὰρ τῆς σφαίρας ὑπὸ τοῦ ἐπιπέδου
tοῦ κατὰ τὴν BD ὀρθοῦ πρὸς τὸν ΑΒΓΔ κύκλον
ἡ ἐπιφάνεια τοῦ ἑτέρου ἡμισφαίριον καὶ ἡ ἐπι-
φάνεια τοῦ σχήματος τοῦ ἐν αὐτῷ ἐγγεγραμμένον
tα αὐτὰ πέρατα ἔχουσιν ἐν ἐνὶ ἐπιπέδῳ ἄμφo-
tέρων γὰρ τῶν ἐπιφανειῶν πέρας ἔστιν τοῦ κύκλον
ἡ περιφέρεια τοῦ περὶ διάμετρον τὴν BD ὀρθοῦ
πρὸς τὸν ΑΒΓΔ κύκλον καὶ εἰσὶν ἄμφοτέραι ἐπὶ
tα αὐτὰ κολλαί, καὶ περιλαμβάνεται αὐτῶν ἡ ἑτέρα
ὑπὸ τῆς ἑτέρας ἐπιφανείας καὶ τῆς ἐπιπέδου τῆς
tα αὐτὰ πέρατα ἔχουσις αὐτή. ὀμοίως δὲ καὶ τοῦ
ἐν τῷ ἑτέρῳ ἡμισφαίριοι σχήματος ἡ ἐπιφάνεια
ἐλάσσων ἐστὶν τῆς τοῦ ἡμισφαίριου ἐπιφανείας,
καὶ ὅλη ὥς ἡ ἐπιφάνεια τοῦ σχήματος τοῦ ἐν τῇ
σφαίρᾳ ἐλάσσων ἔστιν τῆς ἐπιφανείας τῆς σφαίρας.

* Archimedes would not have omitted to make the deduc-
ARCHIMEDES

MN will traverse the surface of a certain cone whose base is the circle about the diameter MH and whose vertex is the point in which ZH, MN produced meet one another and with AT; the sides BH, MA will traverse the surface of a cone whose base is the circle about the diameter BA at right angles to the circle ABGΔ and whose vertex is the point in which BH, AM produced meet one another and with TA; in the same way the sides in the other semicircle will traverse surfaces of cones similar to these. As a result there will be inscribed in the sphere and bounded by the aforesaid surfaces of cones a figure whose surface will be less than the surface of the sphere.

For, if the sphere be cut by the plane through BA at right angles to the circle ABGΔ, the surface of one of the hemispheres and the surface of the figure inscribed in it have the same extremities in one plane; for the extremity of both surfaces is the circumference of the circle about the diameter BA at right angles to the circle ABGΔ; and both are concave in the same direction, and one of them is included by the other surface and the plane having the same extremities with it.a Similarly the surface of the figure inscribed in the other hemisphere is less than the surface of the hemisphere; and therefore the whole surface of the figure in the sphere is less than the surface of the sphere.

a From Postulate 4, that the surface of the figure inscribed in the hemisphere is less than the surface of the hemisphere.
'Η τοῦ ἐγγραφομένου σχῆματος εἰς τὴν σφαῖραν ἐπιφάνεια ἵση ἐστὶ κύκλῳ, οὐ ἢ ἐκ τοῦ κέντρου δύναται τὸ περιεχόμενον ὑπὸ τὲ τῆς πλευρᾶς τοῦ σχῆματος καὶ τῆς ἵσης πάσαις ταῖς ἐπιζευγνυόσαις τὰς πλευρὰς τοῦ πολυγώνου παράλληλοις οὐσίαις τῇ ὑπὸ δύο πλευρὰς τοῦ πολυγώνου ὑποτευνούση εὐθεία.

Ἐστω ἐν σφαίρᾳ μέγιστος κύκλος ὁ ἈΒΓΔ, καὶ ἐν αὐτῷ πολύγωνον ἐγγεγράφθω ἱσόπλευρον, οὐ αἱ πλευραὶ ὑπὸ τετράδος μετροῦνται, καὶ ἀπὸ τοῦ πολυγώνου τὸ ἐγγεγραμμένον νοείσθω τι εἰς τὴν σφαῖραν ἐγγραφέν σχῆμα, καὶ ἐπεζευχθῶσαν αἱ EZ, ΗΘ, ΓΔ, ΚΛ, ΜΝ παράλληλοι οὐσιαὶ τῇ ὑπὸ δύο πλευρὰς ὑποτευνούσῃ εὐθείᾳ, κύκλοις δὲ τις ἐκκεῖσθω ὁ Ξ, οὐ ἢ ἐκ τοῦ κέντρου δυνάσθω τὸ περιεχόμενον ὑπὸ τὲ τῆς ΑΕ καὶ τῆς ἵσης ταῖς EZ, ΗΘ, ΓΔ, ΚΛ, ΜΝ· λέγω, ὅτι ὁ κύκλος οὗτος ἵσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ εἰς τὴν σφαῖραν ἐγγραφομένου σχῆματος.

Ἐκκεῖσθωσαν γὰρ κύκλοι οἱ Ω, Π, Ρ, ΢, Τ, Υ, καὶ τοῦ μὲν ὁ ἢ ἐκ τοῦ κέντρου δυνάσθω τὸ περιεχόμενον ὑπὸ τὲ τῆς ΕΑ καὶ τῆς ἡμισείας τῆς EZ, ἢ δὲ ἐκ τοῦ κέντρου τοῦ Π δυνάσθω τὸ περιεχόμενον ὑπὸ τὲ τῆς ΕΑ καὶ τῆς ἡμισείας τῶν EZ, ΗΘ, ΓΔ, ἢ δὲ ἐκ τοῦ κέντρου τοῦ Ρ δυνάσθω τὸ περιεχόμενον ὑπὸ τὲ τῆς ΕΑ καὶ τῆς ἡμισείας τῶν ΗΘ, ΓΔ, ἢ δὲ ἐκ τοῦ κέντρου τοῦ Σ δυνάσθω τὸ περιεχόμενον ὑπὸ τὲ τῆς ΕΑ καὶ τῆς ἡμισείας τῶν ΓΔ, ΚΛ, ἢ δὲ ἐκ τοῦ κέντρου τοῦ Τ δυνάσθω τὸ περιεχόμενον ὑπὸ τὲ τῆς ΑΕ καὶ τῆς ἡμισείας
The surface of the figure inscribed in the sphere is equal to a circle, the square of whose radius is equal to the rectangle contained by the side of the figure and a straight line equal to the sum of the straight lines joining the angles of the polygon, being parallel to the straight line subtended by two sides of the polygon.

Let $AB\Gamma\Delta$ be the greatest circle in a sphere, and in it let there be inscribed an equilateral polygon, the number of whose sides is divisible by four, and, starting from the inscribed polygon, let there be imagined a figure inscribed in the sphere, and let $EZ$, $H\Theta$, $\Gamma\Delta$, $K\Lambda$, $MN$ be joined, being parallel to the straight line subtended by two sides; now let there be set out a circle $\Xi$, the square of whose radius is equal to the rectangle contained by $AE$ and a straight line equal to the sum of $EZ$, $H\Theta$, $\Gamma\Delta$, $K\Lambda$, $MN$; I say that this circle is equal to the surface of the figure inscribed in the sphere.

For let the circles $O$, $\Pi$, $P$, $\Sigma$, $T$, $\Upsilon$ be set out, and let the square of the radius of $O$ be equal to the rectangle contained by $EA$ and the half of $EZ$, let the square of the radius of $\Pi$ be equal to the rectangle contained by $EA$ and the half of $EZ + H\Theta$, let the square of the radius of $P$ be equal to the rectangle contained by $EA$ and the half of $H\Theta + \Gamma\Delta$, let the square of the radius of $\Sigma$ be equal to the rectangle contained by $EA$ and the half of $\Gamma\Delta + K\Lambda$, let the square of the radius of $T$ be equal to the rectangle
GREEK MATHEMATICS

τῶν ΚΛ, ΜΝ, ἡ δὲ ἐκ τοῦ κέντρου τοῦ Υ δύνασθω τὸ περιεχόμενον ύπὸ τε τῆς ΑΕ καὶ τῆς ἡμισειάς τῆς ΜΝ. διὰ δὴ ταῦτα ὁ μὲν Ο κύκλος ἵσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ ΑΕΖ κώνου, ὁ δὲ Π τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΕΖ, ΗΘ, ὁ δὲ Ρ τῇ μεταξὺ τῶν ΗΘ, ΓΔ, ὁ δὲ Σ τῇ μεταξὺ τῶν

ΔΓ, ΚΛ, καὶ ἐτί ὁ μὲν Τ ἵσος ἐστὶ τῇ ἐπιφανείᾳ τοῦ κώνου τῇ μεταξὺ τῶν ΚΛ, ΜΝ, ὁ δὲ Υ τῇ τοῦ ΜΒΝ κώνου ἐπιφανείᾳ ἵσος ἐστίν· οἱ πάντες ἀρα κύκλοι ἰσοὶ εἰσὶν τῇ τοῦ ἐγγεγραμμένου σχῆματος ἐπιφανείᾳ. καὶ φανερὸν, ὅτι αἱ ἐκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Υ κύκλων δύνανται τὸ περιεχόμενον ύπὸ τε τῆς ΑΕ καὶ δις τῶν ἡμισεων τῆς ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ, αἱ ὅλαι εἰσὶν

98
ARCHIMEDES

contained by $AE$ and the half of $KL + MN$, and let the square of the radius of $Y$ be equal to the rectangle contained by $AE$ and the half of $MN$. Now by these constructions the circle $O$ is equal to the surface of the cone $AEZ$ [Prop. 14], the circle $\Pi$ is equal to the surface of the conical frustum between $EZ$ and $H\Theta$, the circle $P$ is equal to the surface of the conical frustum between $H\Theta$ and $\Gamma\Delta$, the circle $\Sigma$ is equal to the surface of the conical frustum between $\Delta\Gamma$ and $KL$, the circle $T$ is equal to the surface of the conical frustum between $KL$, $MN$ [Prop. 16], and the circle $Y$ is equal to the surface of the cone $MBN$ [Prop. 14]; the sum of the circles is therefore equal to the surface of the inscribed figure. And it is manifest that the sum of the squares of the radii of the circles $O$, $\Pi$, $P$, $\Sigma$, $T$, $Y$ is equal to the rectangle contained by $AE$ and twice the sum of the halves of $EZ$, $H\Theta$, $\Gamma\Delta$, $KL$, $MN$, that is to say, the sum of $EZ$, $99$
αι ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ. αι ἄρα ἐκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Υ κύκλων δύνανται τὸ περιεχόμενον ὑπὸ τῇ ἈΕ καὶ πασῶν τῶν ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ. ἀλλὰ καὶ ἡ ἐκ τοῦ κέντρου τοῦ Ε κύκλου δύναται τὸ ὑπὸ τῆς ΑΕ καὶ τῆς συγκειμένης ἐκ πασῶν τῶν ΕΖ, ΗΘ, ΓΔ, ΚΛ, ΜΝ. ἣ ἄρα ἐκ τοῦ κέντρου τοῦ Ε κύκλου δύναται τὰ ἀπὸ τῶν ἐκ τῶν κέντρων τῶν Ο, Π, Ρ, Σ, Τ, Υ κύκλων· καὶ ὁ κύκλος ἄρα ὁ Ε ἰσος ἐστὶ τοῖς Ο, Π, Ρ, Σ, Τ, Υ κύκλοι ἀπεδείχθησαν ἵσοι τῇ εἰρημένῃ τοῦ σχήματος ἐπιφανείᾳ· καὶ ὁ Ε ἄρα κύκλος ἰσος ἐσται τῇ ἐπιφανείᾳ τοῦ σχήματος.

κε'

Τοῦ ἐγγεγραμμένου σχήματος εἰς τὴν σφαίραν ἡ ἐπιφάνεια ἡ περιεχόμενη ὑπὸ τῶν κωνικῶν ἐπιφανειῶν ἐλάσσων ἐστὶν ἡ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

'Εστω ἐν σφαίρα μέγιστος κύκλος ο ΑΒΓΔ, καὶ ἐν αὐτῷ ἐγγεγράφθω πολύγωνον [ἀρτιόγωνον] ἴσο-πλευρον, οὐ̂ αἱ πλευραὶ ὑπὸ τετράδος μετροῦνται, καὶ ἀπ' αὐτοῦ νοεῖσθω ἐπιφάνεια ἡ ὑπὸ τῶν

* If the radius of the sphere is $a$ this proposition shows that
Surface of inscribed figure $= \pi \cdot AE \cdot (EZ + H\Theta + \Gamma\Delta + K\Lambda + MN)$.

Now $AE = 2a \sin \frac{\pi}{4n}$, and by p. 91 n. b

100
ARCHIMEDES

Therefore the sum of the squares of the radii of the circles $O$, $P$, $\Sigma$, $T$, $Y$ is equal to the rectangle contained by $AE$ and the sum of $EZ$, $H\Theta$, $\Gamma\Delta$, $KL$, $MN$. But the square of the radius of the circle $\Xi$ is equal to the rectangle contained by $AE$ and a straight line made up of $EZ$, $H\Theta$, $\Gamma\Delta$, $KL$, $MN$ [ex hypothesi]; therefore the square of the radius of the circle $\Xi$ is equal to the sum of the squares of the radii of the circles $O$, $P$, $\Sigma$, $T$, $Y$; and therefore the circle $\Xi$ is equal to the sum of the circles $O$, $P$, $\Sigma$, $T$, $Y$. Now the sum of the circles $O$, $P$, $\Sigma$, $T$, $Y$ was shown to be equal to the surface of the aforesaid figure; and therefore the circle $\Xi$ will be equal to the surface of the figure.a

Prop. 25

The surface of the figure inscribed in the sphere and bounded by the surfaces of cones is less than four times the greatest of the circles in the sphere.

Let $\triangle AB\Gamma\Delta$ be the greatest circle in a sphere, and in it let there be inscribed an equilateral polygon, the number of whose sides is divisible by four, and, starting from it, let a surface bounded by surfaces of

\[ EZ + H\Theta + \Gamma\Delta + KL + MN = 2a \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left(2n - 1\right) \frac{\pi}{2n} \right]. \]

\[ = 4na^2 \sin \frac{\pi}{4n} \left\{ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left(2n - 1\right) \frac{\pi}{2n} \right\} \]

\[ = 4na^2 \cos \frac{\pi}{4n} \]

[by p. 91 n. b.]

101
κωνικῶν ἑπιφανειῶν περιεχομένη· λέγω, ὅτι ἡ ἑπιφάνεια τοῦ ἐγγραφέντος ἑλάσσων ἐστὶν ἡ τετραπλασία τοῦ μεγίστου кύκλου τῶν ἐν τῇ σφαίρᾳ.

Ἐπεξεύθυνον γὰρ αἱ ὑπὸ δύο πλευρὰς ὑποτείνουσαι τοῦ πολυγώνου αἱ ΕΙ, ΘΜ καὶ ταύταις παράλληλοι αἱ ΖΚ, ΔΒ, ΗΛ, ἐκκείσθω δὲ τις κύκλος ὁ Ρ, οὗ ἡ ἐκ τοῦ κέντρου δύναται τὸ ὑπὸ τῆς ΕΑ καὶ τῆς ίσης πάσαις ταῖς ΕΙ, ΖΚ, ΒΔ, ΗΛ, ΘΜ· διὰ δὴ τὸ προδιέχθην ἵσος ἐστὶν ὁ κύκλος τῇ τοῦ εἰρημένου σχήματος ἑπιφανεία. καὶ ἐπεὶ ἐδείχθη, ὅτι ἐστὶν, ὡς ἡ ίση πάσαις ταῖς ΕΙ, ΖΚ, ΒΔ, ΗΛ, ΘΜ πρὸς τὴν διάμετρον τοῦ κύκλου τῆς ΑΓ, οὕτως ἢ ΓΕ πρὸς ΕΑ, τὸ ἀρα ὑπὸ τῆς ίσης πάσαις ταῖς εἰρημέναις καὶ τῆς ΕΑ, τούτεστι τὸ ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Ρ κύκλου, ἵσον ἐστὶν τῷ ὑπὸ τῶν ΑΓ, ΓΕ. ἀλλὰ καὶ τὸ ὑπὸ ΑΓ, ΓΕ ἑλάσσων ἐστὶ τοῦ ἀπὸ τῆς ΑΓ· ἑλάσσων ἀρα ἐστὶν τὸ ἀπὸ τῆς ἐκ τοῦ κέντρου τοῦ Ρ τοῦ ἀπὸ τῆς ΑΓ [ἐλάσσων ἀρα ἐστὶν ἢ ἐκ τοῦ κέντρου τοῦ Ρ τῆς ΑΓ· ὥστε ἢ διάμετρος τοῦ Ρ κύκλου ἑλάσσων ἐστὶν ἢ διπλασία τῆς διαμέτρου τοῦ ΑΒΓΔ κύκλου, καὶ δύο ἀρα τοῦ ΑΒΓΔ κύκλου διάμετροι μεῖζοι εἰσὶ τῆς διαμέτρου τοῦ Ρ κύκλου, καὶ τὸ τετράκις ἀπὸ τῆς διαμέτρου τοῦ ΑΒΓΔ κύκλου, τούτεστι τῆς ΑΓ, μεῖζον ἐστὶ τοῦ ἀπὸ τῆς τοῦ Ρ κύκλου διαμέτρου. ὡς δὲ τὸ τετράκις ἀπὸ τῆς ΑΓ πρὸς τὸ ἀπὸ τῆς τοῦ Ρ κύκλου διαμέτρου, οὕτως τέσσαρες κύκλοι οἱ ΑΒΓΔ πρὸς τὸν Ρ κύκλον· τέσσαρες ἀρα κύκλοι οἱ ΑΒΓΔ μεῖζος εἰσίν τοῦ Ρ κύκλου]· ὁ ἀρα κύκλος ὁ Ρ ἑλάσσων ἐστὶν ἡ τετραπλάσιος τοῦ

1 ἑλάσσων ... κύκλου om. Heiberg.
cones be imagined; I say that the surface of the inscribed figure is less than four times the greatest of the circles inscribed in the sphere.

For let $EI$, $OM$, subtended by two sides of the polygon, be joined, and let $ZK$, $AB$, $HA$ be parallel to them, and let there be set out a circle $P$, the square of whose radius is equal to the rectangle contained by $EA$ and a straight line equal to the sum of $EI$, $ZK$, $BA$, $HA$, $OM$; by what has been proved above, the circle is equal to the surface of the aforesaid figure. And since it was proved that the ratio of the sum of $EI$, $ZK$, $BA$, $HA$, $OM$ to $AG$, the diameter of the circle, is equal to the ratio of $GE$ to $EA$ [Prop. 21], therefore

$$EA \cdot (EI + ZK + BA + HA + OM) = AG \cdot GE.$$  

[ex hyp.

But $$AG \cdot GE < AG^2.$$  

[Eucl. iii. 15

Therefore the square on the radius of $P$ is less than the square on $AG$; therefore the circle $P$ is less
GREEK MATHEMATICS

μεγιστον κύκλου. ο δε Ρ κύκλος ίσος έδειχθη τη
eιρημένη επιφανεία του σχήματος· ή άρα επιφάνεια
tου σχήματος έλάσσων εστίν ή τετραπλασία του
μεγιστον κύκλου των εν τη σφαίρα.

κη'

'Εστω εν σφαίρα μέγιστος κύκλος ο ABΓΔ,
περί δε τὸν ABΓΔ κύκλον περιγεγράφθω πολύ-
γωνον ίσοπλευρόν τε καὶ ίσογώνιον, τὸ δὲ πλήθος
tῶν πλευρῶν αὐτοῦ μετρεῖσθω ύπό τετράδος, τὸ
dε περί τὸν κύκλον περιγεγραμμένον πολύγωνον
κύκλος περιγεγραμμένος περιλαμβανότα τοῖ
αὐτὸ κέντρον γινόμενος τῷ ABΓΔ. μενοῦσθαι δὴ
τῆς ΕΗ περιενεχθῆτω τὸ EZΗΘ ἐπίπεδον, ἐν ϑ
τὸ τε πολύγωνον καὶ ὁ κύκλος· δὴλον οὖν, ὅτι ἡ
μὲν περιφέρεια τοῦ ABΓΔ κύκλου κατὰ τῆς ἐπι-
φανείας τῆς σφαίρας οὐσθῆσθαι, ἡ δὲ περιφέρεια
τοῦ EZΗΘ κατ’ ἄλλης ἐπιφανείας σφαίρας τὸ
104
than four times the greatest circle. But the circle P was proved equal to the aforesaid surface of the figure; therefore the surface of the figure is less than four times the greatest of the circles in the sphere.

Prop. 28

Let $AB\Gamma\Delta$ be the greatest circle in a sphere, and about the circle $AB\Gamma\Delta$ let there be circumscribed an equilateral and equiangular polygon, the number of whose sides is divisible by four, and let a circle be described about the polygon circumscribing the circle, having the same centre as $AB\Gamma\Delta$. While $EH$ remains stationary, let the plane $EZ\Theta$, in which lie both the polygon and the circle, be rotated; it is clear that the circumference of the circle $AB\Gamma\Delta$ will traverse the surface of the sphere, while the circumference of $EZ\Theta$ will traverse the surface of another
αὐτὸ κέντρον ἐχούσης τῇ ἐλάσσου οἰσθήσεται, αἱ δὲ ἁφαὶ, καθ’ ἂς ἐπιφανεύσων αἱ πλευραὶ, γράφουσιν κύκλου ὀρθοὺς πρὸς τὸν ἈΒΓΔ κύκλον ἐν τῇ ἐλάσσου οὐσία, αἱ δὲ γωνίας τοῦ πολυγώνου χωρίς τῶν πρὸς τοῖς Ε, Ἡ σημεῖοι κατὰ κύκλων περιφερεῖῶν οἰσθήσονται ἐν τῇ ἐπιφάνεια τῆς μεῖζονος οὐσίας γεγραμμένων ὀρθῶν πρὸς τὸν ΕΖΗΘ κύκλον, αἱ δὲ πλευραὶ τοῦ πολυγώνου κατὰ κωνικῶν ἐπιφανεῖων οἰσθήσονται, καθάπερ ἐπὶ τῶν πρὸ τούτων ἔσται οὖν τὸ σχῆμα τὸ περιεχόμενον ὑπὸ τῶν ἐπιφανείων τῶν κωνικῶν περὶ μὲν τὴν ἐλάσσονα οὐσίαν περιγεγραμμένον, ἐν δὲ τῇ μεῖζον ἐγγεγραμμένον. ὅτι δὲ ἡ ἐπιφάνεια τοῦ περιγεγραμμένου σχῆματος μείζων ἐστὶ τῆς ἐπιφανείας τῆς οὐσίας, οὕτως δειχθήσεται.

"Εστώ γὰρ ἡ ΚΔ διάμετρος κύκλου τινὸς τῶν ἐν τῇ ἐλάσσον οὐσία τῶν Κ, Δ σημείων ὀντῶν, καθ’ ἃ ἀπτονται τοῦ ἈΒΓΔ κύκλου αἱ πλευραὶ τοῦ περιγεγραμμένου πολυγώνου. διηρήμενης δὴ τῆς οὐσίας ὑπὸ τοῦ ετεροῦς τοῦ κατὰ τὴν ΚΔ ὀρθοὺς πρὸς τὸν ἈΒΓΔ κύκλον καὶ ἡ ἐπιφάνεια τοῦ περιγεγραμμένου σχῆματος περὶ τὴν οὐσίαν διαφθορὸς ὑπὸ τοῦ ἐπιπέδου. καὶ φανερὸν, ὅτι τὰ αὐτὰ πέρατα ἔχουσιν ἐν ἐπιπέδῳ ἀμφοτέρων γὰρ τῶν ἐπιπέδων πέρας ἐστὶ τοῦ κύκλου περιφέρεια τοῦ περὶ διάμετρον τῆς ΚΔ ὀρθοῦ πρὸς τὸν ἈΒΓΔ κύκλον καὶ εἰσὶν ἀμφότεροι ἐπὶ τὰ αὐτὰ κολαί, καὶ περιλαμβάνεται ἡ ἐτέρα αὐτῶν ὑπὸ τῆς ἐτέρας ἐπιφανείας καὶ τῆς ἐπιπέδου τῆς τὰ αὐτὰ πέρατα ἐχούσης. ἐλάσσων οὖν ἔστω ἡ περιλαμβανομένη τοῦ τμῆματος τῆς οὐσίας ἐπιφάνεια τῆς ἐπιφανείας τοῦ σχῆματος τοῦ περὶ-
ARCHIMEDES

sphere, having the same centre as the lesser sphere; the points of contact in which the sides touch [the smaller circle] will describe circles on the lesser sphere at right angles to the circle $ABGamma$, and the angles of the polygon, except those at the points $E, H$ will traverse the circumferences of circles on the surface of the greater sphere at right angles to the circle $EZHTheta$, while the sides of the polygon will traverse surfaces of cones, as in the former case; there will therefore be a figure, bounded by surfaces of cones, described about the lesser sphere and inscribed in the greater. That the surface of the circumscribed figure is greater than the surface of the sphere will be proved thus.

Let $KD$ be a diameter of one of the circles in the lesser sphere, $K, Delta$ being points at which the sides of the circumscribed polygon touch the circle $ABGamma$. Now, since the sphere is divided by the plane containing $KD$ at right angles to the circle $ABGamma$, the surface of the figure circumscribed about the sphere will be divided by the same plane. And it is manifest that they $^a$ have the same extremities in a plane; for the extremity of both surfaces $^b$ is the circumference of the circle about the diameter $KD$ at right angles to the circle $ABGamma$; and they are both concave in the same direction, and one of them is included by the other and the plane having the same extremities; therefore the included surface of the segment of the sphere is less than the surface of

$^a$ *i.e.,* the surface formed by the revolution of the circular segment $KDA$ and the surface formed by the revolution of the portion $K . . . E . . . Delta$ of the polygon.

$^b$ In the text *epiteudon* should obviously be *epifaneioun.*
If the radius of the inner sphere is $a$ and that of the outer sphere $a'$, and the regular polygon has $4n$ sides, then

$$a' = a \sec \frac{\pi}{4n}.$$ 

This proposition shows that the area of a figure circumscribed to a circle of radius $a$ is equal to the area of a figure inscribed in a circle of radius $a'$. 

108
the figure circumscribed about it [Post. 4]. Similarly the surface of the remaining segment of the sphere is less than the surface of the figure circumscribed about it; it is clear therefore that the whole surface of the sphere is less than the surface of the figure circumscribed about it.

Prop. 29

The surface of the figure circumscribed about the sphere is equal to a circle, the square of whose radius is equal to the rectangle contained by one side of the polygon and a straight line equal to the sum of all the straight lines joining the angles of the polygon, being parallel to one of the straight lines subtended by two sides of the polygon.

For the figure circumscribed about the lesser sphere is inscribed in the greater sphere [Prop. 28]; and it has been proved that the surface of the figure inscribed in the sphere and formed by surfaces of cones is equal to a circle, the square of whose radius is equal to the rectangle contained by one side of the polygon and a straight line equal to the sum of all the straight lines joining the angles of the polygon, being parallel to one of the straight lines subtended by two sides [Prop. 24]; what was aforesaid is therefore obvious.a

\[ = 4\pi a^2 \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin (2n - 1) \frac{\pi}{2n} \right], \]

or \( 4\pi a^2 \cos \frac{\pi}{4n} \) [by p. 91 n. b]

\[ = 4\pi a^2 \sec^2 \frac{\pi}{4n} \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin (2n - 1) \frac{\pi}{2n} \right], \]

or \( 4\pi a^2 \sec \frac{\pi}{4n}. \)
Τοῦ σχήματος τοῦ περιγραμμένου περὶ τὴν σφαῖραν ἡ ἐπιφάνεια μεῖζων ἐστὶν ἡ τετραπλασία τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαῖρᾳ.

"Ἐστώ γάρ ἡ τε σφαῖρα καὶ ὁ κύκλος καὶ τὰ ἄλλα τὰ αὐτὰ τοῖς πρῶτοι προκειμένοις, καὶ ὁ Λ κύκλος ἰσος τῇ ἐπιφανείᾳ ἐστω τοῦ προκειμένου περιγραμμένου περὶ τὴν ἐλάσσονα σφαῖραν.

'Επεὶ οὖν ἐν τῷ ΕΖΗΘ θύρω εὐκλῆ πολύγωνον

ισόπλευρον ἐγγέγραπται καὶ ἀρτιωγώνιον, αἱ ἐπι-

ζευγνύουσαι τὰς τοῦ πολυγώνου πλευρὰς παράλ-

κῆλος οὐδὲν τῇ ΖΘ πρὸς τὴν ΖΘ τὸν αὐτὸν λόγον

ἐχουσιν, ἂν ἡ ΘΚ πρὸς ΚΖ: ἵσον ἀρα ἐστὶν τὸ

περιεχόμενον σχῆμα ὑπὸ τε μιᾶς πλευρᾶς τοῦ

πολυγώνου καὶ τῆς ἴσης πάσαις ταῖς ἐπιζευγνύο-

σαις τὰς γωνίας τοῦ πολυγώνου τῷ περιεχόμενῳ

ὑπὸ τῶν ΖΘΚ: ὥστε ἡ ἐκ τοῦ κέντρου τοῦ Λ

κύκλου ἵσον δύναται τῷ ὑπὸ ΖΘΚ: μεῖζων ἀρα

110
The surface of the figure circumscribed about the sphere is greater than four times the greatest of the circles in the sphere.

For let there be both the sphere and the circle and the other things the same as were posited before, and let the circle $\Lambda$ be equal to the surface of the given figure circumscribed about the lesser sphere.

Therefore since in the circle $EZ\theta\theta$ there has been inscribed an equilateral polygon with an even number of angles, the [sum of the straight lines] joining the sides of the polygon, being parallel to $Z\Theta$, have the same ratio to $Z\Theta$ as $\Theta K$ to $KZ$ [Prop. 21]; therefore the rectangle contained by one side of the polygon and the straight line equal to the sum of the straight lines joining the angles of the polygon is equal to the rectangle contained by $Z\Theta$, $\Theta K$ [Eucl. vi. 16]; so that the square of the radius of the circle $\Lambda$ is equal to the rectangle contained by $Z\Theta$, $\Theta K$
Εστίν ἡ ἐκ τοῦ κέντρου τοῦ Δ κύκλου τῆς ΘΚ. ἢ δὲ ΘΚ ἵση ἐστὶ τῇ διαμέτρῳ τοῦ AΒΓΔ κύκλου [διπλασία γάρ ἐστὶν τῆς ΧΣ οὐσίας ἐκ τοῦ κέντρου τοῦ AΒΓΔ κύκλου].

Δῆλον οὖν, ὅτι μείζων ἐστὶν ἡ τετραπλάσιος ὁ Δ κύκλος, ουτέστιν ἡ ἐπιφάνεια τοῦ περιγεγραμμένου σχήματος περὶ τὴν ἐλάσσονα σφαῖραν, τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ.

λγ'

Πάσης σφαίρας ἡ ἐπιφάνεια τετραπλάσια ἐστὶ τοῦ μεγίστου κύκλου τῶν ἐν αὐτῇ.

'Εστω γὰρ σφαῖρα τις, καὶ ἐστω τετραπλάσιος τοῦ μεγίστου κύκλου ὁ Α. λέγω, ὅτι ὁ Α ἴσος ἐστὶν τῇ ἐπιφάνειᾳ τῆς σφαίρας.

Εἴ γὰρ μὴ, ἦτοι μείζων ἐστὶν ἡ ἐλάσσων. ἐστώ πρὸτερον μείζων ἡ ἐπιφάνεια τῆς σφαίρας τοῦ κύκλου. ἐστὶ δὴ δύο μεγέθη ἀνίσα ἡ τε ἐπιφάνεια τῆς σφαίρας καὶ ὁ Δ κύκλος· δυνατὸν ἄρα ἐστὶ λαβεῖν δύο εὐθείας ἀνίσους, ὥστε τὴν μείζονα πρὸς τὴν ἐλάσσονα λόγον ἔχειν ἐλάσσονα τοῦ, ὅν ἔχει ἡ 1 διπλασία . . . κύκλου om. Heiberg.

* Because ZΘ>ΘΚ [Eucl. iii. 15].
ARCHIMEDES

[Prop. 29]. Therefore the radius of the circle $\Lambda$ is greater than $\Theta K$. Now $\Theta K$ is equal to the diameter of the circle $\Delta \Gamma \Delta$; it is therefore clear that the circle $\Lambda$, that is, the surface of the figure circumscribed about the lesser sphere, is greater than four times the greatest of the circles in the sphere.

Prop. 33

The surface of any sphere is four times the greatest of the circles in it.

For let there be a sphere, and let $A$ be four times the greatest circle; I say that $A$ is equal to the surface of the sphere.

For if not, either it is greater or less. First, let the surface of the sphere be greater than the circle.

Then there are two unequal magnitudes, the surface of the sphere and the circle $A$; it is therefore possible to take two unequal straight lines so that the greater bears to the less a ratio less than that which the sur-
ἐπιφάνεια τής σφαίρας πρὸς τὸν κύκλον. εἰληφθωσαν αἱ Β, Γ, καὶ τῶν Β, Γ μέση ἀνάλογον ἔστω ἡ Δ, νοείσθω δὲ καὶ ἡ σφαῖρα ἐπιπέδω τετμημένη διά τοῦ κέντρου κατὰ τὸν ΕΖΗΘ κύκλον, νοείσθω δὲ καὶ εἰς τὸν κύκλον ἐγγεγραμμένον καὶ περιγεγραμμένον πολύγωνον, ὅστε ὦμοιον εἶναι τὸ περιγεγραμμένον τῷ ἐγγεγραμμένῳ πολύγωνῷ καὶ τὴν τοῦ περιγεγραμμένου πλευρὰν ἐλάσσονα λόγον ἔχειν τοῦ, ὅν ἔχει ἡ Β πρὸς Δ [καὶ ὁ διπλάσιος ἁρα λόγος τοῦ διπλασίου λόγου ἐστὶν ἐλάσσων. καὶ τοῦ μὲν τῆς Β πρὸς Δ διπλάσιος ἐστὶν ὁ τῆς Β πρὸς τὴν Γ, τῆς δὲ πλευρᾶς τοῦ περιγεγραμμένου πολύγωνον πρὸς τὴν πλευρὰν τοῦ ἐγγεγραμμένου διπλάσιος ὅ τῆς ἐπιφανείας τοῦ περιγεγραμμένου στερεοῦ πρὸς τὴν ἐπιφανείαν τοῦ ἐγγεγραμμένου]1. ἡ ἐπιφάνεια ἁρα τοῦ περιγεγραμμένου σχήματος περὶ τὴν σφαῖραν πρὸς τὴν ἐπιφανείαν τοῦ ἐγγεγραμμένου σχήματος ἐλάσσονα λόγον ἔχει ἡπερ ἡ ἐπιφανεία τῆς σφαίρας πρὸς τὸν Α κύκλον ὅπερ ἀτοπον ἡ μὲν γὰρ ἐπιφάνεια τοῦ περιγεγραμμένου τῆς ἐπιφανείας τῆς σφαίρας μείζων ἐστίν, ἡ δὲ ἐπιφάνεια τοῦ ἐγγεγραμμένου σχήματος τοῦ Α κύκλου ἐλάσσων ἐστὶ [δέδεικται γὰρ ἡ ἐπιφάνεια τοῦ ἐγγεγραμμένου ἐλάσσων τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ τῇ τετραπλασίᾳ, τοῦ δὲ μεγίστου κύκλου τετραπλασίος ἐστὶν ὁ Α κύκλος].2 οὐκ ἁρα ἡ ἐπιφάνεια τῆς σφαίρας μείζων ἐστὶ τοῦ Α κύκλου.

1 καὶ ... ἐγγεγραμμένον om. Heiberg.
2 δέδεικται ... κύκλος “repetitionem inutilem Prop. 25,” om. Heiberg.

114
face of the sphere bears to the circle [Prop. 2]. Let $B$, $\Gamma$ be so taken, and let $\Delta$ be a mean proportional between $B$, $\Gamma$, and let the sphere be imagined as cut through the centre along the [plane of the] circle $EZH\Theta$, and let there be imagined a polygon inscribed in the circle and another circumscribed about it in such a manner that the circumscribed polygon is similar to the inscribed polygon and the side of the circumscribed polygon has [to the side of the inscribed polygon] a ratio less than that which $B$ has to $\Delta$ [Prop. 3]. Therefore the surface of the figure circumscribed about the sphere has to the surface of the inscribed figure a ratio less than that which the surface of the sphere has to the circle $A$; which is absurd; for the surface of the circumscribed figure is greater than the surface of the sphere [Prop. 28], while the surface of the inscribed figure is less than the circle $A$ [Prop. 25]. Therefore the surface of the sphere is not greater than the circle $A$.

*Archimedes would not have omitted: πρὸς τὴν τοῦ ἐγγεγραμμένου.*
GREEK MATHEMATICS

Δέγω δή, οτι οὐδὲ ἐλάσσων. εἰ γὰρ δυνατόν, ἐστω· καὶ ὁμοίως εὐρήσθωσαν αἱ Β, Γ εὐθεῖαι, ὡστε τὴν Β πρὸς Γ ἐλάσσονα λόγον ἔχειν τοῦ, διν ἔχει ο Α κύκλος πρὸς τὴν ἐπιφάνειαν τῆς σφαίρας, καὶ τῶν Β, Γ μέση ἀνάλογον ἡ Δ, καὶ ἐγγεγράφθω καὶ περιγεγράφθω πάλιν, ὡστε τὴν τοῦ περιγεγραμμένου ἐλάσσονα λόγον ἔχειν τοῦ τῆς Β πρὸς Δ [καὶ τὰ διπλάσια ἁρα]. ἡ ἐπιφάνεια ἁρα τοῦ περιγεγραμμένου πρὸς τὴν ἐπιφάνειαν τοῦ ἐγ-

γεγραμμένου ἐλάσσονα λόγον ἔχει ἡπερ [ἡ Β πρὸς Γ. ἡ δὲ Β πρὸς Γ ἐλάσσονα λόγον ἔχει ἡπερ] ἡ

Α κύκλος πρὸς τὴν ἐπιφάνειαν τῆς σφαίρας. ὁπερ ἀτοπον. ἡ μὲν γὰρ τοῦ περιγεγραμμένου ἐπιφάνεια

μείζων ἐστὶ τοῦ Α κύκλου, ἡ δὲ τοῦ ἐγγεγραμμένου ἐλάσσων τῆς ἐπιφανείας τῆς σφαίρας.

Οὐκ ἁρα οὐδὲ ἐλάσσων ἡ ἐπιφάνεια τῆς σφαίρας
tοῦ Α κύκλου. ἐδείχθη δὲ, ὅτι οὐδὲ μείζων ἡ

ἀρα ἐπιφάνεια τῆς σφαίρας ἴση ἐστὶ τῷ Α κύκλῳ,
tουτέστι τῷ τετραπλασίῳ τοῦ μεγίστου κύκλου.

¹ καὶ . . . ἁρα om. Heiberg.
² ἡ Β . . . ἡπερ om. Heiberg.

---

a Archimedes would not have omitted these words.
b On p. 100 n. a it was proved that the area of the inscribed figure is

\[ 4\pi a^2 \sin \frac{\pi}{n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left( 2n - 1 \right) \frac{\pi}{2n} \right], \]

or \[ 4\pi a^2 \cos \frac{\pi}{4n}. \]

On p. 108 n. a it was proved that the area of the circumscribed figure is

\[ 4\pi a^2 \sec^2 \frac{\pi}{4n} \sin \frac{\pi}{4n} \left[ \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \left( 2n - 1 \right) \frac{\pi}{2n} \right], \]

or \[ 4\pi a^2 \sec \frac{\pi}{4n}. \]
ARCHIMEDES

I say now that neither is it less. For, if possible let it be; and let the straight lines B, \( \Gamma \) be similarly found, so that B has to \( \Gamma \) a less ratio than that which the circle A has to the surface of the sphere, and let \( \Delta \) be a mean proportional between B, \( \Gamma \), and let [polygons] be again inscribed and circumscribed, so that the [side] of the circumscribed polygon has [to the side of the inscribed polygon] a less ratio than that of B to \( \Delta \); then the surface of the circumscribed polygon has to the surface of the inscribed polygon a ratio less than that which the circle A has to the surface of the sphere; which is absurd; for the surface of the circumscribed polygon is greater than the circle A, while that of the inscribed polygon is less than the surface of the sphere.

Therefore the surface of the sphere is not less than the circle A. And it was proved not to be greater; therefore the surface of the sphere is equal to the circle A, that is to four times the greatest circle.

When \( n \) is indefinitely increased, the inscribed and circumscribed figures become identical with one another and with the circle, and, since \( \cos \frac{\pi}{4n} \) and \( \sec \frac{\pi}{4n} \) both become unity, the above expressions both give the area of the circle as \( 4\pi a^2 \).

But the first expressions are, when \( n \) is indefinitely increased, precisely what is meant by the integral

\[
4\pi a^2 \cdot \frac{1}{2} \int_0^\pi \sin \phi \, d\phi,
\]

which is familiar to every student of the calculus as the formula for the area of a sphere and has the value \( 4\pi a^2 \).

Thus Archimedes' procedure is equivalent to a genuine integration, but when it comes to the last stage, instead of saying, "Let the sides of the polygon be indefinitely
increased," he prefers to prove that the area of the sphere cannot be either greater or less than $4\pi a^2$. By this double *reductio ad absurdum* he avoids the logical difficulties of dealing with indefinitely small quantities, difficulties that were not fully overcome until recent times.

The procedure by which in this same book Archimedes...
ARCHIMEDES

Prop. 34

Any sphere is four times as great as the cone having a base equal to the greatest of the circles in the sphere and height equal to the radius of the sphere.

For let there be a sphere in which $AB\Gamma\Delta$ is the greatest circle. If the sphere is not four times the aforesaid cone, let it be, if possible, greater than four times; let $\Xi$ be a cone having a base four times the circle $AB\Gamma\Delta$ and height equal to the radius of the sphere; then the sphere is greater than the cone $\Xi$. Accordingly there will be two unequal magnitudes, the sphere and the cone; it is therefore possible to take two unequal straight lines so that finds the surface of the segment of a sphere is equivalent to the integration

$$\pi a^2 \int_0^\alpha 2 \sin \theta \, d\theta = 2\pi a^2 (1 - \cos \alpha).$$

Concurrently Archimedes finds the volumes of a sphere and segment of a sphere. He uses the same inscribed and circumscribed figures, and the procedure is equivalent to multiplying the above formulae by $\frac{1}{4}a$ throughout. Other "integrations" effected by Archimedes are the volume of a segment of a paraboloid of revolution, the volume of a segment of a hyperboloid of revolution, the volume of a segment of a spheroid, the area of a spiral and the area of a segment of a parabola. He also finds the area of an ellipse, but not by a method equivalent to integration. The subject is fully treated by Heath, The Works of Archimedes, pp. cxliii-cliv, to whom I am much indebted in writing this note.

119
Eutocius supplies a proof on these lines. Let the lengths of $K$, $I$, $Θ$, $H$ be $a$, $b$, $c$, $d$. Then $a - b = b - c = c - d$, and it is required to prove that $a : d > a^3 : b^3$.

Take $x$ such that $a : b = b : x$.

Then $a - b : a = b - x : b$,
and since $a > b$, $a - b > b - x$.

But, by hypothesis, $a - b = b - c$.

Therefore $b - c > b - x$,
and so $x > c$. 

1. $σχήματα$ Heiberg, το $σχήμα$ codd.
ARCHIMEDES

the greater will have to the less a less ratio than that which the sphere has to the cone $E$. Therefore let the straight lines $K$, $H$, and the straight lines $I$, $\Theta$, be so taken that $K$ exceeds $I$, and $I$ exceeds $\Theta$ and $\Theta$ exceeds $H$ by an equal quantity; let there be imagined inscribed in the circle $AB\Gamma\Delta$ a polygon the number of whose sides is divisible by four; let another be circumscribed similar to that inscribed so that, as before, the side of the circumscribed polygon has to the side of the inscribed polygon a ratio less than that $K : I$; and let $\Delta \Gamma$, $B\Delta$ be diameters at right angles. Then if, while the diameter $\Delta \Gamma$ remains stationary, the surface in which the polygons lie be revolved, there will result two [solid] figures, one inscribed in the sphere and the other circumscribed, and the circumscribed figure will have to the inscribed the triplicate ratio of that which the side of the circumscribed figure has to the side of the figure inscribed in the circle $AB\Gamma\Delta$ [Prop. 32]. But the ratio of the one side to the other is less than $K : I$ [ex hypothesi]; and so the circumscribed figure has [to the inscribed] a ratio less than $K^3 : I^3$. But $aK : \Pi > K^3 : I^3$; by much more there-

Again, take $y$ such that $b : z = x : y$.

Then, as before $b - x > x - y$.

Therefore, a fortiori, $b - c > x - y$.

But, by hypothesis, $b - c = c - d$.

Therefore $c - d > x - y$.

But $x > c$,

and so $y > d$.

But, by hypothesis, $a : b = b : x = x : y$,

Therefore $a : d > a^3 : b^3$.

[Eucl. v. Def. 10, also vol. i. p. 258 n. b.]

Therefore $a : d > a^3 : b^3$. 121
ἐχει δὲ καὶ ἡ Κ πρὸς Η μεῖζον λόγον ἡ τριπλάσιον τοῦ, ὅν ἐχει ἡ Κ πρὸς Ι [τούτο γὰρ φανερὸν διὰ λημμάτων].¹ πολλῷ ἀρα τὸ περιγραφέν πρὸς τὸ ἐγγραφὲν ἐλάσσονα λόγον ἐχει τοῦ, ὅν ἐχει ἡ Κ πρὸς Η. ἡ δὲ Κ πρὸς Η ἐλάσσονα λόγον ἐχει ἦπερ ἡ σφαῖρα πρὸς τὸν Ξ κώνου καὶ ἐναλλάξῃ ὑπὲρ ἀδύνατον τὸ γὰρ σχήμα τὸ περιγεγραμμένον μεῖζὸν ἐστὶ τῆς σφαίρας, τὸ δὲ ἐγγεγραμμένον ἐλάσσον τοῦ Ξ κώνου [ὅτι δὲ μὲν Ξ κώνος τετραπλάσιος ἐστὶ τοῦ κώνου τοῦ βασιν μὲν ἐχοντος ἴσην τῷ ΑΒΓΔ κύκλῳ, ὑπὸ δὲ ἴσον τῇ ἐκ τοῦ κέντρου τῆς σφαίρας, τὸ δὲ ἐγγεγραμμένον σχήμα ἐλάσσον τοῦ εἰρήμενον κώνου ἡ τετραπλασία τοῦ εἰρήμενου].²

οὐκ ἀρα μεῖζων ἡ τετραπλασία ἡ σφαῖρα τοῦ εἰρήμενου.

Εστω, εἰ δυνατὸν, ἐλάσσων ἡ τετραπλασία ὡστε ἐλάσσων ἐστὶν ἡ σφαῖρα τοῦ Ξ κώνου. εἰλήφθωσαν δή αἱ Κ, Η εὐθείαι, ὡστε τὴν Κ μεῖζον εἶναι τῆς Η καὶ ἐλάσσονα λόγον ἐχει πρὸς αὐτὴν τοῦ, ὅν ἐχει ὃ Ξ κώνος πρὸς τὴν σφαῖραν, καὶ αἱ Θ, Ι ἐκκεῖσθωσαν, καθὼς πρῶτον, καὶ εἰς τὸν ΑΒΓΔ κύκλον νοείσθω πολύγωνον ἐγγεγραμμένον καὶ ἄλλο περιγεγραμμένον, ὡστε τὴν πλευρὰν τοῦ περιγεγραμμένον πρὸς τὴν πλευρὰν τοῦ ἐγγεγραμμένου ἐλάσσονα λόγον ἐχει ἦπερ ἡ Κ πρὸς Ι, καὶ τὰ ἄλλα κατεσκευασμένα τὸν αὐτὸν τρόπον τοῖς πρῶτοι: ἐξει ἀρα καὶ τὸ περιγεγραμμένον στερεὸν σχήμα πρὸς τὸ ἐγγεγραμμένον τριπλασίονα λόγον ἦπερ ἡ πλευρά τοῦ περιγεγραμμένου περὶ τὸν ΑΒΓΔ κύκλον πρὸς τὴν τοῦ ἐγγεγραμμένου. ἡ δὲ πλευρά πρὸς τὴν πλευρὰν ἐλάσσονα λόγον ἐχει

122
fore the circumscribed figure has to the inscribed a ratio less than \( K : H \). But \( K : H \) is a ratio less than that which the sphere has to the cone \( \Xi \) \( [\text{ex hypothesi}] \); [therefore the circumscribed figure has to the inscribed a ratio less than that which the sphere has to the cone \( \Xi \) ] and \( \text{permutando} \), [the circumscribed figure has to the sphere a ratio less than that which the inscribed figure has to the cone] \(^a\); which is impossible; for the circumscribed figure is greater than the sphere [Prop. 28], but the inscribed figure is less than the cone \( \Xi \) [Prop. 27]. Therefore the sphere is not greater than four times the aforesaid cone.

Let it be, if possible, less than four times, so that the sphere is less than the cone \( \Xi \). Let the straight lines \( K, H \) be so taken that \( K \) is greater than \( H \) and \( K : H \) is a ratio less than that which the cone \( \Xi \) has to the sphere [Prop. 2]; let the straight lines \( \Theta, I \) be placed as before; let there be imagined in the circle \( \Lambda \beta \gamma \Delta \) one polygon inscribed and another circumscribed, so that the side of the circumscribed figure has to the side of the inscribed a ratio less than \( K : I \); and let the other details in the construction be done as before. Then the circumscribed solid figure will have to the inscribed the triplicate ratio of that which the side of the figure circumscribed about the circle \( \Lambda \beta \gamma \Delta \) has to the side of the inscribed figure [Prop. 32]. But the ratio of the sides

* A marginal note in one ms. gives these words, which Archimedes would not have omitted.

\(^a\) In one ms. the words "[ex hypothesi]" are written in the margin, which Archimedes would not have done.
Προδεδειγμένων δὲ τούτων φανερῶν, ὅτι πᾶς κύλινδρος βάσιν μὲν ἔχων τὸν μέγιστον κύκλον τῶν ἐν τῇ σφαίρᾳ, ὕψος δὲ ἵσον τῇ διαμέτρῳ τῆς σφαίρας, ἡμιόλιος ἐστὶ τῆς σφαίρας καὶ ἡ ἐπιφάνεια αὐτοῦ μετὰ τῶν βάσεων ἡμιολία τῆς ἐπιφάνειας τῆς σφαίρας.

'Ο μὲν γὰρ κύλινδρος ὁ προειρημένος ἐξαπλάσιος ἐστὶ τοῦ κώνου τοῦ βάσιν μὲν ἔχοντος τὴν αὐτήν, ὕψος δὲ ἵσον τῇ ἐκ τοῦ κέντρου, ὁ δὲ σφαίρα δέδεικται τοῦ αὐτοῦ κώνου τετραπλασία ὀδος. δῆλον δὲ, ὅτι ὁ κύλινδρος ἡμιόλιος ἐστὶ τῆς σφαίρας. πάλιν, ἐπεὶ ἡ ἐπιφάνεια τοῦ κύλινδρου χωρὶς τῶν βάσεων ἵση δέδεικται κύκλω, οὔ ἢ ἐκ

1 πέρισμα. The title is not found in some mss.
ARCHIMEDES

is less than $K : I$ [ex hypothesi]; therefore the circumscribed figure has to the inscribed a ratio less than $K^3 : I^3$. But $K : H > K^3 : I^3$; and so the circumscribed figure has to the inscribed a ratio less than $K : H$. But $K : H$ is a ratio less than that which the cone $\Xi$ has to the sphere [ex hypothesi]; [therefore the circumscribed figure has to the inscribed a ratio less than that which the cone $\Xi$ has to the sphere] a; which is impossible; for the inscribed figure is less than the sphere [Prop. 28], but the circumscribed figure is greater than the cone $\Xi$ [Prop. 31, coroll.]. Therefore the sphere is not less than four times the cone having its base equal to the circle $AB\Gamma\Delta$, and height equal to the radius of the sphere. But it was proved that it cannot be greater; therefore it is four times as great.

[corollary]

From what has been proved above it is clear that any cylinder having for its base the greatest of the circles in the sphere, and having its height equal to the diameter of the sphere, is one-and-a-half times the sphere, and its surface including the bases is one-and-a-half times the surface of the sphere.

For the aforesaid cylinder is six times the cone having the same basis and height equal to the radius [from Eucl. xii. 10], while the sphere was proved to be four times the same cone [Prop. 34]. It is obvious therefore that the cylinder is one-and-a-half times the sphere. Again, since the surface of the cylinder excluding the bases has been proved equal to a circle

a These words, which Archimedes would not have omitted, are given in a marginal note to one ms.
GREEK MATHEMATICS

tοῦ κέντρου μέση ἀνάλογον ἐστὶ τῆς τοῦ κυλίνδρου πλευρᾶς καὶ τῆς διαμέτρου τῆς βάσεως, τοῦ δὲ εἰρημένου κυλίνδρου τοῦ περὶ τὴν σφαῖραν ἡ πλευρά ἵση ἐστὶ τῇ διαμέτρῳ τῆς βάσεως [δῆλον, ὅτι ἡ μέση αὐτῶν ἀνάλογον ἵση γίνεται τῇ διαμέτρῳ τῆς βάσεως], ¹ δὲ κύκλος ὁ τὴν ἐκ τοῦ κέντρου ἔχων ὤσιν τῇ διαμέτρῳ τῆς βάσεως τετραπλάσιος ἐστὶ τῆς βάσεως, τουτέστι τοῦ μεγίστου κύκλου τῶν ἐν τῇ σφαίρᾳ, ἔσται ἄρα καὶ ἡ ἐπιφάνεια τοῦ κυλίνδρου χωρίς τῶν βάσεων τετραπλάσια τοῦ μεγίστου κύκλου. ὅλη ἄρα μετά τῶν βάσεων ἡ ἐπιφάνεια τοῦ κυλίνδρου ἐξαπλασία ἔσται τοῦ μεγίστου κύκλου. ἔστω δὲ καὶ ἡ τῆς σφαίρας ἐπιφάνεια τετραπλασία τοῦ μεγίστου κύκλου. ὅλη ἄρα ἡ ἐπιφάνεια τοῦ κυλίνδρου ἡμιολία ἔστι τῆς ἐπιφάνειας τῆς σφαίρας.

(c) Solution of a Cubic Equation

Archim. De Sphaera et Cyl. ii., Prop. 4, Archim. ed. Heiberg i. 186. 15–192. 6

Τὴν δοθείσαν σφαῖραν τεμεῖν, ὥστε τὰ τμήματα τῆς σφαίρας πρὸς ἄλληλα λόγον ἔχειν τὸν αὐτῶν τῷ δοθέντι.

¹ δῆλον . . . βάσεως om. Heiberg.

* As the geometrical form of proof is rather diffuse, and may conceal from the casual reader the underlying nature of the operation, it may be as well to state at the outset the various stages of the proof. The problem is to cut a given sphere by a plane so that the segments shall have a given ratio, and the stages are:

(a) Analysis of this main problem in which it is reduced to a particular case of the general problem, “so to cut a given straight line ΔΖ at Χ that ΧΖ bears to the given
ARCHIMEDES

whose radius is a mean proportional between the side of the cylinder and the diameter of the base [Prop. 13], and the side of the aforementioned cylinder circumscribing the sphere is equal to the diameter of the base, while the circle having its radius equal to the diameter of the base is four times the base [Eucl. xii. 2], that is to say, four times the greatest of the circles in the sphere, therefore the surface of the cylinder excluding the bases is four times the greatest circle; therefore the whole surface of the cylinder, including the bases, is six times the greatest circle. But the surface of the sphere is four times the greatest circle. Therefore the whole surface of the cylinder is one-and-a-half times the surface of the sphere.

(c) Solution of a Cubic Equation

Archimedes, On the Sphere and Cylinder ii., Prop. 4, Archim. ed. Heiberg i. 186. 15-192. 6

To cut a given sphere, so that the segments of the sphere shall have, one towards the other, a given ratio. *

straight line the same ratio as a given area bears to the square on ΔX */; in algebraical notation, to solve the equation

\[ \frac{a-x}{b} = \frac{c^2}{x^2}, \text{ or } x^2(a-x) = bc^2. \]

(b) Analysis of this general problem, in which it is shown that the required point can be found as the intersection of a parabola \([ax^2 = c^2y]\) and a hyperbola \([(a-x)y = ab]\). It is stated, for the time being without proof, that \(x^2(a-x)\) is greatest when \(x = \frac{1}{3}a\); in other words, that for a real solution \(bc^2 > \frac{1}{37}a^3\).

(c) Synthesis of this general problem, according as \(bc^2\) is greater than, equal to, or less than \(\frac{1}{37}a^3\). If it be greater, there is no real solution; if equal, there is one real solution; if less, there are two real solutions.

(d) Proof that \(x^2(a-x)\) is greatest when \(x = \frac{1}{3}a\), deferred
GREEK MATHEMATICS

*Estw ἡ δοθεῖσα σφαῖρα ἡ ABΓΔ. δεὶ δὴ αὐτὴν τεμεῖν ἐπιπέδω, ὡστε τὰ τμῆματα τῆς σφαῖρας πρὸς ἄλληλα λόγον ἔχειν τὸν δοθέντα.

Τετμήσθω διὰ τῆς ΑΓ ἐπιπέδων λόγον ἀρα τοῦ ΑΔΓ τμῆματος τῆς σφαῖρας πρὸς τὸ ΑΒΓ τμῆμα τῆς σφαῖρας δοθεῖσ. τετμήσθω δὲ ἡ σφαῖρα διὰ τοῦ κέντρου, καὶ ἔστω ἡ τομὴ μέγιστος κύκλος ὁ ABΓΔ, κέντρον δὲ τὸ K καὶ διάμετρος ἡ ΔΒ, καὶ πεποιηθῶ, ως μὲν συναμφότερος ἡ ΚΔΧ πρὸς ΔΧ, οὕτως ἡ PX πρὸς ΧΒ, ως δὲ συναμφότερος ἡ ΚΒΧ πρὸς ΒΧ, οὕτως ἡ ΛΧ πρὸς ΧΔ, καὶ ἐπεζεύχθουσαν αἱ ΑΔ, ΛΓ, ΑΡ, ΡΓ. ἔσοδ ἀρα ἔστιν ὁ μὲν ΑΔΓ κώνος τῷ ΑΔΓ τμῆματι τῆς σφαῖρας, ὃ δὲ ΑΡΓ τῷ ΑΒΓ· λόγος ἀρα καὶ τοῦ ΑΔΓ κώνον πρὸς τὸν ΑΡΓ κώνον δοθείσ. ως δὲ ὁ κώνος πρὸς τὸν κώνον, οὕτως ἡ ΛΧ πρὸς ΧΡ [ἐπείπερ τὴν αὐτὴν βάσιν ἐξουσιν τὸν περὶ διάμετρον τῆς ΑΓ κύκλον]. λόγος ἀρα καὶ τῆς ΛΧ πρὸς ΧΡ δοθείσ. καὶ διὰ ταύτα τοὺς πρό-

1 επείπερ ... κύκλον om. Heiberg.

in (b). This is done in two parts, by showing that (1) if \( x \) has any value less than \( \frac{2}{3}a \), (2) if \( x \) has any value greater than \( \frac{2}{3}a \), then \( a^2(a-x) \) has a smaller value than when \( x = \frac{2}{3}a \).

(e) Proof that, if \( bc^2 < \frac{4}{9}a^2 \), there are always two real solutions.

(f) Proof that, in the particular case of the general problem to which Archimedes has reduced his original problem, there is always a real solution.

(g) Synthesis of the original problem.

Of these stages, (a) and (g) alone are found in our texts of Archimedes; but Eutocius found stages (b)-(d) in an old book, which he took to be the work of Archimedes; and he added stages (e) and (f) himself. When it is considered that all these stages are traversed by rigorous geometrical
ARCHIMEDES

Let $AB\Gamma\Delta$ be the given sphere; it is required so to cut it by a plane that the segments of the sphere shall have, one towards the other, the given ratio.

Let it be cut by the plane $A\Gamma$; then the ratio of the segment $A\Delta\Gamma$ of the sphere to the segment $AB\Gamma$ of the sphere is given. Now let the sphere be cut through the centre [by a plane perpendicular to the plane through $A\Gamma$], and let the section be the great circle $AB\Gamma\Delta$ of centre $K$ and diameter $\Delta B$, and let [$A, P$ be taken on $BA$ produced in either direction so that]

$$K\Delta + \Delta X : \Delta X = PX : XB,$$
$$KB + BX : BX = \Delta X : XD,$$

and let $AA, A\Gamma, AP, P\Gamma$ be joined; then the cone $A\Delta\Gamma$ is equal to the segment $A\Delta\Gamma$ of the sphere, and

the cone $A\Gamma\Delta$ to the segment $AB\Gamma$ [Prop. 2]; therefore the ratio of the cone $A\Delta\Gamma$ to the cone $A\Gamma\Delta$ is given. But cone $A\Delta\Gamma :$ cone $A\Gamma\Delta = \Delta X : XP.$

Therefore the ratio $\Delta X : XP$ is given. And in the methods, the solution must be admitted a veritable tour de force. It is strictly analogous to the modern method of solving a cubic equation, but the concept of a cubic equation did not, of course, come within the purview of the ancient mathematicians.

$^a$ Since they have the same base.
The words καὶ ... πρὸς BX. The words καὶ ... ἀπὸ ΔΧ are shown by Eutocius's comment to be an interpolation. The words πάλιν ... πρὸς BX and καὶ ... πρὸς ZX must also be interpolated, as, in order to prove that ΔΔ : ΔΧ is given, Eutocius first proves that BΖ : ZΧ = ΔΔ : ΔΧ, which he would hardly have done if Archimedes had himself provided the proof.

* This is proved by Eutocius thus:

Since

\[ \text{KB} + \text{BX} : \text{XB} = \Delta \text{Δ} : \text{ΧΔ}, \]

and permutando,

\[ \text{KB} : \text{BP} = \Delta \text{Δ} : \text{ΧΔ}, \]

i.e.,

\[ \text{KB} + \text{BX} : \text{XB} = \Delta \text{Δ} : \text{ΧΔ}, \]

Since

\[ \text{KA} + \Delta \text{Χ} : \Delta \text{Χ} = \text{ΡΧ} : \text{XB}, \]

\[ \text{KA} : \Delta \text{Χ} = \text{ΡΒ} : \text{BX}, \]

\[ \text{KA} : \Delta \text{Χ} = \text{ΡΧ} : \text{XB}, \]

Again, since

\[ \text{KA} : \Delta \text{Χ} = \text{ΡΧ} : \text{XB}, \]
same way as in a previous proposition [Prop. 2], by construction,

\[ \Delta \Delta : K \Delta = KB : BP = \Delta X : XB. \]

And since \[ PB : BK = K \Delta : \Delta \Delta, \quad \text{[Eucl. v. 7, coroll.]} \]
\[ PK : KB = K \Lambda : \Delta \Delta, \quad \text{[Eucl. v. 18]} \]
i.e., \[ PK : K \Delta = K \Lambda : \Delta \Delta. \]
\[ \therefore \quad P \Lambda : K \Lambda = K \Lambda : \Delta \Delta. \quad \text{[Eucl. v. 12]} \]
\[ \therefore \quad P \Lambda : \Delta \Delta = \Lambda K^2. \quad \text{[Eucl. vi. 17]} \]
\[ \therefore \quad P \Lambda : \Delta \Delta = K \Lambda^2 : \Delta \Delta^2. \]

And since \[ \Delta \Delta : \Delta K = \Delta X : XB, \]
\[ \text{inverting et componeando,} \quad K \Lambda : \Delta \Delta = B \Lambda : \Delta X. \quad \text{[Eucl. v. 7, coroll. and v. 18]} \]

Let BZ be placed equal to KB. It is plain that [Z] will fall beyond P. Since the ratio \( \Delta \Delta : \Delta X \) is given, therefore the ratio \( P \Lambda : \Delta X \) is given. Then,

\[ \text{dirimendo et permutando} \quad \Delta X : XB = \Delta \Delta : \Delta K. \]

Now \[ \Delta X : XB = KB : BP. \]

Therefore \[ \Delta \Delta : \Delta K = \Delta X : XB = KB : BP. \]

\[ \therefore \quad BZ > BP. \]
\[ \therefore \quad BZ > BP. \]

As Eutocius’s note shows, what Archimedes wrote was: “Since the ratio \( \Delta \Delta : \Delta X \) is given, and the ratio \( P \Lambda : \Delta X \), therefore the ratio \( P \Lambda : \Delta \Delta \) is also given.” Eutocius’s proof is:

Since \[ KB + BX : BX = \Delta X : X \Delta, \]
\[ \therefore \quad ZX : XB = \Delta X : X \Delta; \]
\[ \therefore \quad XZ : ZB = X \Lambda : \Delta \Delta; \]
\[ \therefore \quad BZ : ZX = \Delta \Delta : \Delta X. \]

But the ratio \( BZ : ZX \) is given because \( ZB \) is equal to the radius of the given sphere and \( BX \) is given. Therefore \( \Delta \Delta : \Delta X \) is given.

Again, since the ratio of the segments is given, the ratio of
GREEK MATHEMATICS

dotheis. ἐπεὶ οὖν ὁ τῆς ΡΛ πρὸς ΛΧ λόγος συν-
ήπται ἐκ τε τοῦ, ὅν ἐχει ἡ ΡΛ πρὸς ΛΔ, καὶ ἡ
ΔΛ πρὸς ΛΧ, ἀλλ' ὡς μὲν ἡ ΡΛ πρὸς ΛΔ, τὸ
ἀπὸ ΔΒ πρὸς τὸ ἀπὸ ΔΧ, ὡς δὲ ἡ ΔΔ πρὸς ΛΧ,
οὔτως ἡ ΒΖ πρὸς ZX, ὁ ἄρα τῆς ΡΛ πρὸς ΛΧ
λόγος συνήπται ἐκ τε τοῦ, ὅν ἐχει τὸ ἀπὸ ΒΔ
πρὸς τὸ ἀπὸ ΔΧ, καὶ ἡ ΒΖ πρὸς ZX. πεποιήθω
δε, ὡς ἡ ΡΛ πρὸς ΛΧ, ἡ ΒΖ πρὸς ΖΘ· λόγος δὲ
tῆς ΡΛ πρὸς ΛΧ dotheis: λόγος ἄρα καὶ τῆς ΒΖ
πρὸς ΖΘ dotheis. dotheisa δἐ ἡ ΒΖ—ίση γάρ ἐστὶ
tῆ ἐκ τοῦ κέντρου· dotheisa ἄρα καὶ ἡ ΖΘ. καὶ
ὁ τῆς ΒΖ ἄρα λόγος πρὸς ΖΘ συνήπται ἐκ τε τοῦ,
ὅν ἐχει τὸ ἀπὸ ΒΔ πρὸς τὸ ἀπὸ ΔΧ, καὶ ἡ ΒΖ
πρὸς ZX. ἀλλ' ὁ ΒΖ πρὸς ΖΘ λόγος συνήπται
ἐκ τε τοῦ τῆς ΒΖ πρὸς ZX καὶ τοῦ τῆς ZX πρὸς
ΖΘ [κοινὸς ἄφθορος ὁ τῆς ΒΖ πρὸς ZX].
λοιπὸν ἄρα ἐστίν, ὡς τὸ ἀπὸ ΒΔ, τοῦτέστι δοθὲν,
πρὸς τὸ ἀπὸ ΔΧ, οὔτως ἡ ΧΖ πρὸς ΖΘ, τοῦτέστι
πρὸς δοθὲν. καὶ ἐστὶν dotheisα ἡ ΖΔ εὐθεία·
eυθείαν ἄρα dotheisαν τὴν ΔΖ τεμεῖν δεῖ κατὰ τὸ
Χ καὶ ποιεῖν, ὡς τὴν ΧΖ πρὸς δοθὲιν [τὴν ΖΘ],
οὔτως τὸ δοθὲν [τὸ ἀπὸ ΒΔ]² πρὸς τὸ ἀπὸ ΔΧ.
tοῦτο οὔτως ἀπλῶς μὲν λεγόμενον ἔχει διορισμὸν,

¹ κοινὸς ... πρὸς ZX. Eutocius's comment shows that
these words are interpolated.
² τὴν ΖΘ, τὸ ἀπὸ ΒΔ. Eutocius's comments show these
words to be glosses.

the cones ΛΛΓ', ΑΡΓ is also given, and therefore the ratio
ΛΧ:ΧΡ. Therefore the ratio ΡΛ:ΛΧ is given. Since
the ratios ΡΛ:ΛΧ and ΛΔ:ΛΧ are given, it follows that
the ratio ΡΛ:ΛΔ is given.

132
since the ratio \( PA : AX \) is composed of the ratios \( PA : \Delta \Delta \) and \( \Delta \Delta : AX \),
and since \( \Delta \Delta = \Delta B^2 : \Delta X^2,^a \)
\( \Delta \Delta : AX = BZ : ZX, \)
therefore the ratio \( PA : AX \) is composed of the ratios \( B\Delta^2 : \Delta X^2 \) and \( BZ : ZX \). Let \( [\Theta be chosen so that] \)
\[ PA : AX = BZ : Z\Theta. \]

Now the ratio \( PA : AX \) is given; therefore the ratio \( ZB : Z\Theta \) is given. Now \( BZ \) is given—for it is equal to the radius; therefore \( Z\Theta \) is also given. Therefore \(^b \) the ratio \( BZ : Z\Theta \) is composed of the ratios \( B\Delta^2 : \Delta X^2 \) and \( BZ : ZX \). But the ratio \( BZ : Z\Theta \) is composed of the ratios \( BZ : ZX \) and \( ZX : Z\Theta \). Therefore, the remainder \(^c \) \( B\Delta^2 : \Delta X^2 = XZ : Z\Theta \), in which \( B\Delta^2 \) and \( Z\Theta \) are given. And the straight line \( Z\Delta \) is given; therefore \textit{it is required so to cut the given straight line} \( \Delta Z \) at \( X \) that \( XZ \) bears to a given straight line the same ratio as a given area bears to the square on \( \Delta X \). When the problem is stated in this general form,\(^d \) it is necessary to investigate the limits of possibility,

\(^a \) For \[ PA : \Delta \Delta = \Delta K^2 : \Delta \Delta = B\Delta^2 : \Delta X. \]

\(^b \) "Therefore" refers to the last equation.

\(^c \) \text{i.e. the remainder in the process given fully by Eutocius as follows:}

\[ (B\Delta^2 : \Delta X^2) \cdot (BZ : ZX) = BZ : Z\Theta = (BZ : ZX) \cdot (XZ : Z\Theta). \]

Removing the common element \( BZ : ZX \) from the extreme terms, we find that the remainder \( B\Delta^2 : \Delta X^2 = XZ : Z\Theta. \)

\(^d \) In algebraic notation, if \( \Delta X = x \) and \( \Delta Z = a \), while the given straight line is \( b \) and the given area is \( c^2 \), then

\[ \frac{a - x}{b} = \frac{c^2}{x^2}, \]

or

\[ x^2(a - x) = bc^2. \]
GREEK MATHEMATICS

προστιθεμένων δὲ τῶν προβλημάτων τῶν ἐνθάδε υπαρχόντων [τουτέστι τοῦ τε διπλασίαν εἶναι τὴν \(\Delta B \) τῆς \(\Delta ZB\) καὶ τοῦ μείζονα τῆς \(\Theta\) τῆς \(\Delta ZB\), ώς κατὰ τὴν ἀνάλυσιν] \(^1\) οὐκ ἔχει διορισμὸν· καὶ ἔσται τὸ πρόβλημα τοιοῦτον· δύο δοθεῖσαν εὐθεῖαν τῶν \(\Delta B\), \(\Delta Z\) καὶ διπλασίας οὕσης τῆς \(\Delta B\) τῆς \(\Delta Z\) καὶ σημείου ἐπὶ τῆς \(\Delta Z\) τοῦ \(\Theta\) τεμεῖν τὴν \(\Delta B\) κατὰ τὸ \(\chi\) καὶ ποιεῖν, ὡς τὸ ἀπὸ \(\Delta B\) πρὸς τὸ ἀπὸ \(\Delta X\), τὴν \(\chi Z\) πρὸς \(\Theta\). ἐκάτερα δὲ ταῦτα ἐπὶ τέλει ἀναλυθήσεται τε καὶ συντεθήσεται.


'Επὶ τέλει μὲν τὸ προρηθὲν ἐπηγγείλατο δείξαι, ἐν οὐδενὶ δὲ τῶν ἀντιγράφων εὐρεῖν ἐνεστὶ τὸ ἐπάγγελμα. ὃθεν καὶ Διονυσόδωρον μὲν εὐρίσκομεν μὴ τῶν αὐτῶν ἐπιτυχόντα, ἀδυνατήσαντα δὲ ἐπιβαλεῖν τῷ καταλειφθέντι λήμματι, ἐφ’ ἑτέραν ὀδὸν τοῦ ὀλού προβλήματος ἔλθειν, ἡντινα ἔξης γράφομεν. Διοκλῆς μέντοι καὶ αὐτὸς ἐν τῷ Περὶ πυρῶν αὐτῶ συγγεγραμμένῳ βιβλίῳ ἐπηγγέλθαι νομίζων τὸν 'Αρχιμήδη, μὴ πεποιηκέναι δὲ τὸ ἐπάγγελμα, αὐτὸς ἀναπληροῦν ἐπιχειρήσειν, καὶ τὸ ἐπιχείρημα ἔξης γράφομεν· ἐστὶν γὰρ καὶ αὐτὸ οὐδένα μὲν ἔχων πρὸς τὰ παραλειμμένα λόγον, ὀμοίως δὲ τῷ Διονυσόδωρῳ δι’ ἑτέρας ἀποδείξεως κατασκευάζον τὸ πρόβλημα. ἐν τινὶ μέντοι παλαιῷ

\(^1\) τουτέστι . . . ἀνάλυσιν. Eutocius’s notes make it seem likely that these words are interpolated.

\(^a\) In the technical language of Greek mathematics, the 134
but under the conditions of the present case no such investigation is necessary. In the present case the problem will be of this nature: Given two straight lines $BA, BZ$, in which $BA = 2BZ$, and a point $\Theta$ upon $BZ$, so to cut $DB$ at $X$ that

$$BA^2 : AX^2 = XZ : Z\Theta;$$

and the analysis and synthesis of both problems will be given at the end.

Eutocius, Commentary on Archimedes' Sphere and Cylinder ii., Archim. ed. Heiberg iii. 130. 17-150. 22

He promised that he would give at the end a proof of what is stated, but the fulfilment of the promise cannot be found in any of his extant writings. Dionysodorus also failed to light on it, and, being unable to tackle the omitted lemma, he approached the whole problem in an altogether different way, which I shall describe in due course. Diocles, indeed, in his work On Burning Mirrors maintained that Archimedes made the promise but had not fulfilled it, and he undertook to supply the omission himself, which attempt I shall also describe in its turn; it bears, however, no relation to the missing discussion, but, like that of Dionysodorus, it solves the problem by a construction reached by a different proof. But general problem requires a diorismos, for which v. vol. i. p. 151 n. 6 and p. 396 n. 9. In algebraic notation, there must be limiting conditions if the equation

$$x^2(a - x) = bc^2$$

is to have a real root lying between 0 and $a$.

Having made this promise, Archimedes proceeded to give the formal synthesis of the problem which he had thus reduced.
GREEK MATHEMATICS

βιβλίω—οὐδὲ γὰρ τῆς εἰς πολλὰ ξητήσεως ἀπέστημεν—ἐντετύχαμεν θεωρήμασι γεγραμμένοις οὐκ ὁλίγην μὲν τὴν ἐκ τῶν πταισμάτων ἐξουσι αὐτοῖς περί τε τὰς καταγραφὰς πολυτρόπως ἡμαρτημένοις, τῶν μέντοι ξητουμένων εἴξον τὴν ὑπόστασιν, ἐν μέρει δὲ τὴν Ἀρχιμήδεϊ φίλην Δωρίδα γλώσσαν ἀπέσωζον καὶ τοὺς συνήθεις τῷ ἀρχαῖῳ τῶν πραγμάτων ὀνόμασιν ἐγέραππε τῆς μὲν παραβολῆς ὀρθογώνιον κάων τομῆς ὀνομαζόμενης, τῆς δὲ ὑπερβολῆς ἀμβλυωνίου κάων τομῆς, ὡς ἐξ αὐτῶν διανοεῖσθαι, μὴ ἄρα καὶ αὐτὰ εἰσπε τῇ ἐν τῷ τέλει ἐπηγγελµένα γράφεσθαι. οἶον σπουδαίτερον ἐντυγχάνοντες αὐτὸ μὲν τῷ ῥήτον, ὡς γέγραπται, διὰ πλῆθος, ὡς εἰρηται, τῶν πταισμάτων δυσχερὲς εὐρόντες τὰς ἐννοιάς κατὰ μικρὸν ἀποσυλήσαντες κοινοτέρα καὶ σαφεστέρα κατὰ τὸ δυνατὸν λέξει γράφομεν. καθόλου δὲ πρῶτον τὸ θεώρημα γραφήσεται, ἵνα τὸ λεγόμενον ὑπ’ αὐτοῦ σαφηνοθῇ περὶ τῶν διορισμῶν· εἰστα καὶ τοὺς ἀναλελυμένους ἐν τῷ προβλήματι προσαρμοσθῆσεται.

"Εὐθείας δοθείσης τῆς ΑΒ καὶ ἐτέρας τῆς ΑΓ καὶ χωρίου τοῦ Δ προκείσθω λαβεῖν ἐπὶ τῆς ΑΒ σημείον ως τὸ Ε, ὅστε εἶναι, ὡς τὴν ΑΕ πρὸς ΑΓ, οὔτω τὸ Δ χωρίον πρὸς τὸ ἀπὸ ΕΒ. "Γεγονέτω, καὶ κείσθω ἡ ΑΓ πρὸς όρθὰς τῆς ΑΒ, καὶ ἐπιζευγχείσα ἡ ΓΕ διήχθω ἐπὶ τὸ Ζ, καὶ ἡχθὼ διὰ τοῦ Γ τῆς ΑΒ παράλληλον ἡ ΓΗ, διὰ δὲ τοῦ Β τῆς ΑΓ παράλληλος ἡ ΖΒΗ συμπίπτουσα
ARCHIMEDES

in a certain ancient book—for I pursued the inquiry thoroughly—I came upon some theorems which, though far from clear owing to errors and to manifold faults in the diagrams, nevertheless gave the substance of what I sought, and furthermore preserved in part the Doric dialect beloved by Archimedes, while they kept the names favoured by ancient custom, the parabola being called a section of a right-angled cone and the hyperbola a section of an obtuse-angled cone; in short, I felt bound to consider whether these theorems might not be what he had promised to give at the end. For this reason I applied myself with closer attention, and, although it was difficult to get at the true text owing to the multitude of the mistakes already mentioned, gradually I routed out the meaning and now set it out, so far as I can, in more familiar and clearer language. In the first place the theorem will be treated generally, in order to make clear what he says about the limits of possibility; then will follow the special form it takes under the conditions of his analysis of the problem.

"Given a straight line AB and another straight line $\Gamma$ and an area $\Delta$, let it be required to find a point $E$ on $AB$ such that $AE : \Gamma = \Delta : EB^2$.

"Suppose it found, and let $\Gamma$ be at right angles to $AB$, and let $\Gamma E$ be joined and produced to $Z$, and through $\Gamma$ let $\Gamma H$ be drawn parallel to $AB$, and through $B$ let $ZB H$ be drawn parallel to $\Gamma$, meeting
GREEK MATHEMATICS

ἐκατέρα τῶν ΓΕ, ΓΗ, καὶ συμπεπληρώσθω τὸ ΗΘ παραλληλόγραμμον, καὶ διὰ τοῦ Ε ὀποτέρα τῶν ΓΘ, ΗΖ παράλληλος ἡχθω ἡ ΚΕΛ, καὶ τῷ Δ ἴσον ἕστω τὸ ὑπὸ ΓΗΜ.

"Επει οὖν ἑστιν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὖτως τὸ Δ πρὸς τὸ ἀπὸ ΕΒ, ὡς δὲ ἡ ΕΑ πρὸς ΑΓ; οὖτως ἡ ΓΗ πρὸς ΗΖ, ὡς δὲ ἡ ΓΗ πρὸς ΗΖ, οὖτως τὸ ἀπὸ ΓΗ πρὸς τὸ ὑπὸ ΓΗΖ, ὡς ἀρα τὸ ἀπὸ ΓΗ πρὸς τὸ ὑπὸ ΓΗζ, οὖτως τὸ Δ πρὸς τὸ ἀπὸ ΕΒ, τουτέστι πρὸς τὸ ἀπὸ ΚΖ· καὶ ἐναλλάξ, ὡς τὸ ἀπὸ ΓΗ πρὸς τὸ Δ, τουτέστι πρὸς τὸ ὑπὸ ΓΗΜ, οὖτως τὸ ὑπὸ ΓΗζ πρὸς τὸ ἀπὸ ΖΚ. ἀλλ' ὡς τὸ ἀπὸ ΓΗ πρὸς τὸ ὑπὸ ΓΗΜ, οὖτως ἡ ΓΗ πρὸς ΗΜ· καὶ ὡς ἀρα ἡ ΓΗ πρὸς ΗΜ, οὖτως τὸ ὑπὸ ΓΗζ πρὸς τὸ ἀπὸ ΖΚ. ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΜ, τῆς ΗΖ κοινοὶ ύψους λαμβανομένης οὖτως τὸ ὑπὸ ΓΗζ πρὸς τὸ ὑπὸ ΜΗζ· ὡς ἀρα τὸ ὑπὸ ΓΗζ πρὸς τὸ ὑπὸ ΜΗζ, οὖτως τὸ ὑπὸ ΓΗζ πρὸς τὸ ἀπὸ ΖΚ· ἴσον ἀρα τὸ ὑπὸ ΜΗζ τὰ ἀπὸ ΖΚ. ἐὰν ἀρα περὶ ἄξονα τῆς ΖΗ

138
both $\Gamma E$ and $\Gamma H$, and let the parallelogram $H\Theta$ be completed, and through $E$ let $KE\Delta$ be drawn parallel to either $\Gamma \Theta$ or $HZ$, and let $[M$ be taken so that]

\[ \Gamma H \cdot HM = \Delta. \]

"Then, since \[ \frac{EA}{AG} = \frac{\Delta}{EB^2} \quad [ex \ hyp.], \]
and \[ \frac{EA}{AG} = \frac{\Gamma H}{HZ}, \]
and \[ \frac{\Gamma H}{HZ} = \frac{\Gamma H^2}{\Gamma H \cdot HZ}, \]

\[ \therefore \quad \frac{\Gamma H^2}{\Gamma H \cdot HZ} = \frac{\Delta}{EB^2} = \frac{\Delta}{KZ^2}; \]

and, permutando, \[ \frac{\Gamma H^2}{\Delta} = \frac{\Gamma H \cdot HZ}{ZK^2}, \]
\[ i.e., \quad \frac{\Gamma H^2}{\Gamma H \cdot HM} = \frac{\Gamma H \cdot HZ}{ZK^2}. \]

But \[ \frac{\Gamma H^2}{\Gamma H \cdot HM} = \frac{\Gamma H}{HM}; \]
\[ \therefore \quad \frac{\Gamma H}{HM} = \frac{\Gamma H \cdot HZ}{ZK^2}. \]

But, by taking a common altitude $HZ$,

\[ \frac{\Gamma H}{HM} = \frac{\Gamma H \cdot HZ}{MH \cdot HZ}; \]
\[ \therefore \quad \frac{\Gamma H \cdot HZ}{MH \cdot HZ} = \frac{\Gamma H}{HZ} = \frac{ZK^2}{ZK^2}; \]
\[ \therefore \quad MH \cdot HZ = ZK^2. \]
GREEK MATHEMATICS

γραφὴ διὰ τοῦ Η παραβολῆ, ὡστε τὰς καταγομένας δύνασθαι παρὰ τὴν ΗΜ, ἢξει διὰ τοῦ Κ, καὶ ἐσται θέσει δεδομένη διὰ τὸ δεδομένην εἶναι τὴν ΗΜ τῷ μεγέθει περιέχουσαν μετὰ τῆς ΗΓ δεδομένης δοθέν τὸ Δ. τὸ ἀρα Κ ἀπτεται θέσει δεδομένης παραβολῆς. γεγράφθω οὐν, ὡς εἰρηται, καὶ ἐστω ὡς ἡ ΗΚ.

"Πάλιν, ἐπειδή τὸ ΘΛ χωρίον ἵσον ἐστὶ τῷ ΓΒ, τουτέστι τὸ ὑπὸ ΘΚΛ τῷ ὑπὸ ΑΒΗ, ἐὰν διὰ τοῦ Β περὶ ἄσυμπτωτος τὰς ΘΓ, ΓΗ γραφὴ ύπερβολῆ, ἢξει διὰ τοῦ Κ διὰ τὴν ἀντιστροφὴν τοῦ ἡ' θεωρῆματος τοῦ δευτέρου βιβλίου τῶν Ἀπολλωνίου Κωνικῶν στοιχείων, καὶ ἐσται θέσει δεδομένη διὰ τὸ καὶ ἐκατέραν τῶν ΘΓ, ΓΗ, ἢτι μὴν καὶ τὸ Β τῇ θέσει δεδόσθαι. γεγράφθω, ὡς εἰρηται, καὶ ἐστω ὡς ἡ ΚΒ· τὸ ἀρα Κ ἀπτεται θέσει δεδομένης ύπερβολῆς. ἦπτετο δὲ καὶ θέσει δεδομένης παραβολῆς· δέδοται ἂρα τὸ Κ. καὶ ἐστὶν ἀπ' αὐτοῦ κάθετος ἡ ΚΕ ἐπὶ θέσει δεδομένην τὴν ΑΒ· δέδοται ἂρα τὸ Ε. ἐπεὶ οὖν ἐστω, ὡς ἡ ΕΑ πρὸς τὴν δοθὲν τὴν ΑΓ, ὀὕτως δοθὲν τὸ Δ πρὸς τὸ ἀπὸ ΕΒ, δύο στερεῶν, ὡν βάσεις τὸ ἀπὸ ΕΒ καὶ τὸ Δ, ὑπη δὲ αἱ ΕΑ, ΑΓ, ἀντιπεπόν.

Let \( AB = a \), \( AG = b \), and \( \Delta = GH \). \( HM = c^2 \), so that \( HM = \frac{c^2}{a} \).

Then if \( \Delta \) be taken as the axis of \( x \) and \( HZ \) as the axis of \( y \), the equation of the parabola is

\[
x^2 = \frac{c^2}{a} y,
\]

and the equation of the hyperbola is

\[
(a - x)y = ab.
\]

Their points of intersection give solutions of the equation

\[
x^2(a - x) = bc^2.
\]

140
ARCHIMEDES

If, therefore, a parabola be drawn through $H$ about the axis $ZH$ with the parameter $HM$, it will pass through $K$ [Apoll. Con. i. 11, converse], and it will be given in position because $HM$ is given in magnitude [Eucl. Data 57], comprehending with the given straight line $HI'$ the given area $\Delta$; therefore $K$ lies on a parabola given in position. Let it then be drawn, as described, and let it be $HK$.

"Again, since the area $\Theta\Lambda = \Gamma B$ [Eucl. i. 43 i.e., $\Theta K \cdot K\Lambda = AB \cdot BH$], if a hyperbola be drawn through $B$ having $\Theta \Gamma, \Gamma H$ for asymptotes, it will pass through $K$ by the converse to the eighth theorem of the second book of Apollonius's Elements of Conics, and it will be given in position because both the straight lines $\Theta \Gamma, \Gamma H$, and also the point $B$, are given in position. Let it be drawn, as described, and let it be $KB$; therefore $K$ lies on a hyperbola given in position. But it lies also on a parabola given in position; therefore $K$ is given. And $KE$ is the perpendicular drawn from it to the straight line $AB$ given in position; therefore $E$ is given. Now since the ratio of $EA$ to the given straight line $\Delta \Gamma$ is equal to the ratio of the given area $\Delta$ to the square on $EB$, we have two solids, whose bases are the square on $EB$ and $\Delta$ and whose altitudes are $EA, \Delta \Gamma$, and the bases are inversely pro-

to which, as already noted, Archimedes had reduced his problem. (N.B.—The axis of $x$ is for convenience taken in a direction contrary to that which is usual; with the usual conventions, we should get slightly different equations.)
In our algebraical notation, \( x^2(a - x) \) is a maximum when \( x = \frac{1}{2}a \). We can easily prove this by the calculus. For, by differentiating and equating to zero, we see that \( x^2(a - x) \) has
portional to the altitudes; therefore the solids are equal [Eucl. xi. 34]; therefore

\[ EB^2 \cdot EA = \Delta \cdot \Gamma A, \]

in which both \( \Delta \) and \( \Gamma A \) are given. But, of all the figures similarly taken upon \( BA, BE^2 \cdot BA \) is greatest when \( BE = 2EA,^a \) as will be proved; it is therefore necessary that the product of the given area and the given straight line should not be greater than

\[ BE^2 \cdot EA. \]

"The synthesis is as follows: Let \( AB \) be the given straight line,\(^c \) let \( \Delta \Gamma \) be any other given straight line, let \( \Delta \) be the given area, and let it be required to cut \( AB \) so that the ratio of one segment to the given straight line \( \Delta \Gamma \) shall be equal to the ratio of the given area \( \Delta \) to the square on the remaining segment."

Let \( AE \) be taken, the third part of \( AB \); then \( \Delta \cdot \Delta \Gamma \) is greater than, equal to or less than \( BE^2 \cdot EA. \)

"If it is greater, no synthesis is possible, as was shown in the analysis; if it is equal, the point \( E \) satisfies the conditions of the problem. For in equal solids the bases are inversely proportional to the altitudes, and \( EA : \Delta \Gamma = \Delta : BE^2. \)

"If \( \Delta \cdot \Delta \Gamma \) is less than \( BE^2 \cdot EA, \) the synthesis is thus accomplished: let \( \Delta \Gamma \) be placed at right angles to \( AB, \) and through \( \Gamma \) let \( \Gamma Z \) be drawn parallel to a stationary value when \( 2ax - 3x^3 = 0, \) \( i.e., \) when \( x = 0 \) (which gives a minimum value) or \( x = \frac{3}{2}a \) (which gives a maximum). No such easy course was open to Archimedes.

\(^a \) See: "not greater than \( BE^2 \cdot EA \) when \( BE = 2EA." \)

\(^c \) Figure on p. 146.
GREEK MATHEMATICS

άλληλος ήχθων ή ΓΖ, διὰ δὲ τοῦ Β τῇ ΑΓ, παράλληλος ήχθων ή ΒΖ καὶ συμπυκνέτω τῇ ΓΕ, εκβιλεθείση κατὰ τὸ Η, καὶ συμπεπληρώσθω τὸ ΖΘ παράλληλόγραμμον, καὶ διὰ τοῦ Ε τῇ ΖΗ παράλληλος ήχθων ή ΚΕΛ. ἐπεὶ οὖν τὸ Δ ἐπὶ τὴν ΑΓ ἐλασσὸν ἠστι τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ, ἠστιν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως τὸ Δ πρὸς ἐλασσὸν τι τοῦ ἀπὸ τῆς ΒΕ, τούτεστι τοῦ ἀπὸ τῆς ΗΚ. ἠστιν οὖν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως τὸ Δ πρὸς τὸ ἀπὸ ΗΜ, καὶ τῷ Δ ἰσον ἠστιν τὸ ὑπὸ ΓΖΝ. ἐπεὶ οὖν ἠστιν, ὡς ἡ ΕΑ πρὸς ΑΓ, οὔτως τὸ Δ, τούτεστι τὸ ὑπὸ ΓΖΝ, πρὸς τὸ ἀπὸ ΗΜ, ἀλλ' ὡς ἡ ΕΑ πρὸς ΛΓ, οὔτως ἡ ΓΖ πρὸς ΖΗ, ὡς δὲ ἡ ΓΖ πρὸς ΖΗ, οὔτως τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΗ, καὶ ὡς ἢ ἰσον τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΗ, οὔτως τὸ ὑπὸ ΓΖΝ πρὸς τὸ ἀπὸ ΗΜ· καὶ ἐναλλάξε, ὡς τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΝ, οὔτως τὸ ὑπὸ ΓΖΗ πρὸς τὸ ἀπὸ ΗΜ. ἀλλ' ὡς τὸ ἀπὸ ΓΖ πρὸς τὸ ὑπὸ ΓΖΝ, ἡ ΓΖ πρὸς ΖΝ, ὡς δὲ ἡ ΓΖ πρὸς ΖΝ, τῆς ΖΗ κοινοῦ ύψους λαμβανομένης οὔτως τὸ ὑπὸ ΓΖΗ πρὸς τὸ ὑπὸ ΝΖΗ· καὶ ὡς ἢ ἰσον τὸ ὑπὸ ΓΖΗ πρὸς τὸ ὑπὸ ΝΖΗ, οὔτως τὸ ὑπὸ ΓΖΗ πρὸς τὸ ἀπὸ ΗΜ. ἰσον ἀρὰ ἠστι τὸ ἀπὸ ΗΜ τῷ ὑπὸ ΗΖΝ.

"Εὰν ἄρα διὰ τοῦ Ζ περὶ ἄξονα τῆς ΖΗ γράψω-μεν παραβολήν, ὡστε τὰς καταγομένας δύνασθαι παρὰ τῆς ΖΝ, ἔξει διὰ τοῦ Μ. γεγράφθω, καὶ ἠστιν ὡς ἡ ΜΕΖ. καὶ ἐπεὶ ἰσον ἠστι τὸ ΔΑ τῷ AZ, τούτεστι τὸ ὑπὸ ΔΚΛ τῷ ὑπὸ ΑΒΖ, ἕαν διὰ
ARCHIMEDES

AB, and through B let BZ be drawn parallel to AΓ, and let it meet ΓΕ produced at H, and let the parallelogram ZΘ be completed, and through E let KEA be drawn parallel to ZH. Now since

\[ Δ \cdot AΓ < BE^2 \cdot EA, \]

\[ \therefore \quad EA : AΓ = Δ : (\text{the square of some quantity less than BE}) = Δ : (\text{the square of some quantity less than HK}). \]

Let \( EA : AΓ = Δ : HM^2, \)
and let \( Δ = ΓZ \cdot ZN. \)
Then \( EA : AΓ = Δ : HM^2 = ΓZ \cdot ZN : HM^2. \)
But \( EA : AΓ = ΓZ : ZH, \)
and \( ΓZ : ZH = ΓZ^2 : ΓZ \cdot ZH; \)
\[ \therefore \quad ΓZ^2 : ΓZ \cdot ZH = ΓZ \cdot ZN : HM^2; \]
and \textit{permutando}, \( ΓZ^2 : ΓZ \cdot ZN = ΓZ \cdot ZH : HM^2. \)
But \( ΓZ^2 : ΓZ \cdot ZN = ΓZ : ZN, \)
and \( ΓZ : ZN = ΓZ \cdot ZH : NZ \cdot ZH, \)
by taking a common altitude ZH;
and \[ ΓZ \cdot ZH : NZ \cdot ZH = ΓZ \cdot ZH : HM^2; \]
\[ \therefore \quad HM^2 = HZ \cdot ZN. \]

"Therefore if we describe through Z a parabola about the axis ZH and with parameter ZN, it will pass through M. Let it be described, and let it be as MΞZ. Then since

\[ ΘΛ =AZ, \]

\[ i.e. \quad ΘK \cdot KA = AB \cdot BZ, \]

[Eucl. i. 43]

145
GREEK MATHEMATICS

tou B peri 'asymptotous tas ΘΓ, ΓΖ gra'iomev uperbolh, 'exei dia tou K dia tηn antistrophi'n

tou 'I thewri'matos twon 'Apolloviou KomiKωn stoixheion. gegrafh, kai estw os η BK tem-
nousa tηn parabolh kata to Ε, kai apto tou Ε
epi tηn AB kathetos 'hchtho η ΕΟΠ, kai dia tou
Ε tη AB parallhlos 'hchtho η ΡΕΣ. epei ovin
uperbolh 'estin η BEK, asymptotoi de ai ΘΓ,
ΓΖ, kai parallhloi hgmniai eisiv ai ΡΕΠ taIs
ABZ, ison 'esti to unpo ΡΕΠ tw tpo ABZ. wste
kai to PO tw OZ. evn ara apto tou Ε epo to
Σ epizevkh' evtheia, 'exei dia tov O. erxevw,
kai estw os η ΓΟΣ. epei ovin estiv, os η ΟΑ
pros ΑΓ, ouths η ΟΒ pros ΒΣ, toutestin η ΓΖ
pros ΖΣ, os de η ΓΖ pros ΖΣ, tηs ΖΝ kouvoi
upsous lamabanomev'os ouths to unpo ΓZN pros
to unpo ΖΖΝ, kai os ara η ΟΑ pros ΑΓ, ouths
to unpo ΓZN pros to unpo ΖΖΝ. kai esti tov
men unpo ΓZN ison to Δ χωριν, tw de unpo ΖΖΝ
isov to apto ΕΕ, toutestin to apto ΒΟ, dia tηn
parabolh'n. os ara η ΟΑ pros ΑΓ, ouths to Δ
146
if we describe through B a hyperbola in the asymptotes $\Theta \Gamma, \Gamma Z$, it will pass through K by the converse of the eighth theorem [of the second book] of Apollonius's *Elements of Conics*. Let it be described, and let it be as BK cutting the parabola in $\Xi$, and from $\Xi$ let $\Xi \Omega \Pi$ be drawn perpendicular to $\Lambda \Pi$, and through $\Xi$ let $\Xi \pi \Sigma$ be drawn parallel to $\Lambda \Pi$. Then since $\Xi \pi K$ is a hyperbola and $\Theta \Gamma, \Gamma Z$ are its asymptotes, while $\Xi \pi, \Xi \Pi$ are parallel to $\Lambda \Pi, \Lambda Z$,

$$\Xi \pi \cdot \Xi \Pi = \Lambda \Pi \cdot \Lambda Z; \quad \text{[Apoll. ii. 12]}$$

$$\therefore \quad \Xi \Omega = \Omega Z.$$

Therefore if a straight line be drawn from $\Gamma$ to $\Sigma$ it will pass through $O$ [Eucl. i. 43, converse]. Let it be drawn, and let it be as $\Gamma O \Sigma$. Then since

$$\Lambda A : \Lambda \Gamma = \Omega B : \Lambda \Sigma \quad \text{[Eucl. vi. 4]}$$

$$= \Gamma Z : \Sigma \Xi,$$

and

$$\Gamma Z : \Sigma \Xi = \Gamma Z \cdot \Sigma N : \Sigma \Xi \cdot \Sigma N,$$

by taking a common altitude $\Sigma N$,

$$\therefore \quad \Lambda A : \Lambda \Gamma = \Gamma Z \cdot \Sigma N : \Sigma \Xi \cdot \Sigma N.$$

And $\Gamma Z \cdot \Sigma N = \Delta, \Sigma \Xi \cdot \Sigma N = \Sigma \Xi^2 = BO^2$, by the property of the parabola [Apoll. i. 11].

$$\therefore \quad \Lambda A : \Lambda \Gamma = \Delta : BO^2;$$
χωρίων πρός τὸ ἀπὸ τῆς ΒΟ. εἰληπταὶ ἀρα τὸ Ο σημειῶν ποιοῦν τὸ πρόβλημα.

"'Οτι δὲ διπλασίας οὔσης τῆς BE τῆς EA τὸ ἀπὸ τῆς BE ἐπὶ τὴν EA μέγιστὸν ἐστὶ πάντων τῶν ὀμοίως λαμβανομένων ἐπὶ τῆς BA, δειχθῆσεται οὕτως. ἔστω γάρ, ὡς ἐν τῇ ἀναλύσει, πάλιν δοθεῖσα εὐθείᾳ πρὸς ὅρθας τῇ AB ἡ AG, καὶ ἐπι-ζευχθεῖσα ἡ GE ἐκβεβλήσθη καὶ συμπιπτέτω τῇ διὰ τοῦ B παραλλήλῳ ἡγμένη τῇ AG κατὰ τὸ Z, καὶ διὰ τῶν Γ, Z παραλληλοι τῇ AB ἡχθωσαν αἱ ΘZ, GH, καὶ ἐκβεβλήσθων ἡ GA ἐπὶ τὸ Θ, καὶ ταύτῃ παραλληλος διὰ τοῦ E ἡχθω ἡ KEL, καὶ γεγονέτω, ὡς ἡ EA πρὸς AG, οὕτως τὸ ὑπὸ ΓHM πρὸς τὸ ἀπὸ EB· τὸ ἄρα ἀπὸ BE ἐπὶ τὴν EA ἦσον ἐστὶ τῷ ὑπὸ ΓHM ἐπὶ τὴν AG διὰ τὸ δυὸ στερεῶν ἀντιπεποθθέναι τὰς βάσεις τοῖς ύψεσιν. λέγον οὖν, ὧτι τὸ ὑπὸ ΓHM ἐπὶ τὴν AG μέγιστὸν ἐστὶ πάντων τῶν ὀμοίως ἐπὶ τῆς BA λαμβανομένων. "Γεγράφθω γάρ διὰ τοῦ Η περὶ ἄξονα τῆς ZH παραβολῆς, ὡστε τὰς καταγομένας δύνασθαι παρὰ τὴν HM· ἦσει δὴ διὰ τοῦ K, ὡς ἐν τῇ ἀναλύσει δέδεικται, καὶ συμπεσεῖται ἐκβαλλομένη τῇ ΘΓ παραλλήλῳ οὔσῃ τῇ διαμέτρῳ τῆς Τομῆς διὰ τὸ ἐβδομον καὶ εἰκοστὸν θεώρημα τοῦ πρῶτου βιβλίου τῶν 'Απολλωνίου Κωνικῶν στοιχείων. ἐκβεβλήσθω καὶ συμπιπτέτως κατὰ τὸ N, καὶ διὰ τοῦ B περὶ ἀσυμπτῶτους τὰς ΝΓΗ γεγράφθων ὑπερ-βολῆς· ἦσει ἀρα διὰ τοῦ K, ὡς ἐν τῇ ἀναλύσει εὑρηται. ἐρχόταθι οὖν ὡς ἡ BK, καὶ ἐκβληθείσῃ τῇ ZH ὅση κείσαθω ἡ ΗΕ, καὶ ἐπεξεύχθω ἡ ΕΚ

148
therefore the point O has been found satisfying the conditions of the problem.

"That $BE^2 \cdot EA$ is the greatest of all the figures similarly taken upon $BA$ when $BE = 2EA$ will be thus proved. Let there again be, as in the analysis, a given straight line $AI$ at right angles to $AB$, and let $IE$ be joined and let it, when produced, meet at $Z$ the line through $B$ drawn parallel to $AI$, and through $I$, $Z$ let $\Theta Z$, $\Gamma H$ be drawn parallel to $AB$, and let $IA$ be produced to $\Theta$, and through $E$ let $KEA$ be drawn parallel to it, and let

$$EA : IA = \Gamma H \cdot HM : EB^2;$$

then $BE^2 \cdot EA = (\Gamma H \cdot HM) \cdot IA$,

owing to the fact that in two [equal] solids the bases are inversely proportional to the altitudes. I assert, then, that $(\Gamma H \cdot HM) \cdot IA$ is the greatest of all the figures similarly taken upon $BA$.

"For let there be described through $H$ a parabola about the axis $ZH$ and with parameter $HM$; it will pass through $K$, as was proved in the analysis, and, if produced, it will meet $\Theta I$, being parallel to the axis of the parabola, by the twenty-seventh theorem of the first book of Apollonius's *Elements of Conics*. Let it be produced, and let it meet at $N$, and through $B$ let a hyperbola be drawn in the asymptotes $NI'$, $\Gamma H$; it will pass through $K$, as was shown in the analysis. Let it be described as $BK$, and let $ZH$ be produced to $\Xi$ so that $ZH = H\Xi$, and let $\Xi K$ be joined.

---

*a* Figure on p. 151.

*b* Lit. "diameter," in accordance with Archimedes' usage.

*c* Apoll. i. 26 in our texts.
καὶ ἐκβεβλήσθω ἐπὶ τὸ Ὀ. φανερὸν ἄρα, ὅτι ἐφάπτεται τῆς παραβολῆς διὰ τὴν ἀντιστροφὴν τοῦ τετάρτου καὶ τριακοστοῦ θεωρήματος τοῦ πρώτου βιβλίου τῶν Ἀπολλωνίου Κωνικῶν στοιχείων. ἔπει οὖν διπλῇ ἐστιν ἡ BE τῆς EA—οὗτως γὰρ ὑπόκειται—τούτεστιν ἡ ZK τῆς KΘ,

* Apoll. i. 33 in our texts.
and produced to $O$; it is clear that it will touch
the parabola by the converse of the thirty-fourth

\[ \text{ARCHIMEDES} \]

\[ \text{theorem of the first book of Apollonius's } \textit{Elements of}
\]

\[ \text{Conics}.^a \text{ Then since } BE = 2EA—\text{for this hypothesis}
\]

\[ \text{has been made—therefore } ZK = 2K\theta, \text{ and the triangle} \]

151
καὶ ἔστιν ὄμοιον τὸ ὈΘΚ τρίγωνον τῷ ἙΚ τριγώνῳ, ὑπελασία ἔστι καὶ ἡ ἙΚ τῆς ΚΩ. ἔστιν δὲ καὶ ἡ ἙΚ τῆς ΚΙΠ διπλὴ διὰ τὸ καὶ τὴν ἙΖ τῆς ΧΗ καὶ παράλληλον εἶναι τὴν ΠΗ τῇ ΚΖ. ἵνα ἀρα ἡ ὈΚ τῇ ΚΙΠ. ἡ ἀρα ΟΚΠ ψαύσοσα τῆς ύπερβολῆς καὶ μεταξὺ οὐσα τῶν ἀσυμπτωτῶν δίχα τείνεται· ἐφάπτεται ἀρα τῆς ύπερβολῆς διὰ τὴν ἀντιστροφὴν τοῦ τρίτου θεωρήματος τοῦ δευτέρου βιβλίου τῶν 'Απολλωνίου Κωνικῶν στοιχείων. ἐφήπτετο δὲ καὶ τῆς παραβολῆς κατὰ τὸ αὐτὸ Κ. ἡ ἀρα παραβολὴ τῆς ύπερβολῆς ἐφάπτεται κατὰ τὸ Κ.

"Νενοήσωθω οὖν καὶ ἡ ύπερβολή προσεκβαλλομένη ὡς ἐπὶ τὸ ρ, καὶ εἰλήφθω ἐπὶ τῆς ΑΒ τυχὸν σημείον τὸ Σ, καὶ διὰ τοῦ Σ τῇ ΚΑ παράλληλος ηχθῶ η ΤΣΥ καὶ συμβαλλέτω τῇ ύπερβολή κατὰ τὸ Τ, καὶ διὰ τοῦ Τ τῇ ΓΗ παράλληλος ηχθῶ η ΦΧ. ἐπεὶ οὖν διὰ τὴν ύπερβολήν καὶ τὰς ἀσυμπτωτὰς ἵσον ἐστὶ τὸ ΦΥ τῷ ΓΒ, κοινοῦ ἀφαιρεθέντος τοῦ ΓΣ ἵσον γίνεται τὸ ΦΣ τῷ ΣΗ, καὶ διὰ τοῦτο ἡ ἀπὸ τοῦ Γ ἐπὶ τὸ Χ ἐπιζευγνυμένη εὐθεία ἥξει διὰ τοῦ Σ. ἐρχέσθω καὶ ἔστω ὡς η ΓΣΧ. καὶ ἐπεὶ τὸ ἀπὸ ΨΧ ἵσον ἐστὶ τῷ ὑπὸ ΧΗΜ διὰ τὴν παραβολήν, τὸ ἀπὸ ΤΧ ἔλασσον

a In the same notation as before, the condition $BE^{2} = (ΓΗ \cdot HM)$. $AG = \frac{4}{27}a^{3} = be^{2}$; and Archimedes has proved that, when this condition holds, the parabola $x^{2} = \frac{c^{2}}{a}y$ touches the hyperbola $(a - x)y = ab$ at the point $\left(\frac{2}{3}a, 3b\right)$ because they both touch at this point the same straight line, that is the 152
ARCHIMEDES

OδΚ is similar to the triangle ΕΖΚ, so that ΕΚ = 2ΚΟ. But ΕΚ = 2ΚΠ because ΕΖ = 2ΕΗ and ΠΗ is parallel to ΚΖ; therefore ΟΚ = ΚΠ. Therefore ΟΚΠ, which meets the hyperbola and lies between the asymptotes, is bisected; therefore, by the converse of the third theorem of the second book of Apollonius’s *Elements of Conics*, it is a tangent to the hyperbola. But it touches the parabola at the same point Κ. Therefore the parabola touches the hyperbola at Κ.a

"Let the hyperbola be therefore conceived as produced to Ρ, and upon ΑΒ let any point Σ be taken, and through Σ let ΤΣΥ be drawn parallel to ΚΛ and let it meet the hyperbola at Τ, and through Τ let \( ΦΤΧ \) be drawn parallel to \( ΤΗ \). Now by virtue of the property of the hyperbola and its asymptotes, \( ΦΥ=ΓΒ \), and, the common element \( ΓΣ \) being subtracted, \( ΦΣ=ΣΗ \), and therefore the straight line drawn from \( Γ \) to \( Χ \) will pass through \( Σ \) [Eucl. i. 43, conv.]. Let it be drawn, and let it be as \( ΓΣΧ \). Then since, in virtue of the property of the parabola,

\[ \PsiΧ^2=ΧΗ \cdot ΗΜ, \]  

[Apoll. i. 11 line 96x - ay - 3ab = 0, as may easily be shown. We may prove this fact in the following simple manner. Their points of intersection are given by the equation

\[ x^2(a-x)=bc^2, \]

which may be written

\[ x^2 - ax^2 + \frac{4}{27}a^3 = \frac{4}{27}a^3 - bc^2, \]

or

\[ (x - \frac{2}{3}a)(x + \frac{a}{3}) = \frac{4}{27}a^3 - bc^2. \]

Therefore, when \( bc^2 = \frac{4}{27}a^3 \) there are two coincident solutions, \( x = \frac{2}{3}a \), lying between 0 and \( a \), and a third solution \( x = -\frac{a}{3} \), outside that range.

153
ἐστι τὸν ὑπὸ ΧΗΜ. γεγονέτω οὖν τῷ ἀπὸ ΤΧ ἴσον τὸ ὑπὸ ΧΗΩ. ἐπεὶ οὖν ἔστω, ὥς ἡ ΣΑ πρὸς ΑΓ, οὕτως ἡ ΓΗ πρὸς ΗΧ, ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΧ, τῆς ΗΩ κοινοῦ ύψους λαμβανομένης οὕτως τὸ ὑπὸ ΓΗΩ πρὸς τὸ ὑπὸ ΧΗΩ καὶ πρὸς τὸ ἴσον αὐτῷ τὸ ἀπὸ ΧΤ, τούτεστι τὸ ἀπὸ ΒΣ, τὸ ἁρὰ ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ ἴσον ἔστι τῷ ὑπὸ ΓΗΩ ἐπὶ τὴν ΓΑ. τὸ δὲ ὑπὸ ΓΗΩ ἐπὶ τὴν ΓΑ ἔλασσον ἔστι τοῦ ὑπὸ ΓΗΜ ἐπὶ τὴν ΓΑ· τὸ ἁρὰ ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ ἔλασσον ἔστι τοῦ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ. ὅμοιως δὴ δειχθήσεται καὶ ἐπὶ πάντων τῶν σημείων τῶν μεταξὺ λαμβανομένων τῶν Ε, Β.

"Ἀλλὰ δὴ εἰλήφθω μεταξὺ τῶν Ε, Α σημείου τὸ 5. λέγω, ὅτι καὶ οὕτως τὸ ἀπὸ τῆς ΒΕ ἐπὶ τὴν ΕΑ μειώσθ' ἐστι τοῦ ἀπὸ ΒΣ ἐπὶ τὴν 5Α.

"Τῶν γὰρ αὐτῶν κατεσκευασμένων ἥχων διὰ τοῦ 5 τῇ ΚΑ παράλληλος ἡ ζΠ καὶ συμβαλλέτω τῇ ύπερβολῇ κατὰ τὸ Ρ. συμβαλεῖ γὰρ αὐτή διὰ τὸ παράλληλος εἶναι τῇ ἀσυμπτωτῷ· καὶ διὰ τοῦ Ρ παράλληλος ἀχθείσαι τῇ ΑΒ ή ΑΡΒ' συμβαλλέτω τῇ ΗΖ ἐκβαλλομένη κατά τὸ Β'. καὶ ἐπεὶ πάλιν διὰ τὴν ύπερβολὴν ἴσον ἔστι τὸ Γ' τῷ ΑΗ, ἢ ἀπὸ τοῦ Γ ἐπὶ τὸ Β' ἐπιζευγνυμένη εὐθεία ἥξει διὰ τοῦ 5. ἔρχεσθω καὶ ἐστώ ὡς ἡ ΓΩΒ'. καὶ ἐπεὶ πάλιν διὰ τὴν παραβολὴν ἴσον ἔστι τὸ ἀπὸ ΑΒ' τῷ ὑπὸ Β'ΗΜ, τὸ ἁρὰ ἀπὸ ΡΒ' ἔλασσον ἔστι τοῦ ὑπὸ Β'ΗΜ. γεγονέτω τὸ ἀπὸ ΡΒ' ἴσον τῷ ὑπὸ Β'ΗΩ. ἐπεὶ οὖν ἔστω, ὡς ἡ 5Α πρὸς ΑΓ, οὕτως ἡ ΓΗ πρὸς ΗΒ', ἀλλ' ὡς ἡ ΓΗ πρὸς ΗΒ', τῆς

a Figure on p. 156.
ARCHIMEDES

\[ \therefore \quad TX^2 < \chi \cdot H \cdot HM. \]

Let \[ TX^2 = \chi \cdot H \cdot \omega. \]

Then since \[ \Sigma A : AG = \Gamma H : HX, \]
while \[ \Gamma H : HX = \Gamma H : H \Omega : \chi H : H \Omega, \]
by taking a common altitude \( H \Omega, \)
\[ = \Gamma H : H \Omega : \chi T^2 \]
\[ = \Gamma H : H \Omega : B \Sigma^2, \]
\[ \therefore \quad B \Sigma^2 : \Sigma A = (\Gamma H : H \Omega) \cdot \Gamma A. \]

But \[ (\Gamma H : H \Omega) \cdot \Gamma A < (\Gamma H : H \Omega) \cdot \Gamma A; \]
\[ \therefore \quad B \Sigma^2 : \Sigma A < B \varepsilon \cdot \varepsilon. \]

This can be proved similarly for all points taken between \( E, B. \)

"Now let there be taken a point \( \varphi \) between \( E, A. \) I assert that in this case also \( B \varepsilon \cdot \varepsilon. \]

"With the same construction, let \( \varphi \varphi \cdot \varphi \) be drawn \( a \)
through \( \varphi \) parallel to \( K \Lambda \) and let it meet the hyper-
bola at \( P; \) it will meet the hyperbola because it is
parallel to an asymptote [Apoll. ii. 13]; and through \( P \) let \( A ' \cdot \varepsilon \cdot \varepsilon \) be drawn parallel to \( A B \) and let it meet
HZ produced in \( B ' . \) Since, in virtue of the property
of the hyperbola, \( \Gamma ' \cdot \varphi = \Lambda H, \) the straight line drawn
from \( \Gamma ' \) to \( B ' \) will pass through \( \varphi \) [Eucl. i. 43, conv.].
Let it be drawn and let it be as \( \Gamma ' \cdot B ' . \) Again, since, in
virtue of the property of the parabola,
\[ A ' \cdot \varepsilon \cdot \varepsilon = B ' \cdot H \cdot \varepsilon \cdot \varepsilon, \]
\[ \therefore \quad B ' \cdot \varepsilon \cdot \varepsilon < B ' \cdot H \cdot \varepsilon \cdot \varepsilon. \]

Let \[ B ' \cdot \varepsilon \cdot \varepsilon = B ' \cdot H \cdot \omega. \]

Then since \[ \varphi A : AG = \Gamma H : H \varepsilon \cdot \varepsilon, \]
while \[ \Gamma H : H \varepsilon = \Gamma H : H \varepsilon : B ' H : H \omega, \]
GREEK MATHEMATICS

Ἡ Ω κοινοῦ ὕψους λαμβανομένης οὗτως τὸ ὑπὸ ΓΗΩ πρὸς τὸ ὑπὸ Β’ΗΩ, τούτεστι πρὸς τὸ ἀπὸ ΒΒ', τούτεστι πρὸς τὸ ἀπὸ ΒΣ, τὸ ἄρα ἀπὸ ΒΣ ἐπὶ τὴν SA ἵσον ἐστὶ τῷ ὑπὸ ΓΗΩ ἐπὶ τὴν ΓΑ, καὶ μεῖζον τὸ ὑπὸ ΓΗΜ τοῦ ὑπὸ ΓΗΩ· μεῖζον ἄρα καὶ τὸ ἀπὸ ΒΕ ἐπὶ τὴν ΕΑ τοῦ ἀπὸ ΒΣ ἐπὶ 156
ARCHIMEDES

by taking a common altitude $H\Omega$,

$$= \Gamma H \cdot H\Omega : PB'^2$$

$$= \Gamma H \cdot H\Omega : B^2,$$

$$\therefore B^2 \cdot \sigma A = (\Gamma H \cdot H\Omega) \cdot \Gamma A.$$ 

And $\Gamma H \cdot HM > \Gamma H \cdot H\Omega$;

$$\therefore BE^2 \cdot EA > B^2 \cdot \sigma A.$$
GREEK MATHEMATICS

tὴν σ. ὁμοῖος δὴ δειχθήσεται καὶ ἐπὶ πάντων τῶν σημείων τῶν μεταξὺ τῶν Ε, Α λαμβανομένων. ἔδειξθε δὲ καὶ ἐπὶ πάντων τῶν μεταξὺ τῶν Ε, Β· πάντων ἄρα τῶν ἐπὶ τῆς AB ὁμοῖος λαμβανομένων μέγιστον ἔστω τὸ ἀπὸ τῆς BE ἐπὶ τῆς ΕΑ, ὅταν ἡ διπλασία ἡ BE τῆς EA.''

'Επιστήσαι δὲ χρή καὶ τοῖς ἀκολουθοῦσιν κατὰ τὴν εἰρημένην καταγραφὴν. ἐπεὶ γὰρ δεδεικταὶ τὸ ἀπὸ BS ἐπὶ τὴν ΣΑ καὶ τὸ ἀπὸ Βς ἐπὶ τὴν 5Α ἔλασσον τοῦ ἀπὸ BE ἐπὶ τὴν ΕΑ, δυνατόν ἐστὶ καὶ τοῦ δοθέντος χωρίου ἐπὶ τὴν δοθείσαν ἑλάσσονος οὖντος τοῦ ἀπὸ τῆς BE ἐπὶ τῆς ΕΑ κατὰ δύο σημεῖα τὴν AB τεμνομένην ποιεῖν τὸ ἔς ἀρχής πρόβλημα. τούτῳ δὲ γίνεται, εἰ νοήσαμεν περὶ διάμετρον τὴν ΧΗ γραφομένην παραβολὴν, ὥστε τὰς καταγομένας δύνασθαι παρὰ τὴν ΗΩ· ἡ γὰρ θοιαύτη παραβολὴ πάντως ἔρχεται διὰ τοῦ Τ. καὶ ἐπειδὴ ἀνάγκη αὐτὴν συμπίπτειν τῇ ΙΝ παραλλήλως οὖσῃ τῇ διαμέτρῳ, δήλον, ὅτι τέμνει τὴν ὑπερβολὴν καὶ κατ’ ἄλλο σημείον ἀνωτέρω τοῦ Κ, ὥς ἐνταῦθα κατὰ τὸ P, καὶ ἀπὸ τοῦ P ἐπὶ τὴν AB κάθετος ἀγομένη, ὥς ἐνταῦθα ἡ Ρς, τέμνει τὴν AB κατὰ τὸ 5, ὥστε τὸ 5 σημείον ποιεῖν τὸ πρό-

---

a There is some uncertainty where the quotation from Archimedes ends and Eutocius's comments are resumed. Heiberg, with some reason, makes Eutocius resume his comments at this point.

b In the mss. the figures on pp. 150 and 156 are com-
This can be proved similarly for all points taken between E, A. And it was proved for all points between E, B; therefore for all figures similarly taken upon AB, BE^2. EA is greatest when BE = 2EA.

The following consequences should also be noticed in the aforementioned figure. Inasmuch as it has been proved that

$$B^2 \cdot \Sigma A < BE^2.$$  

and

$$B^2 \cdot \Sigma A < BE^2 \cdot EA,$$

if the product of the given space and the given straight line is less than BE^2 . EA, it is possible to cut AB in two points satisfying the conditions of the original problem. This comes about if we conceive a parabola described about the axis XH with parameter HΩ; for such a parabola will necessarily pass through T. And since it must necessarily meet ΠN, being parallel to a diameter [Apoll. Con. i. 26], it is clear that it cuts the hyperbola in another point above K, as at P in this case, and a perpendicular drawn from P to AB, as P' in this case, will cut AB in P', so that the point P' satisfies the conditions of the

bined; in this edition it is convenient, for the sake of clarity, to give separate figures.

With the same notation as before this may be stated: when $bc^2 < \frac{4}{27}a^3$, there are always two real solutions of the cubic equation $x^2(a - x) = bc^2$ lying between 0 and a. If the cubic has two real roots it must, of course, have a third real root as well, but the Greeks did not recognize negative solutions.

By Apoll. i. 11, since $TX^2 = XH \cdot HΩ$.  

159
GREEK MATHEMATICS

...βλημα, καὶ ἵσον γίνεσθαι τὸ ἀπὸ ΒΣ ἐπὶ τὴν ΣΑ
tῷ ἀπὸ Βς ἐπὶ τὴν ΣΑ, ὡς ἐστὶ διὰ τῶν προειρη-
μένων ἀποδείξεων ἐμφανεῖς. ὥστε δυνατοῦ δυνοῖ
ἐπὶ τῆς ΒΑ δύο σημεῖα λαμβάνειν ποιοῦντα τὸ
ξητούμενον, ἔξεστιν, ὁπότερον τις βούλοιτο, λαμ-
βάνειν ἡ τὸ μεταξὺ τῶν Ε, Β ἡ τὸ μεταξὺ τῶν Ε, Α.
εἰ μὲν γὰρ τὸ μεταξὺ τῶν Ε, Β, ὡς εἰρήσκα,
tῆς διὰ τῶν Η, ὧς σημεῖων γραφομένης παραβολῆς
κατὰ δύο σημεῖα τεμνοῦσθαι τὴν ύπερβολὴν τὸ μὲν
ἐγγύτερον τοῦ Η, τούτεστι τοῦ ἄξονος τῆς παρα-
βολῆς, εὑρίσκει τὸ μεταξὺ τῶν Ε, Β, ὡς ἑνταῦθα
τὸ Τ εὑρίσκει τὸ Σ, τὸ δὲ ἀπτέρω τὸ μεταξὺ
tῶν Ε, Α, ὡς ἑνταῦθα τὸ Ρ εὑρίσκει τὸ 5.

Καθόλου μὲν οὖν οὕτως ἀναλέλυται καὶ συντε-
θεται τὸ πρόβλημα: ἵνα δὲ καὶ τοῖς Ἀρχιμήδειοις
ῥήμασιν ἐφαρμοσθῆ, νενοήθω ὡς ἐν αὐτῇ τῇ τοῦ
ῥητοῦ καταγραφῆ διάμετρος μὲν τῆς σφαίρας ἡ
ΔΒ, ἢ δὲ ἐκ τοῦ κέντρου ἡ ΒΖ, καὶ ἡ δεδομένη
ἡ ΖΘ. κατηντησαμεν ἀρα, φησίν, εἰς τὸ "τὴν
ΔΖ τεμεῖν κατὰ τὸ Χ, ὥστε εἶναι, ὡς τὴν ΧΖ
πρὸς τὴν δοθεῖσαν, οὕτως τὸ δοθέν πρὸς τὸ ἀπὸ
tῆς ΔΧ. τούτῳ δὲ ἀπλῶς μὲν λεγόμενον ἔχει
διορισμὸν." εἰ γὰρ τὸ δοθέν ἐπὶ τὴν δοθεῖσαν
μεῖζον ἔτυγχανον τοῦ ἀπὸ τῆς ΔΒ ἐπὶ τὴν ΒΖ,
ἀδύνατον ἢν τὸ πρόβλημα, ὡς δέδεικται, εἰ δὲ
ἰσον, τὸ Β σημεῖον ἐποίει τὸ πρόβλημα, καὶ οὐτως
δὲ οὐδὲν ἢν πρὸς τὴν ἐς ἀρχῆς ᾿Αρχιμήδους πρό-
θεσιν. ἡ γὰρ σφαίρα οὐκ ἑτέμοιτο εἰς τὸν δοθέντα

---

a Archimedes' figure is re-drawn (v. page 162) so that B, Z come on the left of the figure and Δ on the right, instead of B, Z on the right and Δ on the left.
b v. supra, p. 133.
ARCHIMEDES

problem, and $BZ^2 \cdot \Sigma A = B^2 \cdot \Sigma A$, as is clear from the above proof. Inasmuch as it is possible to take on BA two points satisfying what is sought, it is permissible to take whichever one wills, either the point between E, B or that between E, A. If we choose the point between E, B, the parabola described through the points H, T will, as stated, cut the hyperbola in two points; of these the one nearer to H, that is to the axis of the parabola, will determine the point between E, B, as in this case T determines $\Sigma$, while the point farther away will determine the point between E, A, as in this case P determines $\Sigma$.

The analysis and synthesis of the general problem have thus been completed; but in order that it may be harmonized with Archimedes' words, let there be conceived, as in Archimedes' own figure, a diameter $\Delta B$ of the sphere, with radius [equal to] BZ, and a given straight line Z\( \Theta \). We are therefore faced with the problem, he says, "so to cut $\Delta Z$ at $X$ that $XZ$ bears to the given straight line the same ratio as the given area bears to the square on $\Delta X$. When the problem is stated in this general form, it is necessary to investigate the limits of possibility." If therefore the product of the given area and the given straight line chanced to be greater than $\Delta B^2 \cdot BZ$, the problem would not admit a solution, as was proved, and if it were equal the point $B$ would satisfy the conditions of the problem, which also would be of no avail for the purpose Archimedes set himself at the outset; for the sphere would not be

---

\(^a\) For $\Delta B = \frac{3}{2} \Delta Z$ [ex hyp.], and so $\Delta B$ in the figure on p. 162 corresponds with $BE$ in the figure on p. 146, while $BZ$ in the figure on p. 162 corresponds with $EA$ in the figure on p. 146.
Eutocius proceeds to give solutions of the problem by Dionysodorus and Diocles, by whose time, as he has explained, Archimedes' own solution had already disappeared. Dionysodorus solves the less general equation by means of the intersection of a parabola and a rectangular hyperbola; Diocles solves the general problem by the intersection of an ellipse with a rectangular hyperbola, and his proof is both ingenious and intricate. Details may be consulted in Heath, *H.G.M.* ii. 46-49 and more fully in Heath, 162
ARCHIMEDES

cut in the given ratio. Therefore when the problem was stated generally, an investigation of the limits of possibility was necessary as well; "but under the conditions of the present case," that is, if $\Delta B = 2ZB$ and $BZ > Z\theta$, "no such investigation is necessary." For the product of the given area $\Delta B^2$ into the given straight line $Z\theta$ is less than the product of $\Delta B^2$ into $BZ$ by reason of the fact that $BZ$ is greater than $Z\theta$, and we have shown that in this case there is a solution, and how it can be effected.\(^a\)

The Works of Archimedes, pp. cxxiii-cxli, which deals with the whole subject of cubic equations in Greek mathematical history. It is there pointed out that the problem of finding mean proportionals is equivalent to the solution of a pure cubic equation, $a^3 = \frac{a}{x^2}$, and that Menaechmus's solution, by the intersection of two conic sections (v. vol. i. pp. 278-283), is the precursor of the method adopted by Archimedes, Dionysodorus and Diocles. The solution of cubic equations by means of conics was, no doubt, found easier than a solution by the manipulation of parallelepipeds, which would have been analogous to the solution of quadratic equations by the application of areas (v. vol. i. pp. 186-215). No other examples of the solution of cubic equations have survived, but in his preface to the book On Conoids and Spheroids Archimedes says the results there obtained can be used to solve other problems, including the following, "from a given spheroidal figure or conoid to cut off, by a plane drawn parallel to a given plane, a segment which shall be equal to a given cone or cylinder, or to a given sphere" (Archim. ed. Heiberg i. 258. 11-15); the case of the paraboloid of revolution does not lead to a cubic equation, but those of the spheroid and hyperboloid of revolution do lead to cubics, which Archimedes may be presumed to have solved. The conclusion reached by Heath is that Archimedes solved completely, so far as the real roots are concerned, a cubic equation in which the term in $x$ is absent: and as all cubic equations can be reduced to this form, he may be regarded as having solved geometrically the general cubic.
GREEK MATHEMATICS

(d) CONOIDS AND SPHEROIDS

(i.) Preface


'Αρχιμήδης Δοσιθέω εὑρίσκειν.
'Αποστέλλω τοι γράφας εν τῷ δε τῷ βιβλίῳ τῶν
τε λοιπῶν θεωρημάτων τὰς ἀποδείξεις, ὅν οὐκ
εἶχες εν τοῖς πρότερον ἀπεσταλμένοις, καὶ ἄλλων
ὑστερον ποτεξερημένων, ἂ πρότερον μὲν ἦδη
πολλάκις ἐγχειρήσας ἐπισκέπτεσθαι δύσκολον ἔχειν
τι φανείσας μοι τὰς εὑρέσεις αὐτῶν ἀπόρησα,
διότι οὔδε συνεξεδόθεν τοῖς ἄλλοις αὐτὰ τὰ προ-
βεβλημένα. Ὑστερον δὲ ἐπιμελέστερον ποτ' αὐτοῖς
γενόμενος ἔξευρον τὰ ἀπορηθέντα. ἢν δὲ τὰ μὲν
λοιπὰ τῶν προτέρων θεωρημάτων περὶ τοῦ ὀρθο-
γωνίου κυκλοειδεὸς προβεβλημένα, τὰ δὲ νῦν ἐντὶ
ποτεξερημένα περὶ τε ἀμβλυγωνίου κυκλοειδεὸς
καὶ περὶ σφαιροειδέων σχημάτων, ὅν τὰ μὲν
παραμάκεα, τὰ δὲ ἐπιπλατέα καλέω.

(ii.) Two Lemmas

Ibid., Lemma ad Prop. 1, Archim. ed. Heiberg i. 260. 17-24

Εἰ κα ἐώςτι μεγέθεια ὀποσαοῦν τῷ ἵσῳ ἄλλαλων

---

a In the books On the Sphere and Cylinder, On Spirals and on the Quadrature of a Parabola.
b i.e., the paraboloid of revolution.
c i.e., the hyperboloid of revolution.
d An oblong spheroid is formed by the revolution of an
(d) Conoids and Spheroids

(i.) Preface

Archimedes, On Conoids and Spheroids, Preface, Archim. ed. Heiberg i. 246. 1-14

Archimedes to Dositheus greeting.

I have written out and now send you in this book the proofs of the remaining theorems, which you did not have among those sent to you before, and also of some others discovered later, which I had often tried to investigate previously but their discovery was attended with some difficulty and I was at a loss over them; and for this reason not even the propositions themselves were forwarded with the rest. But later, when I had studied them more carefully, I discovered what had left me at a loss before. Now the remainder of the earlier theorems were propositions about the right-angled conoid; but the discoveries now added relate to an obtuse-angled conoid and to spheroidal figures, of which I call some oblong and others flat.

(ii.) Two Lemmas

Ibid., Lemma to Prop. 1, Archim. ed. Heiberg i. 260. 17-24

If there be a series of magnitudes, as many as you please, in which each term exceeds the previous term by an ellipse about its major axis, a flat spheroid by the revolution of an ellipse about its minor axis.

In the remainder of our preface Archimedes gives a number of definitions connected with those solids. They are of importance in studying the development of Greek mathematical terminology.
ἐπερέχοντα, ἢ δὲ ἡ ὑπεροχὰ ἵσα τῷ ἑλαχίστῳ, καὶ ἄλλα μεγέθεα τῷ μὲν πλήθει ἵσα τούτοις, τῷ δὲ μεγέθει ἐκαστὸν ἵσον τῷ μεγίστῳ, πάντα τὰ μεγέθεα, ὡς ἐστὶν ἐκαστὸν ἵσον τῷ μεγίστῳ, πάντων μὲν τῶν τῷ ἵσῳ ὑπερεχόντων ἑλάσσονα ἐσούνται ἡ διπλάσια, τῶν δὲ λοιπῶν χωρὶς τοῦ μεγίστου μείζονα ἡ διπλάσια. ἃ δὲ ἀπόδειξις τοῦτον φανερά.

Ibid., Prop. 1, Archim. ed. Heiberg l. 260. 26-261. 22

Εἴ κα μεγέθεα ὁποσαοῦ ἄλλοις μεγέθεσιν ἵσοις τῷ πλήθει κατὰ δύο τὸν αὐτὸν λόγον ἐχωντι τὰ ὁμοίως τεταγμένα, λέγεται δὲ τὰ τε πρῶτα μεγέθεα ποτ' ἄλλα μεγέθεα ἡ πάντα ἡ τίνα αὐτῶν ἐν λόγοις ὁποιοισοῦν, καὶ τὰ ύστερον ποτ' ἄλλα μεγέθεα τὰ ὁμόλογα ἐν τοῖς αὐτοῖς λόγοις, πάντα τὰ πρῶτα μεγέθεα ποτὶ πάντα, ἄ λέγονται, τὸν αὐτὸν ἐξούντι λόγον, ὅν ἐχωντι πάντα τὰ ύστερον μεγέθεα ποτὶ πάντα, ἄ λέγονται.

"Εστω τινὰ μεγέθεα τὰ Α, Β, Γ, Δ, Ε, Ζ ἄλλοις μεγέθεσιν ἵσοις τῷ πλήθει τοῖς, Η, Θ, Ι, Κ, Λ, Μ

* If $h$ is the common difference, the first series is $h$, $2h$, $3h$ . . . $nh$, and the second series is $nh$, $nh$ . . . to $n$ terms, its sum obviously being $n^2h$. The lemma asserts that

$$2(h + 2h + 3h + \ldots + n - 1h) < n^2h < 2(h + 2h + 3h + \ldots + nh).$$

It is proved in the book *On Spirals*, Prop. 11. The proof is geometrical, lines being placed side by side to represent the 166
equal quantity, which common difference is equal to the least term, and if there be a second series of magnitudes, equal to the first in number, in which each term is equal to the greatest term [in the first series], the sum of the magnitudes in the series in which each term is equal to the greatest term is less than double of the sum of the magnitudes differing by an equal quantity, but greater than double of the sum of those magnitudes less the greatest term. The proof of this is clear.\textsuperscript{a}

\textit{Ibid.}, Prop. 1, Archim. ed. Heiberg i. 260. 26–261. 22

If there be a series of magnitudes, as many as you please, and it be equal in number to another series of magnitudes, and the terms have the same ratio two by two, and if any or all of the first series of magnitudes form any proportion with another series of magnitudes, and if the second series of magnitudes form the same proportion with the corresponding terms of another series of magnitudes, the sum of the first series of magnitudes bears to the sum of those with which they are in proportion the same ratio as the sum of the second series of magnitudes bears to the sums of the terms with which they are in proportion.

Let the series of magnitudes $A, B, \Gamma, \Delta, E, Z$ be equal in number to the series of magnitudes $H, \Theta, I$, terms of the arithmetical progression and produced until each is equal to the greatest term. It is equivalent to this algebraical proof:

Let \[ S_n = h + 2h + 3h + \ldots + nh. \]
Then \[ S_n = nh + (n - 1)h + (n - 2)h + \ldots + h. \]
Adding, \[ 2S_n = n(n + 1)h, \]
and so \[ 2S_{n-1} = (n - 1)nh. \]
Therefore \[ 2S_{n-1} < n^2h < 2Sn. \]
Greek Mathematics

κατὰ δύο τὸν αὐτὸν ἔχοντα λόγον, καὶ ἐχέτω τὸ μὲν Α ποτὶ τὸ Β τὸν αὐτὸν λόγον, ὁν τὸ Η ποτὶ τὸ Θ, τὸ δὲ Β ποτὶ τὸ Γ, ὁν τὸ Θ ποτὶ τὸ Ι, καὶ τὰ ἄλλα ὁμοίως τοῦτοις, λεγέσθω δὲ τὰ μὲν Α, Β, Γ, Δ, Ε, Ζ μεγέθεα ποτί ἄλλα μεγέθεα τὰ Ν, Ξ, Ο, Π, Ρ, Σ ἐν λόγοις ὁποιοισοῦν, τὰ δὲ Η, Θ, Ι, Κ, Λ, Μ ποτὶ ἄλλα τὰ Τ, Υ, Φ, Χ, Ψ, Ω, τὰ ὁμόλογα ἐν τοῖς αὐτοῖς λόγοις, καὶ ὃν μὲν ἔχει λόγον τὸ Α ποτὶ τὸ Ν, τὸ Η ἐχέτω ποτὶ τὸ Τ, ὁν δὲ λόγον ἔχει τὸ Β ποτὶ τὸ Ξ, τὸ Θ ἐχέτω ποτὶ τὸ Υ, καὶ τὰ ἄλλα ὁμοίως τοῦτοις: δεικτέον, ὅτι πάντα τὰ Α, Β, Γ, Δ, Ε, Ζ ποτὶ πάντα τὰ Ν, Ξ, Ο, Π, Ρ, Σ τὸν αὐτὸν ἔχοντι λόγον, ὅτι πάντα τὰ Η, Θ, Ι, Κ, Λ, Μ ποτὶ πάντα τὰ Τ, Υ, Φ, Χ, Ψ, Ω.

Since

\[ N : A = T : H, \quad A : B = H : \Theta, \quad \text{[ex hyp.]} \]

.: ex aequo

\[ N : B = T : \Theta. \quad \text{[Eucl. v. 22]} \]

But

\[ B : \Xi = \Theta : \Upsilon; \quad \text{[ex hyp.]} \]

.: ex aequo

\[ N : \Xi = T : \Upsilon. \quad \text{[Eucl. v. 22]} \]

Similarly

\[ \Xi : O = \Upsilon : \Phi, \quad O : \Pi = \Phi : \chi, \quad \Pi : P = \chi : \Psi, \quad P : \Sigma = \Psi : \Omega. \]

Now since

\[ A : B = H : \Theta, \quad \text{[ex hyp.]} \]

.: componendo

\[ A + B : A = H + \Theta : H, \quad \text{[Eucl. v. 18]} \]

i.e., permutando

\[ A + B : H + \Theta = A : H. \quad \text{[Eucl. v. 16]} \]

But since

\[ N : A = T : H, \quad \text{[ex hyp.]} \]

.: \[ A : H = N : T \]

\[ = \Xi : \Upsilon \quad \text{[ibid.]} \]

\[ = O : \Phi \quad \text{[ibid.]} \]

\[ = \Gamma : I. \quad \text{[ibid.]} \]

.: \[ A + B : H + \Theta = \Gamma : I. \quad \text{[Eucl. v. 18]} \]

.: \[ A + B + \Gamma : H + \Theta + I = \Gamma : I \]

\[ = O : \Phi \quad \text{[Eucl. v. 16]} \]

168
ARCHIMEDES

K, Λ, M, and let them have the same ratio two by two, so that

\[ \frac{A}{B} = \frac{H}{\Theta}, \frac{B}{\Gamma} = \frac{\Theta}{I}, \]

and so on, and let the series of magnitudes \( A, B, \Gamma, \Delta, E, Z \) form any proportion with another series of magnitudes \( N, \Xi, \Omega, \Pi, P, \Sigma \), and let \( H, \Theta, I, K, \Lambda, M \) form the same proportion with the corresponding terms of another series, \( T, \Upsilon, \Phi, X, \Psi, \Omega \) so that

\[ \frac{A}{N} = \frac{H}{T}, \frac{B}{\Xi} = \frac{\Theta}{\Upsilon}, \]

and so on; it is required to prove that

\[ \frac{A + B + \Gamma + \Delta + E + Z}{N + \Xi + \Omega + \Pi + P + \Sigma} = \frac{H + \Theta + I + K + \Lambda + M}{T + \Upsilon + \Phi + X + \Psi + \Omega} \]

\[ = \Pi : \chi \] [ibid.]
\[ = \Delta : K. \] [ibid.]

By pursuing this method it may be proved that

\[ A + B + \Gamma + \Delta + E + Z : H + \Theta + I + K + \Lambda + M = A : H, \]

or, permutando,

\[ A + B + \Gamma + \Delta + E + Z : A = H + \Theta + I + K + \Lambda + M : H. \] (1)

Now

\[ N : \Xi = T : \Upsilon; \]

:. componendo et permutando,

\[ N + \Xi : T + \Upsilon = \Xi : \Upsilon \] [Eucl. v. 18, v. 16
\[ = O : \Phi; \] [Eucl. v. 16

whence \( N + \Xi + O : T + \Upsilon + \Phi = O : \Phi, \) [Eucl. v. 18

and so on until we obtain

\[ N + \Xi + O + \Pi + P + \Sigma : T + \Upsilon + \Phi + X + \Psi + \Omega = N : T. \] (2)

But

\[ A : N = H : T; \] [ex hyp.

:. by (1) and (2),

\[ \frac{A + B + \Gamma + \Delta + E + Z}{N + \Xi + O + \Pi + P + \Sigma} = \frac{H + \Theta + I + K + \Lambda + M}{T + \Upsilon + \Phi + X + \Psi + \Omega}. \]

Q.E.D.

169
GREEK MATHEMATICS

(iii.) Volume of a Segment of a Paraboloid of Revolution

Ibid., Prop. 21, Archim. ed. Heiberg i. 344. 21–354. 20

Πάν τμάμα ὀρθογωνίου κωνοειδεὸς ἀποτετμαμένον ἐπιπέδῳ ὀρθῷ ποτὶ τὸν ἄξονα ἡμιόλιον ἐστὶ τοῦ κώνου τοῦ βάσιν ἔχοντος τὰν αὐτὰν τῷ τμάματι καὶ ἄξονα.

"Εστω γὰρ τμάμα ὀρθογωνίου κωνοειδεὸς ἀποτετμαμένον ὀρθῷ ἐπιπέδῳ ποτὶ τὸν ἄξονα, καὶ τμαθέντος αὐτοῦ ἐπιπέδῳ ἄλλῳ διὰ τοῦ ἄξονος τὰς μὲν ἐπιφανείας τομὰ ἐστὼ ἀ ΑΒΓ ὀρθογωνίου κόνων τομά, τοῦ δὲ ἐπιπέδου τοῦ ἀποτέμνοντος τὸ τμάμα ἀ ΓΑ εὐθεία, ἄξων δὲ ἔστω τοῦ τμάματος ἀ ΒΔ, ἔστω δὲ καὶ κώνος τὰν αὐτὰν βάσιν ἔχων τῷ τμάματι καὶ ἄξονα τὸν αὐτὸν, οὐ κορυφὰ τὸ Β. θεικτέον, ὅτι τὸ τμάμα τοῦ κωνοειδεὸς ἡμιόλιον ἐστὶ τοῦ κώνου τούτου.

Εκκείσθω γὰρ κώνος ὁ Ψ ἡμιόλιος ἐὼν τοῦ κώνου, οὐ βάσις ὁ περὶ διάμετρον τὰν ΑΓ, ἄξων δὲ ἀ ΒΔ, ἔστω δὲ καὶ κύλινδρος βάσιν μὲν ἔχων 170
(iii.) Volume of a Segment of a Paraboloid of Revolution


Any segment of a right-angled conoid cut off by a plane perpendicular to the axis is one-and-a-half times the cone having the same base as the segment and the same axis.

For let there be a segment of a right-angled conoid cut off by a plane perpendicular to the axis, and let it be cut by another plane through the axis, and let the section be the section of a right-angled cone $\Delta \Gamma$, and let $\Gamma A$ be a straight line in the plane cutting off the segment, and let $B\Delta$ be the axis of the segment, and let there be a cone, with vertex $B$, having the same base and the same axis as the segment. It is required to prove that the segment of the conoid is one-and-a-half times this cone.

For let there be set out a cone $\Psi$ one-and-a-half times as great as the cone with base about the diameter $A\Gamma$ and with axis $B\Delta$, and let there be a

* It is proved in Prop. 11 that the section will be a parabola.  

171
GREEK MATHEMATICS

tον κύκλον τὸν περὶ διάμετρον τὰν ΑΓ, ἀξόνα δὲ τὰν ΒΔ. ἐσσεῖται οὖν ὁ Ψ' κώνος ἡμίσεος τοῦ κυλίνδρου [ἐπείπερ ἡμιόλιος ἐστιν ὁ Ψ' κώνος τοῦ αὐτοῦ κώνου].¹ λέγω, ὅτι τὸ τμάμα τοῦ κωνοειδεὸς ἴσον ἐστὶ τῷ Ψ' κώνῳ.

Εἰ γὰρ μὴ ἐστὶν ἴσον, ἦτοι μείζον ἐντὶ ἦ ἐλασσόν. ἐστώ δὴ πρῶτον, εἰ δυνατόν, μεῖζον. ἐγγεγράφθω δὴ σχῆμα στερεὸν εἰς τὸ τμάμα, καὶ ἀλλο περιγεγράφθω ἐκ κυλίνδρων ύψος ἴσον ἐχόντων συγκείμενον, ὥστε τὸ περιγραφὲν σχῆμα τοῦ ἐγγραφέντος ύπερέχειν ἐλάσσον, ἢ ἀλίκω ὑπερέχει τὸ τοῦ κωνοειδεὸς τμᾶμα τοῦ Ψ' κώνου, καὶ ἐστὼ τῶν κυλίνδρων, εξ ὧν σύγκειται τὸ περιγραφὲν σχῆμα, μέγιστος μὲν ὁ βάσιν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΑΓ, ἀξόνα δὲ τὰν ΕΔ, ἐλάχιστος δὲ ὁ βάσιν μὲν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΣΤ, ἀξόνα δὲ τὰν ΒΙ, τῶν δὲ κυλίνδρων, εξ ὧν σύγκειται τὸ ἐγγραφὲν σχῆμα, μέγιστος μὲν ἐστὼ ὁ βάσιν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΚΛ, ἀξόνα δὲ τὰν ΔΕ, ἐλάχιστος δὲ ὁ βάσιν μὲν ἔχων τὸν κύκλον τὸν περὶ διάμετρον τὰν ΣΤ, ἀξόνα δὲ τὰν ΘΙ, ἐκβεβλήσθω δὲ τὰ ἐπίπεδα πάντων τῶν κυλίνδρων ποτὶ τὰν

¹ ἐπείπερ . . . κώνου om. Heiberg.

* For the cylinder is three times, and the cone Ψ one-and-a-
cylinder having for its base the circle about the diameter $\Gamma \Delta$ and for its axis $\Xi \Delta$; then the cone $\Psi$ is one-half of the cylinder; I say that the segment of the conoid is equal to the cone $\Psi$.

If it be not equal, it is either greater or less. Let it first be, if possible, greater. Then let there be inscribed in the segment a solid figure and let there be circumscribed another solid figure made up of cylinders having an equal altitude, in such a way that the circumscribed figure exceeds the inscribed figure by a quantity less than that by which the segment of the conoid exceeds the cone $\Psi$ [Prop. 19]; and let the greatest of the cylinders composing the circumscribed figure be that having for its base the circle about the diameter $\Gamma \Delta$ and for axis $\Xi \Delta$, and let the least be that having for its base the circle about the diameter $\Sigma \Theta$ and for axis $\Theta \Gamma$; and let the planes of all the cylinders be half times, as great as the same cone; but because $\tau \omega \alpha \tau \omega \kappa \omega \nu \omega \nu$ is obscure and $\epsilon \pi \epsilon \iota \pi \epsilon \rho$ often introduces an interpolation, Heiberg rejects the explanation to this effect in the text.

Archimedes has used those inscribed and circumscribed figures in previous propositions. The paraboloid is generated by the revolution of the parabola $\Delta \Gamma \Gamma$ about its axis $\Xi \Delta$. Chords $\Gamma \Delta \ldots \Sigma \Theta$ are drawn in the parabola at right angles to the axis and at equal intervals from each other. From the points where they meet the parabola, perpendiculars are drawn to the next chords. In this way there are built up inside and outside the parabola "staggered" figures consisting of decreasing rectangles. When the parabola revolves, the rectangles become cylinders, and the segment of the paraboloid lies between the inscribed set of cylinders and the circumscribed set of cylinders.
GREEK MATHEMATICS

ἐπιφάνειαν τοῦ κυλίνδρου τοῦ βάσιν ἔχοντος τοῦ κύκλου τὸν περὶ διάμετρον τὰν ΑΓ, ἄξονα δὲ τὰν ΒΔ· ἐσσεῖται δὴ ὁ ὀλὸς κυλίνδρος διηγημένος εἰς κυλίνδρους τῷ μὲν πλήθει ἴσους τοῖς κυλίνδροις τοῖς ἐν τῷ περιγεγραμμένῳ σχήματι, τῷ δὲ με-γέθει ἴσους τῷ μεγίστῳ αὐτῶν. καὶ ἐπεὶ τὸ περιγεγραμμένον σχήμα περὶ τὸ τμάμα ἐλάσσον ὑπερέχει τοῦ ἐγγεγραμμένου σχήματος ἢ τὸ τμάμα τοῦ κόνου, δὴ λοιπόν, ὅτι καὶ τὸ ἐγγεγραμμένον σχήμα ἐν τῷ τμάματι μείζον ἐστὶ τοῦ Ψ κόνου. ὁ δὴ πρῶτος κυλίνδρος τῶν ἐν τῷ ὄλῳ κυλίνδρῳ ὁ ἔχων ἄξονα τὰν ΔΕ ποτὶ τὸν πρῶτον κυλίνδρον τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι τὸν ἔχοντα ἄξονα τὰν ΔΕ τὸν αὐτὸν ἔχει λόγον, δὲν ἀ ΔΑ ποτὶ τὰν ΚΕ δυνάμει· οὕτως δὲ ἐστὶν ὁ αὐτὸς τῷ, δὲν ἔχει ἀ ΒΔ ποτὶ τὰν ΒΕ, καὶ τῷ, δὲν ἔχει ἀ ΔΑ ποτὶ τὰν ΕΣ. ὁμοίως δὲ δειχθῆσαι καὶ ὁ δεύτερος κυλίνδρος τῶν ἐν τῷ ὄλῳ κυλίνδρῳ ὁ ἔχων ἄξονα τὸν ΕΣ ποτὶ τὸν δεύτερον κυλίνδρον τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι τὸν αὐτὸν ἔχει λόγον, δὲν ἀ ΠΕ, τούτου ἀ ΔΑ, ποτὶ τὰν ΖΟ, καὶ τῶν ἄλλων κυλίνδρων ἐκαστὸς τῶν ἐν τῷ ὄλῳ κυλίνδρῳ ἄξονα ἔχοντων ἴσον τὰ ΔΕ ποτὶ ἐκαστὸν τῶν κυλίνδρων τῶν ἐν τῷ ἐγγεγραμμένῳ σχήματι ἄξονα ἔχοντων τὸν αὐτὸν ἔχει τοῦτον τὸν λόγον, δὲν ἀ ἡμίσεια τὸς διαμέτρου τᾶς βάσιος αὐτοῦ ποτὶ τῶν ἀπολελαμμένων ἀπ' αὐτὰς μεταξὺ τῶν ΑΒ, ΒΔ εὐθείων· καὶ πάντες οὖν οἱ κυλίνδροι οἱ ἐν τῷ κυλίνδρῳ, οὐ βάσις μὲν ἐστὶν ὁ κύκλος ὁ περὶ διάμετρον τὰν ΑΓ, ἄξονα δὲ [ἔστω] ἀ ΔΙ εὐθεία, ποτὶ πάντας τοὺς κυλίνδρους τοὺς ἐν τῷ ἐγ- γεγραμμένῳ σχήματι τὸν αὐτὸν ἔξοντι λόγον, δὲν 174.
produced to the surface of the cylinder having for its base the circle about the diameter $\Delta\Gamma$ and for axis $\Gamma\Delta$; then the whole cylinder is divided into cylinders equal in number to the cylinders in the circumscribed figure and in magnitude equal to the greatest of them. And since the figure circumscribed about the segment exceeds the inscribed figure by a quantity less than that by which the segment exceeds the cone, it is clear that the figure inscribed in the segment is greater than the cone $\Psi$.\(^a\) Now the first cylinder of those in the whole cylinder, that having $\Delta E$ for its axis, bears to the first cylinder in the inscribed figure, which also has $\Delta E$ for its axis, the ratio $\Delta A^2:KE^2$ [Eucl. xii. 11 and xii. 2]; but $\Delta A^2:KE^2=BD:BE=\Delta A:E\Xi$. Similarly it may be proved that the second cylinder of those in the whole cylinder, that having $E\Sigma$ for its axis, bears to the second cylinder in the inscribed figure the ratio $PE:ZO$, that is, $\Delta A:ZO$, and each of the other cylinders in the whole cylinder, having its axis equal to $\Delta E$, bears to each of the cylinders in the inscribed figure, having the same axis in order, the same ratio as half the diameter of the base bears to the part cut off between the straight lines $AB$, $B\Delta$; and therefore the sum of the cylinders in the cylinder having for its base the circle about the diameter $\Delta\Gamma$ and for axis the straight line $\Delta I$ bears to the sum of the cylinders in the inscribed figure the same ratio as the sum of

\(^a\) Because the circumscribed figure is greater than the segment.
\(^b\) By the property of the parabola; v. Quadr. parab. 3.

\(^1\) ἔστιν om. Heiberg.
πάσαι αἱ εὐθείαι αἱ ἐκ τῶν κέντρων τῶν κύκλων, οἱ ἐντὶ βάσιες τῶν εἰρημένων κυλίνδρων, ποτὲ πάσαι τὰς εὐθείας τὰς ἀπολελαμμένας ἀπ' αὐτῶν μεταξὺ τῶν ΑΒ, ΒΔ. αἱ δὲ εἰρημέναι εὐθείαι τῶν εἰρημένων χωρίς τὰς ΔΔ μεῖζονες ἐντὶ ἡ διπλάσιαι: ὥστε καὶ οἱ κυλίνδροι πάντες οἱ ἐν τῷ κυλίνδρῳ, οὗ ἄξων ὁ ΔΙ, μεῖζονες ἐντὶ ἡ διπλάσιοι τοῦ ἐγγεγραμμένου σχήματος· πολλῷ ἄρα καὶ ὁ ὅλος κυλίνδρος, οὗ ἄξων ὁ ΔΒ, μεῖζων ἐντὶ ἡ διπλασίων τοῦ ἐγγεγραμμένου σχήματος. τοῦ δὲ Ψ κώνου ἢν διπλασίων ἐλάσσον ἄρα τὸ ἐγγεγραμμένον σχῆμα τοῦ Ψ κώνου· ὀπερ ἀδύνατον· ἐδείχθη γὰρ μεῖζον. οὖκ ἄρα ἐστὶν μεῖζον τὸ κωνοεἰδὲς τοῦ Ψ κώνου.

'Ομοίως δὲ οὐδὲ ἐλάσσον· πάλιν γὰρ ἐγγεγράφθω τὸ σχῆμα καὶ περιγεγράφθω, ὡστε ὑπερέχειν [ἐκαστὸν] ἐλάσσον, ἡ ἀλίκω ὑπερέχει ὁ Ψ κώνος τοῦ κωνοειδέος, καὶ τὰ ἄλλα τὰ αὐτὰ τῶν πρότερον κατεσκευάσθω. ἐπεὶ οὖν ἐλάσσον ἔστι τὸ ἐγγεγραμμένον σχῆμα τοῦ τμάματος, καὶ τὸ ἐγγραφέν τοῦ περιγραφέντος ἐλάσσον λείπεται ἡ τὸ τμάμα τοῦ Ψ κώνου, δῆλον, ὡς ἐλάσσον ἔστι τὸ περιγραφέν σχῆμα τοῦ Ψ κώνου. πάλιν δὲ ὁ ἐκαστὸν om. Heiberg, ἐκαστὸν ἐκάστου Torelli (for ἐκάτερον ἐκατέρου).

<table>
<thead>
<tr>
<th>i.e.,</th>
<th>First cylinder in whole cylinder</th>
<th>ΑΔ</th>
</tr>
</thead>
<tbody>
<tr>
<td>First cylinder in inscribed figure</td>
<td>ΕΞ</td>
<td></td>
</tr>
<tr>
<td>Second cylinder in whole cylinder</td>
<td>ΕΠ</td>
<td></td>
</tr>
<tr>
<td>Second cylinder in inscribed figure</td>
<td>ΖΟ</td>
<td></td>
</tr>
</tbody>
</table>

and so on.

<table>
<thead>
<tr>
<th>Whole cylinder</th>
<th>ΑΔ + ΕΠ + ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inscribed figure</td>
<td>ΕΞ + ΖΟ + ...</td>
</tr>
</tbody>
</table>
the radii of the circles, which are the bases of the aforesaid cylinders, bears to the sum of the straight lines cut off from them between $AB, BD$. But the sum of the aforesaid straight lines is greater than double of the aforesaid straight lines without $AD$; so that the sum of the cylinders in the cylinder whose axis is $DI$ is greater than double of the inscribed figure; therefore the whole cylinder, whose axis is $DB$, is greater by far than double of the inscribed figure. But it was double of the cone $Ψ$; therefore the inscribed figure is less than the cone $Ψ$; which is impossible, for it was proved to be greater. Therefore the conoid is not greater than the cone $Ψ$.

Similarly [it can be shown] not to be less; for let the figure be again inscribed and another circumscribed so that the excess is less than that by which the cone $Ψ$ exceeds the conoid, and let the rest of the construction be as before. Then because the inscribed figure is less than the segment, and the inscribed figure is less than the circumscribed by some quantity less than the difference between the segment and the cone $Ψ$, it is clear that the circumscribed figure is less than the cone $Ψ$. Again, the first

This follows from Prop. 1, for

\[
\frac{\text{First cylinder in whole cylinder}}{\text{Second cylinder in whole cylinder}} = 1 = \frac{ΔA}{ΕΠ}.
\]

and so on, and thus the other condition of the theorem is satisfied.

For $ΔA, ΕΞ, ZO \ldots$ is a series diminishing in arithmetical progression, and $ΔA, ΕΠ \ldots$ is a series, equal in number, in which each term is equal to the greatest in the arithmetical progression. Therefore, by the Lemma to Prop. 1,

\[ΔA + ΕΠ + \ldots > 2(ΕΞ + ZO + \ldots).\]
As before,

First cylinder in whole cylinder \( \Delta A \)
First cylinder in circumscribed figure \( \Delta A \)
Second cylinder in whole cylinder \( \Delta A = E II \)
Second cylinder in circumscribed figure \( E \Xi = E \Xi \)

and so on.
cylinder of those in the whole cylinder, having $\Delta E$ for its axis, bears to the first cylinder of those in the circumscribed figure, having the same axis $E\Delta$, the ratio $A\Delta^2 : A\Delta^2$; the second cylinder in the whole cylinder, having $EZ$ for its axis, bears to the second cylinder in the circumscribed figure, having $EZ$ also for its axis, the ratio $\Delta A^2 : KE^2$; this is the same as $B\Delta : BE$, and this is the same as $\Delta A : E\Xi$; and each of the other cylinders in the whole cylinder, having its axis equal to $\Delta E$, will bear to the corresponding cylinder in the circumscribed figure, having the same axis, the same ratio as half the diameter of the base bears to the portion cut off from it between the straight lines $AB$, $B\Delta$; and therefore the sum of the cylinders in the whole cylinder, whose axis is the straight line $B\Delta$, bears to the sum of the cylinders in the circumscribed figure the same ratio as the sum of the one set of straight lines bears to the sum of the other set of straight lines.\textsuperscript{a} But the sum of the radii of the circles which are the bases of the cylinders is less than double of the sum of the straight lines cut off from them together with $A\Delta$; it is therefore clear

\[
\frac{\text{First cylinder in whole cylinder}}{\text{Second cylinder in whole cylinder}} = 1 = \frac{\Delta A}{E\Pi},
\]

and so on.

Therefore the conditions of Prop. 1 are satisfied and

\[
\frac{\text{Whole cylinder}}{\text{Circumscribed figure}} = \frac{\Delta A + E\Pi + \ldots}{\Delta A + E\Xi + \ldots}.
\]

\textsuperscript{b} As before, $\Delta A$, $E\Xi$ \ldots is a series diminishing in arithmetical progression, and $\Delta A$, $E\Pi$ \ldots is a series, equal in number, in which each term is equal to the greatest in the arithmetical progression.

Therefore, by the Lemma to Prop. 1,

\[
\Delta A + E\Pi + \ldots < 2(\Delta A + E\Xi + \ldots).
\]

179
GREEK MATHEMATICS

Archimedes' proof may be shown to be equivalent to an integration, as Heath has done (The Works of Archimedes, cxlvii-cxlviii).

For, if \( n \) be the number of cylinders in the whole cylinder, and \( \Delta \Delta = nh \), Archimedes has shown that

\[
\text{Whole cylinder} \quad \frac{n^2h}{\text{Inscribed figure}} = \frac{n^2h}{h + 2h + 3h + \ldots + nh} > 2, \quad \text{[Lemma to Prop. 1]}
\]

and

\[
\text{Whole cylinder} \quad \frac{n^2h}{\text{Circumscribed figure}} = \frac{n^2h}{n + 2h + 3h + \ldots + nh} < 2. \quad \text{[ibid.}]
\]

In Props. 19 and 20 he has meanwhile shown that, by increasing \( n \) sufficiently, the inscribed and circumscribed figures can be made to differ by less than any assigned volume.

180
that the sum of all the cylinders in the whole cylinder is less than double of the cylinders in the circumscribed figure; therefore the cylinder having for its base the circle about the diameter $\Delta \Gamma$ and for axis $B \Delta$ is less than double of the circumscribed figure. But it is not, for it is greater than double; for it is double of the cone $\Psi$, and the circumscribed figure was proved to be less than the cone $\Psi$. Therefore the segment of the conoid is not less than the cone $\Psi$. But it was proved not to be greater; therefore it is one-and-a-half times the cone having the same base as the segment and the same axis.\footnote{\textsection 3}

When $n$ is increased, $h$ is diminished, but their product remains constant; let $nh = c$.

Then the proof is equivalent to an assertion that, when $a$ is indefinitely increased,

\[ \lim_{n \to \infty} n(h+2h+3h+\ldots+(n-1)h) = \frac{1}{2}c^2, \]

which, in the notation of the integral calculus reads,

\[ \int_0^c xdx = \frac{1}{2}c^2. \]

If the paraboloid is formed by the revolution of the parabola $y^2 = ax$ about its axis, we should express the volume of a segment as

\[ \int_0^c \pi y^2dx, \]

or

\[ \pi a \int_0^c xdx. \]

The constant does not appear in Archimedes' proof because he merely compares the volume of the segment with the cone, and does not give its absolute value. But his method is seen to be equivalent to a genuine integration.

As in other cases, Archimedes refrains from the final step of making the divisions in his circumscribed and inscribed figures indefinitely large; he proceeds by the orthodox method of \textit{reductio ad absurdum}.
(e) The Spiral of Archimedes

(1.) Definitions

44. 17–46. 21

α'. Εἰ καὶ εὐθείᾳ ἐπιζευχθῇ γραμμά ἐν ἑπιπέδῳ καὶ μένοντος τοῦ ἐτεροῦ πέρατος αὐτᾶς ἱσοταξέως περιενεχθεῖσα ὀσκίσοις ἀποκατασταθῇ πάλιν, ὅθεν ἄρμασεν, ἀμα δὲ τὰ γραμμὰ περιαγομένα φέρηται τι σαμεῖον ἱσοταξέως αὐτὸ ἑαυτῷ κατὰ τᾶς εὐθείας ἀρξάμενον ἀπὸ τοῦ μένοντος πέρατος, τὸ σαμεῖον ἔλικα γράφει ἐν τῷ ἑπιπέδῳ.

β'. Καλεῖσθω οὖν τὸ μὲν πέρας τᾶς εὐθείας τὸ μένον περιαγομένας αὐτᾶς ἀρχὰ τᾶς ἔλικος.

γ'. 'Α δὲ θέσις τᾶς γραμμᾶς, ἀφ' ἀς ἀρξατο αἱ εὐθεία περιφέρεσθαι, ἀρχὰ τῆς περιφορᾶς.

δ'. Εὐθεία, ἀν μὲν ἐν τὰ πρῶτα περιφορᾶ δια- πορευθῇ τὸ σαμεῖον τὸ κατὰ τᾶς εὐθείας φέρο- μενον, πρώτα καλεῖσθω, ἀν δ' ἐν τὰ δευτέρα περιφορᾶ τὸ αὐτὸ σαμεῖον διανύσῃ, δευτέρα, καὶ αἱ ἄλλαι ὁμοίως ταῦτας ὁμονύμως ταῖς περι- φοραῖς καλεῖσθωσαν.

ε'. Τὸ δὲ χωρὶον τὸ περιλαθέν ὑπὸ τε τὰς ἔλικος τᾶς ἐν τὰ πρῶτα περιφορᾶ γραφεῖσας καὶ τᾶς εὐθείας, ἀ ἐστιν πρῶτα, πρῶτον καλεῖσθω, τὸ δὲ περιλαθέν ὑπὸ τε τὰς ἔλικος τᾶς ἐν τὰ δευτέρα περιφορᾶ γραφεῖσας καὶ τᾶς εὐθείας τᾶς δευτέρας δεύτερον καλεῖσθω, καὶ τὰ ἄλλα ἔξῆς οὕτω καλεῖσθω.

ζ'. Καὶ εἰ καὶ ἀπὸ τοῦ σαμείου, ὃ ἐστιν ἀρχὰ τὰς ἔλικος, ἀνθῇ τις εὐθεία γραμμά, τᾶς εὐθείας ταύτας

182
1. If a straight line drawn in a plane revolve uniformly any number of times about a fixed extremity until it return to its original position, and if, at the same time as the line revolves, a point move uniformly along the straight line, beginning at the fixed extremity, the point will describe a spiral in the plane.

2. Let the extremity of the straight line which remains fixed while the straight line revolves be called the origin of the spiral.

3. Let the position of the line, from which the straight line began to revolve, be called the initial line of the revolution.

4. Let the distance along the straight line which the point moving along the straight line traverses in the first turn be called the first distance, let the distance which the same point traverses in the second turn be called the second distance, and in the same way let the other distances be called according to the number of turns.

5. Let the area comprised between the first turn of the spiral and the first distance be called the first area, let the area comprised between the second turn of the spiral and the second distance be called the second area, and let the remaining areas be so called in order.

6. And if any straight line be drawn from the origin, let [points] on the side of this straight line in
GREEK MATHEMATICS

tά ἐπὶ τά αὐτά, ἐφ' ἂν κα ἂ περιφορὰ γένηται, προαγούμενα καλείσθω, τά δέ ἐπὶ άτερα ἐπόμενα.
ζ' ὁ τε γραφεῖς κύκλος κέντρω μὲν τῷ σαμείῳ, ὃ ἐστιν ἀρχὰ τάς ἐλικος, διαστήματι δέ τά εὐθεία, ἀ ἐστιν πρῶτα, πρῶτος καλείσθω, ὃ δέ γραφεῖς κέντρω μὲν τῷ αὐτῷ, διαστήματι δέ τά διπλασία εὐθεία δεύτερος καλείσθω, καὶ οἱ ἄλλοι δέ ἐξῆς τούτοις τόν αὐτόν τρόπον.

(ii.) Fundamental Property

Ibid., Prop. 14, Archim. ed. Heiberg ii. 50. 9–52. 15

Εἰ κα ποτὶ τᾶν ἐλικα τᾶν ἐν τά πρῶτα περιφορὰ γεγραμμέναν ποτιπέσωντι δύο εὐθείαι ἀπὸ τοῦ σαμείου, ὃ ἐστὶν ἀρχὰ τᾶς ἐλικος, καὶ ἐκβληθέωντι ποτὶ τᾶν πρῶτον κύκλου περιφέρειαν, τῶν αὐτῶν ἔξοντι λόγον αι ποτὶ τᾶν ἐλικα ποτιπίπτουσαι ποτ' ἄλλας, ὅν αἱ περιφέρειαι τοῦ κύκλου αἱ μεταξὺ τοῦ πέρατος τᾶς ἐλικος καὶ τῶν περάτων τῶν ἐκβληθεισάν εὐθεῖαν τῶν ἐπὶ τᾶς περιφέρειας γινομένων, ἐπὶ τὰ προαγούμενα λαμβανομενὰν τῶν περιφέρειαν ἀπὸ τοῦ πέρατος τᾶς ἐλικος.

'Εστω ἔλιξ ἂ ΑΒΓΔΕΘ ἐν τά πρῶτα περιφορὰ γεγραμμένα, ἀρχὰ δὲ τᾶς μὲν ἐλικος ἐστὶν τῷ Ἄ σαμείου, ἄ δε ΘΑ εὐθεία ἀρχὰ τᾶς περιφορᾶς ἐστω, κἀ κύκλος ὃ ΘΚΗ ἐστω ὁ πρῶτος, ποτιπιπτόντων δὲ ἀπὸ τοῦ Ἄ σαμείου ποτὶ τᾶν ἐλικα αἱ ΑΕ, ΑΔ κἀ ἐκπίπτοντων ποτὶ τῶν τοῦ κύκλου περιφέρειαν ἐπὶ τὰ Ζ, Η. δεικτέων, ὅτι τῶν αὐτῶν ἔχοντι λόγον ἂ ΑΕ ποτὶ τῶν ΑΔ, ὅν ἂ ΘΚΖ περιφέρεια ποτὶ τῶν ΘΚΗ περιφέρειαι.

Περιαγομένας γὰρ τᾶς ΑΘ γραμμᾶς δῆλον, ὡς 184
ARCHIMEDES

the direction of the revolution be called forward, and let those on the other side be called rearward.

7. Let the circle described with the origin as centre and the first distance as radius be called the first circle, let the circle described with the same centre and double of the radius of the first circle be called the second circle, and let the remaining circles in order be called after the same manner.

(ii.) Fundamental Property

Ibid., Prop. 14, Archim. ed. Heiberg ii. 50. 9–52. 15

If, from the origin of the spiral, two straight lines be drawn to meet the first turn of the spiral and produced to meet the circumference of the first circle, the lines drawn to the spiral will have the same ratio one to the other as the arcs of the circle between the extremity of the spiral and the extremities of the straight lines produced to meet the circumference, the arcs being measured in a forward direction from the extremity of the spiral.

Let $AB\Gamma\Delta\Theta$ be the first turn of a spiral, let the point $A$ be the origin of the spiral, let $\Theta A$ be the initial line, let $\Theta K H$ be the first circle, and from the point $A$ let $A E, A \Delta$ be drawn to meet the spiral and be produced to meet the circumference of the circle at $Z, H$. It is required to prove that $AE : \Delta = \text{arc } \Theta K Z : \text{arc } \Theta K H$.

When the line $A \Theta$ revolves it is clear that the point

* i.e., with radius equal to the sum of the radii of the first and second circles.
GREEK MATHEMATICS

to μέν Θ σαμειον κατὰ τὰς τοῦ ΘKH κύκλου περιφερείας ἐννεγμένον ἐστὶν ἴσοτάχεως, τὸ δὲ

A κατὰ τὰς εὐθείας ψερόμενον τὰν ΑΘ γραμμάν πορεύεται, καὶ τὸ Θ σαμείον κατὰ τὰς τοῦ κύκλου περιφερείας ψερόμενον τὰν ΘΚΖ περιφέρειαν, τὸ δὲ Α τὰν ΑΕ εὐθείαν, καὶ πάλιν τὸ τε Α σαμείον τὰν ΑΔ γραμμάν καὶ τὸ Θ τὰν ΘKH περιφέρειαν, ἐκάτερον ἴσοτάχεως αὐτὸ ἐαυτῷ ψερόμενον δήλον οὖν, ὅτι τὸν αὐτὸν ἑξοντι λόγον ὁ ἈΕ ποτὶ τὰν ΑΔ, ὅν ἡ ΘΚΖ περιφέρεια ποτὶ τὰν ΘKH περιφέρειαν δεδεκταί γὰρ τοῦτο ἔξω ἐν τοῖς πρῶτοις.1

'Ὅμοιως δὲ δειχθήσεται, καὶ εἴ καὶ ἡ ἑτέρα τὰν ποτιπτουσᾶν ἐπὶ τὸ πέρας τὰς ἐλικοὺς ποτιπτῆς, ὅτι τὸ αὐτὸ συμβαίνει.

(iii.) A Verging

Ibid., Prop. 7, Archim. ed. Heiberg ii. 22. 14–24. 7

Τῶν αὐτῶν δεδομένων καὶ τὰς ἐν τῷ κύκλῳ εὐθείας ἐκβεβλημένας δυνατὸν ἐστὶν ἀπὸ τοῦ t

186
ARCHIMEDES

\( \Theta \) moves uniformly round the circumference \( \Theta KH \) of the circle while the point \( A \), which moves along the straight line, traverses the line \( A\Theta \); the point \( \Theta \) which moves round the circumference of the circle traverses the arc \( \Theta KZ \) while \( A \) traverses the straight line \( AE \); and furthermore the point \( A \) traverses the line \( \Delta A \) in the same time as \( \Theta \) traverses the arc \( \Theta KH \), each moving uniformly; it is clear, therefore, that

\[
AE : A\Delta = \text{arc } \Theta KZ : \text{arc } \Theta KH \quad \text{[Prop. 2]}
\]

Similarly it may be shown that if one of the straight lines be drawn to the extremity of the spiral the same conclusion follows.\(^a\)

(iii.) \textit{A Verging} \(^b\)

\textit{Ibid., Prop. 7, Archim. ed. Heiberg ii. 22. 14–24. 7}

With the same data and the chord in the circle produced,\(^c\) it is possible to draw a line from the centre to meet

- In Prop. 15 Archimedes shows (using different letters, however) that if \( AE, A\Delta \) are drawn to meet the second turn of the spiral, while \( AZ, AH \) are drawn, as before, to meet the circumference of the first circle, then

\[
AE : A\Delta = \text{arc } \Theta KZ + \text{circumference of first circle : arc } \Theta KH + \text{circumference of first circle},
\]

and so on for higher turns.

In general, if \( E, \Delta \) lie on the \( n \)th turn of the spiral, and the circumference of the first circle is \( c \), then

\[
AE : A\Delta = \text{arc } \Theta KZ + n - 1c : \text{arc } \Theta KH + n - 1c.
\]

These theorems correspond to the equation of the curve \( r = a\theta \) in polar co-ordinates.

\(^b\) This theorem is essential to the one that follows.

\(^c\) See n. \( a \) on this page.

\(^a\) \( \delta \epsilon \delta \epsilon i k t a i . . . \pi \rho \omega t o i s \) om. Heiberg.
κέντρου ποτιβαλείν ποτὲ τὰν ἐκβεβλημέναν, ὡστε
τὰν μεταξὺ τάς περιφερείας καὶ τάς ἐκβεβλημένας
ποτὲ τὰν ἐπὶζευκθεῖσαν ἀπὸ τοῦ πέρατος τὰς
ἐναπολαφθεῖσας ποτὲ τὸ πέρας τὰς ἐκβεβλημένας
tὸν ταχθέντα λόγον ἔχεω, εἴ καὶ ὁ δοθεὶς λόγος
μείζων ή τοῦ, ὃν ἔχει ἀ ἡμίσεια τᾶς ἐν τῷ κύκλῳ
dεδομένας ποτὲ τὰν ἀπὸ τοῦ κέντρου κάθετον ἐπ᾿
αὐτὰν ἀγμέναν.

Δεδοσθω τὰ αὐτά, καὶ ἔστω ἀ ἐν τῷ κύκλῳ
gραμμὰ ἐκβεβλημένα, ὃ δὲ δοθεὶς λόγος ἔστω, ὃν
ἔχει ἀ Ζ ποτὲ τὰν ἤ, μείζων τοῦ, ὃν ἔχει ἀ ΓΘ
ποτὲ τὰν ΘΚ. μείζων οὖν ἐσσείαν καὶ τοῦ, ὃν ἔχει
ἀ ΚΓ ποτὲ ΓΛ. ὃν δὴ λόγον ἔχει ἀ Ζ ποτὲ Ἡ
τοῦτον ἔχει ἀ ΚΓ ποτὲ ἐλάσσονα τὰς ΓΛ. ἔχεται
ποτὶ ΙΝ νεύονσαν ἐπὶ τὸ Γ—δυνατὸν δὲ ἔστω
οὕτως τέμνει—καὶ πεσεῖται ἐντὸς τὰς ΓΛ, ἐπειδὴ
ἐλάσσον ἐστὶ τὰς ΓΛ. ἐπεὶ οὖν τὸν αὐτὸν ἔχει
λόγον ἀ ΚΓ ποτὲ IN, ὃν ἀ Ζ ποτὲ Ἡ, καὶ ἀ ΕΙ
ποτὶ ΙΓ τὸν αὐτὸν ἔχει λόγον, ὃν ἀ Ζ ποτὶ τὰν Ἡ.

* ΑΓ is a chord in a circle of centre K, and BN is the
diameter drawn parallel to AG and produced. From K,
ΚΘ is drawn perpendicular to AG, and ΓΛ is drawn per-
pendicular to KG so as to meet the diameter in Λ. Archimedes
asserts that it is possible to draw KE to meet the circle in I
and AG produced in E so that ΕΙ : ΙΓ = Ζ : Η, an assigned
ratio, provided that Ζ : Η > ΘΚ : ΘΚ. The straight line ΙΓ
meets ΒΛ in Ν. In Prop. 5 Archimedes has proved a similar
proposition when AG is a tangent, and in Prop. 6 he has
proved the proposition for the case where the positions of I,
Γ are reversed.

b For triangle ΓΠΕ is similar to triangle KΙΝ, and therefore
ΚΙ : IN = ΕΙ : ΙΓ [Eucl. vi. 4]; and ΚΙ = KG.

c The type of problem known as νεύοσεις, vergings, has
already been encountered (vol. i. p. 244 n. a). In this pro-
position, as in Props. 5 and 6, Archimedes gives no hint how
188
the produced chord so that the distance between the circumference and the produced chord shall bear to the distance between the extremity of the line intercepted [by the circle] and the extremity of the produced chord an assigned ratio, provided that the given ratio is greater than that which half of the given chord in the circle bears to the perpendicular drawn to it from the centre.

Let the same things be given, and let the chord in the circle be produced, and let the given ratio be \(Z : H\), and let it be greater than \(\Gamma \Theta : \Theta K\); therefore it will be greater than \(\Gamma \Gamma : \Gamma \Lambda\) [Eucl. vi. 4]. Then \(Z : H\) is equal to the ratio of \(\Gamma \Gamma\) to some line less than \(\Gamma \Lambda\) [Eucl. v. 10]. Let it be to \(IN\) verging upon \(\Gamma\)—for it is possible to make such an intercept—and \(IN\) will fall within \(\Gamma \Lambda\), since it is less than \(\Gamma \Lambda\). Then since \(\Gamma \Gamma : IN = Z : H\), therefore \(EI : II = Z : H\).

the construction is to be accomplished, though he was presumably familiar with a solution.

In the figure of the text, let \(T\) be the foot of the perpen-
GREEK MATHEMATICS

(iv.) Property of the Subtangent

Ibid., Prop. 20, Archim. ed. Heiberg ii. 72. 4-74. 26

Εἰ κα τάς ἔλικος τᾶς ἐν ταῖ πρώτα περιφορὰ γεγραμμένας εὐθεία γραμμὰ ἐπιψαύη μὴ κατά τὸ πέρας τᾶς ἔλικος, ἀπὸ δὲ τᾶς ἄφας ἐπὶ τὰν ἄρχαν τᾶς ἔλικος εὐθεία ἐπιζευγχῇ, καὶ κέντρῳ μὲν τὰ ἄρχα τᾶς ἔλικος, διαστήματι δὲ τὰ ἐπιζευγχθεῖσα κύκλος γραφῇ, ἀπὸ δὲ τᾶς ἄρχας τᾶς ἔλικος ἅχθῃ τοῦ ὀρθᾶς τὰ ἀπὸ τᾶς ἄφας ἐπὶ τὰν ἄρχαν τᾶς ἔλικος ἐπιζευγχθεῖσα, συμπεσεῖται αὕτα ποτὶ τὰν ἐπιψαύονσαν, καὶ ἐσοεῖται ὁ μεταξὺ εὐθεία τᾶς τε συμπτῶσις καὶ τὰς ἄρχας τᾶς ἔλικος ἵσα τὰ περιφερεῖα τοῦ γραφέντος κύκλου τὰ μεταξὺ τᾶς ἄφας καὶ τὰς τομᾶς, καθ' ἂν τέμνῃ ὁ γραφεῖς κύκλος τὰν ἄρχαν τὰς περιφορὰς, ἐπὶ τὰ προαγούμενα λαμβανομένας τᾶς περιφερεῖας ἀπὸ τοῦ σαμείου τοῦ ἐν τὰ ἄρχα τὰς περιφορὰς.

Εστὶν ἔλικα, ἐφ' ὡς ἄ ΑΒΓΔ, ἐν τά πρώτα περιφορὰ γεγραμμένα, καὶ ἐπιψαυνέτω τις αὐτάς εὐθεία ἐς ΕΖ κατὰ τὸ Δ, ἀπὸ δὲ τοῦ Δ ποτὶ τὰν δικαλὰ from Γ to ΒΛ, and let Δ be the other extremity of the diameter through Β. Let the unknown length KN=x, let ΓΤ=a, KT=b, ΒΔ=2c, and let IN=k, a given length.
Then

\[ k \sqrt{a^2 + (x-b)^2} = (x-c)(x+c), \]

which, after rationalization, is an equation of the fourth degree in x.

Alternatively, if we denote \( \text{NG} \) by \( y \), we can determine \( x \) and \( y \) by the two equations

\[ y^2 = a^2 + (x-b)^2, \]
\[ ky = x^2 - c^2, \]
(iv.) Property of the Subtangent

Ibid., Prop. 20, Archim. ed. Heiberg ii. 72. 4–74. 26

If a straight line touch the first turn of the spiral other than at the extremity of the spiral, and from the point of contact a straight line be drawn to the origin, and with the origin as centre and this connecting line as radius a circle be drawn, and from the origin a straight line be drawn at right angles to the straight line joining the point of contact to the origin, it will meet the tangent, and the straight line between the point of meeting and the origin will be equal to the arc of the circle between the point of contact and the point in which the circle cuts the initial line, the arc being measured in the forward direction from the point on the initial line.

Let $AB\Gamma\Delta$ lie on the first turn of a spiral, and let the straight line $EZ$ touch it at $\Delta$, and from $\Delta$ let $\Delta\Delta$ so that values of $x$ and $y$ satisfying the conditions of the problem are given by the points of intersection of a certain parabola and a certain hyperbola.

The whole question of vergings, including this problem, is admirably discussed by Heath, The Works of Archimedes, c-cxxii.
GREEK MATHEMATICS

ἀρχὰν τᾶς ἔλικος ἐπεξεύχθω ἀ ΛΔ, καὶ κέντρῳ μὲν τῷ Α, διαστήματι δὲ τῷ ΑΔ κύκλῳ γεγράφῳ ὁ ΔΜΝ, τεμνέτω δὲ οὕτως τὰν ἀρχὰν τᾶς περιφορᾶς κατὰ τὸ Κ, ἄχθω δὲ ἀ ΖΑ ποτὶ τὰν ΑΔ ὀρθά. ὅτι μὲν οὖν αὕτα συμπίπτει, δὴλον· ὅτι δὲ καὶ ίσα ἐστὶν ἀ ΖΑ εὐθείᾳ τὰ ΚΜΝΔ περιφερείᾳ, δεικτέον.

Εἰ γὰρ μὴ, ἦτοι μεῖζων ἐστὶν ἡ ἐλάσσων. ἐστὶν, εἰ δυνατὸν, πρότερον μεῖζων, λελάφθω δὲ τις ἀ ΛΔ τὰς μὲν ΖΑ εὐθείας ἐλάσσων, τὰς δὲ ΚΜΝΔ περιφερείας μεῖζων. πάλιν δὴ κύκλος ἐστὶν ὁ ΚΜΝ καὶ ἐν τῷ κύκλῳ γραμμὰ ἐλάσσων τὰς διαμέτρου ἀ ΔΝ καὶ λόγος, ὃν ἔχει ἀ ΔΑ ποτὶ ΑΛ, μεῖζων τοῦ, ὃν ἔχει ἡ ἡμίσεια τὰς ΔΝ ποτὶ τὰν ἀπὸ τοῦ Α κάθετον ἐπ' αὐτάν ἁγμέναν· δυνατον οὖν ἐστὶν ἀπὸ τοῦ Α ποτιβαλεὶν τὰν ΑΕ ποτὶ τὰν ΝΔ ἐκβεβλημέναν, ὡστε τὰν ΕΡ ποτὶ τὰν ΔΡ τὸν αὐτὸν ἔχειν λόγον, ὃν ἀ ΔΑ ποτὶ τὰν ΑΛ· δέδεικται γὰρ τοῦτο δυνατὸν ἐόν· ἔχει οὖν καὶ ἀ ΕΡ ποτὶ τὰν ΔΡ τὸν αὐτὸν λόγον, ὃν ἀ ΔΡ ποτὶ τὰν ΑΛ. ἀ δὲ ΔΡ ποτὶ τὰν ΑΛ ἐλάσσονα λόγον ἔχει ἡ ἀ ΔΡ περιφέρεια ποτὶ τὰν ΚΜΔ περιφέρειαν, ἐπεὶ ἀ μὲν ΔΡ ἐλάσσων ἐστὶ τὰς ΔΡ περιφερείας, ἀ δὲ ΑΛ μεῖζων τᾶς ΚΜΔ περιφερείας· ἐλάσσονα οὖν λόγον ἔχει ἀ ΕΡ εὐθείᾳ ποτὶ ΡΑ ἡ ἀ ΔΡ περιφέρεια ποτὶ τὰν ΚΜΔ περιφέρειαν· ῥᾴστε καὶ ἀ ΑΕ ποτὶ ΔΡ ἐλάσσονα λόγον ἔχει ἡ ἀ ΚΜΡ περιφέρεια ποτὶ τὰν ΚΜΔ περι-

+a For in Prop. 16 the angle ΑΔΖ was shown to be acute.
+b For ΔΝ touches the spiral and so can have no part within the spiral, and therefore cannot pass through Α; therefore it is a chord of the circle and less than the diameter.
+c For, if a perpendicular be drawn from Α to ΔΝ, it bisects
ARCHIMEDES

be drawn to the origin, and with centre $A$ and radius $AA$ let the circle $\Delta MN$ be described, and let this circle cut the initial line at $K$, and let $ZA$ be drawn at right angles to $AA$. That it will meet $[Z\Delta]$ is clear; it is required to prove that the straight line $ZA$ is equal to the arc $KMN\Delta$.

If not, it is either greater or less. Let it first be, if possible, greater, and let $AA$ be taken less than the straight line $ZA$, but greater than the arc $KMN\Delta$ [Prop. 4]. Again, $KMN$ is a circle, and in this circle $\Delta N$ is a line less than the diameter, and the ratio $AA : AA$ is greater than the ratio of half $\Delta N$ to the perpendicular drawn to it from $A$; it is therefore possible to draw from $A$ a straight line $AE$ meeting $N\Delta$ produced in such a way that

$$EP : \Delta P = \Delta A : AA;$$

for this has been proved possible [Prop. 7]; therefore

$$EP : AP = \Delta P : AA.$$

But

$$\Delta P : AA < \text{arc} \ \Delta P : \text{arc} \ KM\Delta,$$

since $\Delta P$ is less than the arc $\Delta P$, and $AA$ is greater than the arc $KM\Delta$;

therefore

$$EP : PA < \text{arc} \ \Delta P : \text{arc} \ KM\Delta;$$

$$AE : AP < \text{arc} \ KMP : \text{arc} \ KM\Delta.$$  

[Eucl. v. 18]

$\Delta N$ [Eucl. iii. 3] and divides triangle $\Delta AZ$ into two triangles of which one is similar to triangle $\Delta AZ$ [Eucl. vi. 8]; therefore

$$\Delta A : AZ = \frac{1}{2} N\Delta : (\text{perpendicular from } A \text{ to } N\Delta).$$

But

$$AZ > AA;$$

therefore

$$\Delta A : AA > \frac{1}{2} N\Delta : (\text{perpendicular from } A \text{ to } N\Delta).$$

For $\Delta A = AP$, being a radius of the same circle; and the proportion follows permutando.
This part of the proof involves a *verging* assumed in Prop. 8, just as the earlier part assumed the *verging* of Prop. 7. The *verging* of Prop. 8 has already been described (vol. i. p. 350 n. b) in connexion with Pappus’s comments on it.

Archimedes goes on to show that the theorem is true even if the tangent touches the spiral in its second or some higher turn, not at the extremity of the turn; and in Props. 18 and 19 he has shown that the theorem is true if the tangent should touch at an extremity of a turn.

194
ARCHIMEDES

Now \( \text{arc KMP} : \text{arc KMD} = \text{XA} : \text{AD} \); \[\text{Prop. 14} \]

\[ \therefore \text{EA} : \text{AP} < \text{AX} : \text{DA} ; \]

which is impossible. Therefore \( \text{ZA} \) is not greater than the arc \( \text{KMD} \). In the same way as above it may be shown to be not less \(^a\); therefore it is equal. \(^b\)

\((f)\) Semi-Regular Solids

Pappus, Collection v. 19, ed. Hultsch i. 352. 7–354. 10

Although many solid figures having all kinds of surfaces can be conceived, those which appear to be regularly formed are most deserving of attention. Those include not only the five figures found in the godlike Plato, that is, the tetrahedron and the cube, the octahedron and the dodecahedron, and fifthly the icosahedron,\(^c\) but also the solids, thirteen in number, which were discovered by Archimedes\(^d\) and are contained by equilateral and equiangular, but not similar, polygons.

As Pappus (ed. Hultsch 302. 14-18) notes, the theorem can be established without recourse to propositions involving \textit{solid loci} (for the meaning of which see vol. i. pp. 348-349), and proofs involving only “plane” methods have been developed by Tannery, \textit{Mémoires scientifiques}, i., 1912, pp. 300-316 and Heath, \textit{H.G.M.} ii. 556-561. It must remain a puzzle why Archimedes chose his particular method of proof, especially as Heath’s proof is suggested by the figures of Props. 6 and 9; Heath (\textit{loc. cit.}, p. 557) says “it is scarcely possible to assign any reason except his definite predilection for the form of proof by \textit{reductio ad absurdum} based ultimately on his famous ‘Lemma’ or Axiom.”

\(^a\) For the five regular solids, see vol. i. pp. 216-225.

\(^b\) Heron (\textit{Definitions} 104, ed. Heiberg 66. 1-9) asserts that two were known to Plato. One is that described as \(P_2\) below, but the other, said to be bounded by eight squares and six triangles, is wrongly given.
GREEK MATHEMATICS

Τὸ μὲν γὰρ πρῶτον ὀκτάεδρον ἑστὶν περιεχόμενον ὑπὸ τριγώνων δ' καὶ ἕξαγώνων δ'.

Τρία δὲ μετὰ τοῦτο τετσαρεσκαίδεκαέδρα, ὅν τὸ μὲν πρῶτον περιέχεται τριγώνοις η' καὶ τετραγώνοις ε', τὸ δὲ δεύτερον τετραγώνοις ε' καὶ ἕξαγώνοις η', τὸ δὲ τρίτον τριγώνοις η' καὶ ὀκταγώνοις ε'.

Μετὰ δὲ ταῦτα ἑκκαιεικοσάεδρα ἑστὶν δύο, ὅν τὸ μὲν πρῶτον περιέχεται τριγώνοις η' καὶ τετραγώνοις ιη', τὸ δὲ δεύτερον τετραγώνοις ιβ', ἕξαγώνοις η' καὶ ὀκταγώνοις ε'.

Μετὰ δὲ ταῦτα δυοκαιτρικοντάεδρα ἑστὶν τρία, ὅν τὸ μὲν πρῶτον περιέχεται τριγώνοις ι' καὶ πενταγώνοις ιβ', τὸ δὲ δεύτερον πενταγώνοις ιβ καὶ ἕξαγώνοις ι', τὸ δὲ τρίτον τριγώνοις ι' καὶ δεκαγώνοις ιβ'.

Μετὰ δὲ ταῦτα ἑν ἑστὶν ὀκτωκαιτρικοντάεδρον περιεχόμενον ὑπὸ τριγώνων λβ καὶ τετραγώνων ε'.

Μετὰ δὲ τοῦτο δυοκαιεξηκοντάεδρα ἑστὶν δύο, ὅν τὸ μὲν πρῶτον περιέχεται τριγώνοις ι' καὶ τετραγώνοις λ καὶ πενταγώνοις ιβ, τὸ δὲ δεύτερον τετραγώνοις λ καὶ ἕξαγώνοις ι' καὶ δεκαγώνοις ιβ'.

Μετὰ δὲ ταῦτα τελευταίον ἑστὶν δυοκαιενηκοντάεδρον, ὅ περιέχεται τριγώνοις π καὶ πενταγώνοις ιβ'.

---

a For the purposes of n. b, the thirteen polyhedra will be designated as $P_1, P_2 \ldots P_{13}$.

b Kepler, in his Harmonice mundi (Opera, 1864, v. 123-126), appears to have been the first to examine these figures systematically, though a method of obtaining some is given in a scholium to the Vatican ms. of Pappus. If a solid angle of a regular solid be cut by a plane so that the same length is cut off from each of the edges meeting at the solid angle,
ARCHIMEDES

The first is a figure of eight bases, being contained by four triangles and four hexagons \([P_4]\)^a.

After this come three figures of fourteen bases, the first contained by eight triangles and six squares \([P_5]\), the second by six squares and eight hexagons \([P_6]\), and the third by eight triangles and six octagons \([P_7]\).

After these come two figures of twenty-six bases, the first contained by eight triangles and eighteen squares \([P_8]\), the second by twelve squares, eight hexagons and six octagons \([P_9]\).

After these come three figures of thirty-two bases, the first contained by twenty triangles and twelve pentagons \([P_{10}]\), the second by twelve pentagons and twenty hexagons \([P_{11}]\), and the third by twenty triangles and twelve decagons \([P_{12}]\).

After these comes one figure of thirty-eight bases, being contained by thirty-two triangles and six squares \([P_{13}]\).

After this come two figures of sixty-two bases, the first contained by twenty triangles, thirty squares and twelve pentagons \([P_{14}]\), the second by thirty squares, twenty hexagons and twelve decagons \([P_{15}]\).

After these there comes lastly a figure of ninety-two bases, which is contained by eighty triangles and twelve pentagons \([P_{16}]\)^b.

the section is a regular polygon which is a triangle, square or pentagon according as the solid angle is composed of three, four or five plane angles. If certain equal lengths be cut off in this way from all the solid angles, regular polygons will also be left in the faces of the solid. This happens (i) obviously when the cutting planes bisect the edges of the solid, and (ii) when the cutting planes cut off a smaller length from each edge in such a way that a regular polygon is left in each face with double the number of sides. This method gives (1) from the tetrahedron, \(P_1\); (2) from the

197
GREEK MATHEMATICS

(g) System of expressing Large Numbers

Archim. Aren. 3, Archim. ed. Heiberg ii. 236. 17–240. 1

"A μὲν οὖν ὑποτίθεμαι, ταῦτα· χρήσιμον δὲ εἰμεν ὕπολαμβάνω τὰν κατονύμαξιν τῶν ἀριθμῶν ῥηθήμεν, ὡς καὶ τῶν ἄλλων οἱ τῶν βιβλίων μὴ περιτετευχότες τῷ ποτὶ Ζεύξιππον γεγραμμένῳ μὴ πλανώνται διὰ τὸ μηδὲν εἰμεν ὑπὲρ αὐτῶς ἐν τῶδε τῷ βιβλίῳ προειρημένον. συμβαίνει δὴ τὰ ὀνόματα τῶν ἀριθμῶν ἐστὶ τὸ μὲν τῶν μυρίων ὑπάρχειν ἀμῶν παραδεδομένα, καὶ ὑπὲρ τὸ τῶν μυρίων [μὲν] ἀποχρέοντως γυγινῶσκομες μυριάδων ἀριθμῶν λέγοντες ἐστε ποτὶ τὰς μυρίας μυριάδας. ἐστων οὖν ἀμῶν οἱ μὲν νῦν εἰρημένοι ἀριθμοὶ ἐστὶ τὰς μυρίας μυριάδας πρῶτοι καλουμένοι, τῶν δὲ πρῶτων ἀριθμῶν αἱ μύριαι μυριάδες μονὰς καλείσθω δευτέρων ἀριθμῶν, καὶ ἀριθμεῖσθων τῶν δευτέρων μονάδων καὶ ἐκ τῶν μονάδων δεκάδες καὶ ἐκατοντάδες καὶ χιλιάδες καὶ μυριάδες ἐστὶ τὰς μυρίας μυριάδας. πάλιν δὲ καὶ αἱ μύριαι μυριάδες τῶν δευτέρων ἀριθμῶν μονὰς καλείσθω τρίτων ἀριθμῶν, καὶ ἀριθμεῖσθων τῶν τρίτων ἀριθμῶν μονάδων καὶ ἀπὸ τῶν μονάδων δεκάδες καὶ ἐκατοντάδες καὶ χιλιάδες καὶ μυριάδες ἐστὶ τὰς μυρίας μυριάδας. τὸν αὐτὸν δὲ τρόπον καὶ τῶν τρίτων ἀριθμῶν μύρια μυριάδες μονὰς καλείσθω τετάρτων ἀριθμῶν,

1 μὲν om. Heiberg.

cube, $P_2$ and $P_4$; (3) from the octahedron, $P_2$ and $P_5$; (4) from the icosahedron, $P_7$ and $P_8$; (5) from the dodecahedron, $P_7$ and $P_9$. It was probably the method used by Plato.

Four more of the semi-regular solids are obtained by first cutting all the edges symmetrically and equally by planes parallel to the edges, and then cutting off angles. This

198
Such are then the assumptions I make; but I think it would be useful to explain the naming of the numbers, in order that, as in other matters, those who have not come across the book sent to Zeuxippus may not find themselves in difficulty through the fact that there had been no preliminary discussion of it in this book. Now we already have names for the numbers up to a myriad \(10^3\), and beyond a myriad we can count in myriads up to a myriad myriads \(10^8\). Therefore, let the aforesaid numbers up to a myriad myriads be called *numbers of the first order* [numbers from 1 to \(10^8\)], and let a myriad myriads of numbers of the first order be called a unit of *numbers of the second order* [numbers from \(10^8\) to \(10^{16}\)], and let units of the numbers of the second order be enumerable, and out of the units let there be formed tens and hundreds and thousands and myriads up to a myriad myriads. Again, let a myriad myriads of numbers of the second order be called a unit of *numbers of the third order* [numbers from \(10^{16}\) to \(10^{24}\)], and let units of numbers of the third order be enumerable, and from the units let there be formed tens and hundreds and thousands and myriads up to a myriad myriads. In the same manner, let a myriad myriads of numbers of the third order be
gives (1) from the cube, \(P_5\) and \(P_6\); (2) from the icosahedron, \(P_{11}\); (3) from the dodecahedron, \(P_{12}\).

The two remaining solids are more difficult to obtain; \(P_{10}\) is the *snub cube* in which each solid angle is formed by the angles of four equilateral triangles and one square; \(P_{13}\) is the *snub dodecahedron* in which each solid angle is formed by the angles of four equilateral triangles and one regular pentagon.
GREEK MATHEMATICS

καὶ αἱ τῶν τετάρτων ἀριθμῶν μὺριαὶ μυριάδες
μονὰς καλείσθω πέμπτων ἀριθμῶν, καὶ άεὶ οὕτως
προάγοντες οἱ ἀριθμοὶ τὰ ὄνοματα ἔχοντων ἐς τὰς
μυριακισμυριστῶν ἀριθμῶν μυρίας μυριάδας.

Ἀποχρέοντι μὲν οὖν καὶ ἐπὶ τοσοῦτον οἱ ἀριθμοὶ
γιγνωσκόμενοι, ἔξεστι δὲ καὶ ἐπὶ πλέον προάγειν.
ἔστων γὰρ οἱ μὲν νῦν εἰρημένοι ἀριθμοὶ πρῶτας
περίοδου καλομένοι, ὃ δὲ ἐσχατὸς ἀριθμὸς τὰς
πρώτας περίοδου μονὰς καλείσθω δευτέρας περι-
όδου πρῶτων ἀριθμῶν. πάλιν δὲ καὶ αἱ μύριαι
μυριάδες τὰς δευτέρας περίοδου πρώτων ἀριθμῶν
μονὰς καλείσθω τὰς δευτέρας περίοδου δευτέρων
ἀριθμῶν. ὡμοίως δὲ καὶ τούτων ὁ ἐσχατὸς μονὰς
καλείσθω δευτέρας περίοδου τρίτων ἀριθμῶν, καὶ
ἄεὶ οὕτως οἱ ἀριθμοὶ προάγοντες τὰ ὄνοματα
ἔχοντων τὰς δευτέρας περίοδου ἐς τὰς μυριακι-
κυσμυριστῶν ἀριθμῶν μυρίας μυριάδας.

Πάλιν δὲ καὶ ὁ ἐσχατὸς ἀριθμὸς τὰς δευτέρας
περίοδου μονὰς καλείσθω τρίτων περίοδου πρώτων
ἀριθμῶν, καὶ ἄεὶ οὕτως προαγόντων ἐς τὰς μυρια-
κυσμυριστὰς περίοδου μυριακυσμυριστῶν ἀριθμῶν
μυρίας μυριάδας.

* Expressed in full, the last number would be 1 followed
by 80,000 million millions of ciphers. Archimedes uses this
system to show that it is more than sufficient to express the
number of grains of sand which it would take to fill the
universe, basing his argument on estimates by astronomers
of the sizes and distances of the sun and moon and their
relation to the size of the universe and allowing a wide margin
for safety. Assuming that a poppy-head (for so μῆκος is here
to be understood, not “poppy-seed,” v. D’Arcy W. Thompson,
The Classical Review, lvi. (1942), p. 75) would contain not
more than 10,000 grains of sand, and that its diameter is
not less than a finger’s breadth, and having proved that the
200
called a unit of numbers of the fourth order [numbers from $10^{24}$ to $10^{32}$], and let a myriad myriads of numbers of the fourth order be called a unit of numbers of the fifth order [numbers from $10^{32}$ to $10^{40}$], and let the process continue in this way until the designations reach a myriad myriads taken a myriad myriad times $[10^8 \cdot 10^8]$.

It is sufficient to know the numbers up to this point, but we may go beyond it. For let the numbers now mentioned be called numbers of the first period [1 to $10^8 \cdot 10^8$], and let the last number of the first period be called a unit of numbers of the first order of the second period $[10^8 \cdot 10^8$ to $10^8 \cdot 10^8 \cdot 10^8]$. And again, let a myriad myriads of numbers of the first order of the second period be called a unit of numbers of the second order of the second period $[10^8 \cdot 10^8 \cdot 10^8$ to $10^8 \cdot 10^8 \cdot 10^8 \cdot 10^8]$. Similarly let the last of these numbers be called a unit of numbers of the third order of the second period $[10^8 \cdot 10^8 \cdot 10^8 \cdot 10^8$ to $10^8 \cdot 10^8 \cdot 10^8 \cdot 10^8 \cdot 10^8]$, and let the process continue in this way until the designations of numbers in the second period reach a myriad myriads taken a myriad myriad times $[10^8 \cdot 10^8 \cdot 10^8 \cdot 10^8$, or $(10^8 \cdot 10^8)^2$].

Again, let the last number of the second period be called a unit of numbers of the first order of the third period $[(10^8 \cdot 10^8)^2$ to $(10^8 \cdot 10^8)^3 \cdot 10^8]$, and let the process continue in this way up to a myriad myriad units of numbers of the myriad myriadth order of the myriad myriadth period $[(10^8 \cdot 10^8)^{10^8}$ or $10^8 \cdot 10^{16}]$.

sphere of the fixed stars is less than $10^7$ times the sphere in which the sun's orbit is a great circle, Archimedes shows that the number of grains of sand which would fill the universe is less than "10,000,000 units of the eighth order of numbers," or $10^{68}$. The work contains several references important for the history of astronomy.
GREEK MATHEMATICS

(h) INDETERMINATE ANALYSIS: THE CATTLE PROBLEM


Πρόβλημα

ὅπερ Ἀρχιμήδης ἐν ἐπιγράμμασιν εὐρών τοῖς ἐν Ἀλεξανδρείᾳ περὶ ταῦτα πραγματευομένους ἔρημον ἀπέστειλεν ἐν τῇ πρὸς Ἐρατοσθένην τὸν Κυρηναίου ἐπιστολῇ.

Πληθὺν ᾿Ηελίου βοῶν, ὃ ἐζεῖνε, μέτρησον
φροντίδ’ ἐπιστήμασις, εἰ μετέχεις σοφίας,
πόση ἄρ’ ἐν πεδίοις Σικελίας ποτ’ ἐβόσκετο νήσον
Ορινάχης τετραχή στίφεα δασσαμένη
χροῆν ἀλλάσσοντα· τὸ μὲν λευκοῦ γάλακτος,
κυανέω δ’ ἐτερον χρώματι λαμπάομενον,
ἀλλ’ ἄρ’ ἐπὶ νεῦρον τὸ ποικίλον. ἐν δὲ ἐκάστῳ
στίφει ἐσαν ταῦροι πλῆθει βριθόμενοι
συμμετρίας τοιηδεὶς τετευχότες· ἀργότριχας μὲν
κυανέων ταῦρων ἡμίσει ἢδὲ τρίτῳ
καὶ εἰσαχθός σύμπασαν ἵσους, ὃ ζεῖνε, νόησον,
αὐτῶς κυανέους τῷ τετράτῳ τε μέρει
μικτοχρόων καὶ πέμπτῳ, ἐτὶ εἰσαχθός τε πάσιν.
τοὺς δ’ ὄποιοι ποικιλόχρωτος ἄρθρει
ἀργεννών ταῦρων ἐκτῶν μέρει ἐβδομάτω τε
καὶ εἰσαχθός αὐτῶς πάσιν ἱσαζομένους.
θηλείασι δὲ βοῶν τάδ’ ἐπλετό· λευκότριχες μὲν
ἵσαν συμπάσας κυανέως ἀγέλης
τῷ τριτάτῳ τε μέρει καὶ τετράτῳ ἀτρικές ἵσαι.
αὐτῶς κυανέαι τῷ τετράτῳ τε πάλιν
μικτοχρών καὶ πέμπτῳ ὑμοῦ μέρει ἱσαζοντο
σὺν ταῦροις πάσαις εἰς νομὸν ἐρχομέναις.

202
which Archimedes solved in epigrams, and which he communicated to students of such matters at Alexandria in a letter to Eratosthenes of Cyrene.

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, the third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now

It is unlikely that the epigram itself, first edited by G. E. Lessing in 1773, is the work of Archimedes, but there is ample evidence from antiquity that he studied the actual problem. The most important papers bearing on the subject have already been mentioned (vol. i. p. 16 n. c), and further references to the literature are given by Heiberg ad loc.
GREEK MATHEMATICS

ξανθοτρίχων δ' ἀγέλης πέμπτω μέρει ἕδε καὶ ἐκτὸς ποικίλαι ἰσάριθμον πλήθος ἔχων τετραχή.
ξανθαὶ δ' ἤρθιμεντο μέρους τρίτου ἡμίσει ἵσαι ἀργεννῆς ἀγέλης ἐβδομάτῳ τε μέρει.
ξεῖνε, σὺ δ', 'Ηλείοιο βοῖς πόσαι, ἀτρεκὲς εἰπὼν, χωρὶς μὲν ταύρων ἑστρεφέων ἀριθμόν, χωρὶς δ' αὖ, θύλειαι ὡσαι κατὰ χροῖν ἔκασται, οὐκ ἀιδρίς κε λέγοι' οὔτ' ἀριθμῶν ἀδαίν, οὐ μὴν πώ γε σοφοίς ἐναριθμοῖς. ἀλλ' ἰθι φράζευ καὶ τάδε πάντα βοῶν 'Ηλείοιο πάθη.
ἀργότριχες ταύροι μὲν ἐπεὶ μιξαῖάτο πλήθων κυνάειοι, ἱσταντ' ἐμπεδὸν ἱσομετροὶ εἰς βάθος εἰς εὗρος τε, τὰ δ' αὖ περιμήκεα πάντη πίμπλαντο πλήθους. Θυρακίης πεδία.
ξανθοὶ δ' αὖτ' εἰς ἐν καὶ ποικίλιοι ἀθροισθέντες ἱσταντ' ἀμβολάδην εἰς ἐνὸς ἀρχόμενοι σχῆμα τελειοῦντες τὸ τρικράσπεδον οὔτε προσόντων ἀλλοχρῶν ταύρων οὔτ' ἐπιλειπομένων.
ταῦτα συνεξευρών καὶ ἐνὶ πραπίδεσσων ἀθροίσας καὶ πληθέων ἀποδοὺς, ξεῖνε, τὰ πάντα μέτρα ἔρχεο κυδώμων νυκτηρός ἵσθι τε πάντως κεκριμένος ταύτῃ γ' ὀμπνιος ἐν σοφίᾳ.

1 πλήθους Krumbiegel, πλίνθου cod.

*a* i.e. a fifth and a sixth both of the males and of the females.

*b* At a first glance this would appear to mean that the sum of the number of white and black bulls is a square, but this makes the solution of the problem intolerably difficult. There is, however, an easier interpretation. If the bulls are packed together so as to form a square figure, their number need not be a square, since each bull is longer than it is broad. The simplified condition is that the sum of the number of white and black bulls shall be a rectangle.

204
the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise. But come, understand also all these conditions regarding the cows of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

If
\[ X, x \text{ are the numbers of white bulls and cows respectively,} \]
\[ Y, y \text{ " black " } \]
\[ Z, z \text{ " yellow } \]
\[ W, w \text{ " dappled } \]

the first part of the epigram states that
\[ X = (\frac{1}{4} + \frac{1}{8}) Y + Z \quad \text{(1)} \]
\[ Y = (\frac{1}{4} + \frac{1}{8}) W + Z \quad \text{(2)} \]
\[ W = (\frac{1}{4} + \frac{1}{8}) X + Z \quad \text{(3)} \]
GREEK MATHEMATICS

(i) Mechanics: Centres of Gravity

(i.) Postulates

Heiberg ii. 124. 3–126. 3

α'. Αἰτούμεθα τὰ ἵσα βάρεα ἀπὸ ἰσων μακέων ἰσορροπεῖν, τὰ δὲ ἵσα βάρεα ἀπὸ τῶν ἀνίσων μακέων μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος τὸ ἀπὸ τοῦ μείζονος μάκεος.

(b)

\[ x = (\frac{1}{2} + \frac{i}{2}) (Y + y) \quad \ldots \quad (4) \]
\[ y = (\frac{1}{2} + \frac{i}{2}) (W + w) \quad \ldots \quad (5) \]
\[ w = (\frac{1}{2} + \frac{i}{2}) (Z + z) \quad \ldots \quad (6) \]
\[ z = (\frac{1}{2} + \frac{i}{2}) (X + x) \quad \ldots \quad (7) \]

The second part of the epigram states that

\[ X + Y = \text{a rectangular number} \quad \ldots \quad (8) \]
\[ Z + W = \text{a triangular number} \quad \ldots \quad (9) \]

This was solved by J. F. Wurm, and the solution is given by A. Amthor, Zeitschrift für Math. u. Physik. (Hist.-litt. Abtheilung), xxy. (1880), pp. 153-171, and by Heath, The Works of Archimedes, pp. 319-326. For reasons of space, only the results can be noted here.

Equations (1) to (7) give the following as the values of the unknowns in terms of an unknown integer \( n \):

\[ X = 10366482n \quad x = 7206360n \]
\[ Y = 7460514n \quad y = 4893246n \]
\[ Z = 4149387n \quad z = 5439213n \]
\[ W = 7358060n \quad w = 3515820n \]

We have now to find a value of \( n \) such that equation (9) is also satisfied—equation (8) will then be simultaneously satisfied. Equation (9) means that

\[ Z + W = \frac{p(p + 1)}{2}, \]

where \( p \) is some positive integer, or

\[ (4149387 + 7358060)n = \frac{p(p + 1)}{2}. \]
ARCHIMEDES

(i) Mechanics: Centres of Gravity

(i.) Postulates

Archimedes, On Plane Equilibriums, a Definitions,
Archim. ed. Heiberg ii. 124. 3-126. 3

1. I postulate that equal weights at equal distances balance, and equal weights at unequal distances do not balance, but incline towards the weight which is at the greater distance.

\[ 2471 \cdot 4657n = \frac{p(p+1)}{2} \]

This is found to be satisfied by \( n = 3^3 \cdot 4349 \), and the final solution is

\[
X = 1217263415886 \quad x = 846192410280 \\
Y = 876035935422 \quad y = 574579625058 \\
Z = 487233469701 \quad z = 638688708099 \\
W = 864005479380 \quad w = 412838131860
\]

and the total is 5916837175686.

If equation (8) is taken to be that \( X + Y \) a square number, the solution is much more arduous; Amthor found that in this case,

\[ W = 1598 \cdot 206541 \]

where \( \cdot 206541 \) means that there are 206541 more digits to follow, and the whole number of cattle = 7766 \( \cdot 206541 \). Merely to write out the eight numbers, Amthor calculates, would require a volume of 660 pages, so we may reasonably doubt whether the problem was really framed in this more difficult form, or, if it were, whether Archimedes solved it.

a This is the earliest surviving treatise on mechanics; it presumably had predecessors, but we may doubt whether mechanics had previously been developed by rigorous geometrical principles from a small number of assumptions. References to the principle of the lever and the parallelogram of velocities in the Aristotelian Mechanics have already been given (vol. i. pp. 430-433).
GREEK MATHEMATICS

β'. εἰ καὶ βαρέων ἰσορροπεόντων ἀπὸ τινῶν μακέων ποτὶ τὸ ἔτερον τῶν βαρέων ποτιτεθῇ, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος ἐκεῖνο, ὥστε ἡ εἰδήσης.

γ'. Ὁμοίως δὲ καὶ, εἰ καὶ ἀπὸ τοῦ ἔτερον τῶν βαρέων ἀφαίρεθῇ τι, μὴ ἰσορροπεῖν, ἀλλὰ ῥέπειν ἐπὶ τὸ βάρος, ἀφ' ὅς οὖν ἀφιρρέθη.

δ'. Τῶν ἵσων καὶ ὁμοίων σχημάτων ἐπιπέδων ἐφαρμοζομένων ἐπὶ ἀλλαλα καὶ τὰ κέντρα τῶν βαρέων ἐφαρμόζει ἐπὶ ἀλλαλα.

ε'. Τῶν δὲ ἄνισων, ὁμοίων δὲ, τὰ κέντρα τῶν βαρέων ὁμοίως ἐσσεῖται κείμενα. ὁμοίως δὲ λέγομες σαμείᾳ κέσθαι ποτὶ τὰ ὁμοία σχήματα, ἀφ' ὅν ἐπὶ τὰς ἵσας γωνίας ἀγόμεναι εὑθεὶα ποιέοντι γωνίας ἵσας ποτὶ τὰς ὁμολόγους πλευράς.

ζ'. Εἰ καὶ μεγέθεα ἀπὸ τινῶν μακέων ἰσορροπεῖντι, καὶ τὰ ἵσα αὐτοῖς ἀπὸ τῶν αὐτῶν μακέων ἰσορροπήσει.

ζ'. Παντὸς σχήματος, οὐ καὶ αὐτὸς ἐπὶ τὰ αὐτὰ κοίλα ἡ, τὸ κέντρον τοῦ βάρεος ἐντὸς εἰμεν δεῖ τοῦ σχήματος.

(ii.) Principle of the Lever

Ibid., Props. 6 et 7, Archim. ed. Heiberg ii. 132. 13–138. 8

ζ'

Τὰ σύμμετρα μεγέθεα ἰσορροπεύοντι ἀπὸ μακέων ἀντιπεπουθότως τὸν αὐτὸν λόγον ἐχόντων τοῖς βάρεσιν.

Εἰστώ σύμμετρα μεγέθεα τὰ A, B, ὅπερ κέντρα τὰ A, B, καὶ μάκος ἐστώ τι τὸ EΔ, καὶ ἐστώ, ὥστε τὸ A ποτὶ τὸ B, οὕτως τὸ ΔΓ μάκος ποτὶ τὸ ΓΕ 208
ARCHIMEDES

2. If weights at certain distances balance, and something is added to one of the weights, they will not remain in equilibrium, but will incline towards that weight to which the addition was made.

3. Similarly, if anything be taken away from one of the weights, they will not remain in equilibrium, but will incline towards the weight from which nothing was subtracted.

4. When equal and similar plane figures are applied one to the other, their centres of gravity also coincide.

5. In unequal but similar figures, the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures, I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides.

6. If magnitudes at certain distances balance, magnitudes equal to them will also balance at the same distances.

7. In any figure whose perimeter is concave in the same direction, the centre of gravity must be within the figure.

(ii.) Principle of the Lever

_Ibid._, Props. 6 and 7, Archim. ed. Heiberg

ii. 132. 13–138. 8

Prop. 6

Commensurable magnitudes balance at distances reciprocally proportional to their weights.

Let \( A, B \) be commensurable magnitudes with centres [of gravity] \( A, B \), and let \( E\Delta \) be any distance, and let \( A : B = \Delta \Gamma : \Gamma E \);

\[ 209 \]
GREEK MATHEMATICS

μάκος· δεικτέον, ὅτι τοῦ ἐξ ἀμφοτέρων τῶν Α, Β συγκεκριμένου μεγέθεος κέντρον ἐστὶ τοῦ βάρεως τοῦ Γ.

'Επει γάρ ἐστιν, ὡς τὸ Α ποτὶ τὸ Β, οὕτως τὸ ΔΓ ποτὶ τὸ ΓΕ, τὸ δὲ Α τῷ Β σύμμετρον, καὶ τὸ ΓΔ ἀρα τῷ ΓΕ σύμμετρον, τούτεστιν εὐθεία τᾷ εὐθείᾳ· ὥστε τῶν ΕΓ, ΓΔ ἐστὶ κοινὸν μέτρον. ἐστω δὴ τὸ Ν, καὶ κεῖσθω τὰ μὲν ΕΓ ἵσα ἐκατέρα τῶν ΔΗ, ΔΚ, τὰ δὲ ΔΓ ἵσα ἄ ΕΛ. καὶ ἐπεὶ ἵσα

\[ \begin{array}{c}
\text{A} \\
\hline
\text{B} \\
\hline
\text{Z}
\end{array} \]

\[ \begin{array}{cccccccc}
\Lambda & E & \Gamma & H & \Delta & K \\
\end{array} \]

\[ \text{Ν} \]

ἀ ΔΗ τᾶ ΓΕ, ἵσα καὶ ἀ ΔΓ τᾶ ΕΗ· ὥστε καὶ ἀ ΔΕ ἵσα τᾶ ΕΗ. διπλασία ἀρα ἀ μὲν ΛΗ τᾶς ΔΓ, ἀ δὲ ΗΚ τᾶς ΓΕ· ὥστε τὸ Ν καὶ ἐκατέραν τῶν ΛΗ, ΗΚ μετρεῖ, ἐπειδήπερ καὶ τὰ ἡμισει αὐτῶν. καὶ ἐπεὶ ἐστιν, ὡς τὸ Α ποτὶ τὸ Β, οὕτως ἀ ΔΓ ποτὶ ΓΕ, ὡς δὲ ἀ ΔΓ ποτὶ ΓΕ, οὕτως ἀ ΔΗ ποτὶ ΗΚ—διπλασία γὰρ ἐκατέρα ἐκατέρας—καὶ ὡς ἀρα τὸ Α ποτὶ τὸ Β, οὐτως ἀ ΔΗ ποτὶ ΗΚ. ὅσπλασίων δὲ ἐστὶν ἀ ΔΗ τᾶς Ν, τοσαυ-

210
ARCHIMEDES

it is required to prove that the centre of gravity of
the magnitude composed of both \( A, B \) is \( \Gamma \).

Since \( A : B = \Delta \Gamma : \Gamma E \),
and \( A \) is commensurate with \( B \), therefore \( \Gamma \Delta \) is com-
mensurate with \( \Gamma E \), that is, a straight line with a
straight line [Eucl. x. 11]; so that \( \Gamma E, \Gamma \Delta \) have a
common measure. Let it be \( N \), and let \( \Delta H, \Delta K \) be
each equal to \( \Gamma \Gamma' \), and let \( \Gamma \Lambda \) be equal to \( \Delta \Gamma \). Then
since \( \Delta H = \Gamma E \), it follows that \( \Delta \Gamma = \Delta \Lambda \); so that
\( \Delta \Gamma E = \Delta H \). Therefore \( \Delta H = 2 \Delta \Gamma \) and \( \Delta K = 2 \Gamma E \); so
that \( N \) measures both \( \Delta \Lambda \) and \( \Delta K \), since it measures
their halves [Eucl. x. 12]. And since

\[ A : B = \Delta \Gamma : \Gamma E, \]

while \( \Delta \Gamma : \Gamma E = \Delta \Lambda : \Delta K \)

for each is double of the other—

therefore \( A : B = \Delta \Lambda : \Delta K \).

Now let \( Z \) be the same part of \( A \) as \( N \) is of \( \Delta \Lambda \);
тαπλασίων ἐστὶ καὶ τὸ Α τοῦ Ζ· ἐστὶν ἅρα, ώσ ἀ ΛΗ ποτὶ Ν, οὕτως τὸ Α ποτὶ Ζ. ἐστὶ δὲ καὶ, ὡς ἀ ΚΗ ποτὶ ΛΗ, οὕτως τὸ Β ποτὶ Α· δι’ ἵσου ἅρα ἐστὶν, ὡς ἀ ΚΗ ποτὶ Ν, οὕτως τὸ Β ποτὶ Ζ· ἵσακις ἅρα πολλαπλασίων ἐστὶν ἀ ΚΗ τὰς Ν καὶ τὸ Β τοῦ Ζ. ἐδείχθη δὲ τοῦ Ζ καὶ τὸ Α πολλαπλάσιον ἐόν· ὦστε τὸ Ζ τῶν Α, Β κοινὸν ἐστὶ μέτρον. διαφρεδίσεις οὖν τὰς μὲν ΛΗ εἰς τὰς τὰ Ν ἵσας, τοῦ δὲ Α εἰς τὰ τῷ Ζ ἵσα, τὰ ἐν τὰ ΛΗ τμάματα ἰσομεγέθεα τὰ Ν ἵσα ἐσσεῖται τῷ πλήθει τοῖς ἐν τῷ Α τμαμάτεσσον ἰσοις ἐδόσα τῷ Ζ. ὦστε, ἀν ἐφ’ ἐκαστον τῶν τμαμάτων τῶν ἐν τὰς ΛΗ ἐπιτεθη μέγεθος ἵσον τῷ Ζ τὸ κέντρον τοῦ βάρεος ἔχων ἐπὶ μέσον τοῦ τμάματος, τὰ τε πάντα μεγέθεα ὑσα ἐντὶ τῶν Α, καὶ τοῦ ἐκ πάντων συγκειμένου κέντρον ἐσσείται τοῦ βάρεος τῷ Ε· ἀρτιά τε γάρ ἐστι τὰ πάντα τῷ πλήθει, καὶ τὰ ἐφ’ ἐκάτερα τοῦ Ε ἵσα τῷ πλήθει διὰ τὸ ἴσαν εἴμεν τὰν ΛΕ τὰς ΗΕ.

Ὀμοίως δὲ δεικθήσεται, ότι κἂν, εἰ καὶ ἐφ’ ἐκαστον τῶν ἐν τὰς ΚΗ τμαμάτων ἐπιτεθῇ μέγεθος ἵσον τῷ Ζ κέντρον τοῦ βάρεος ἔχων ἐπὶ τοῦ μέσου τοῦ τμάματος, τὰ τε πάντα μεγέθεα ὑσα ἐσσεῖται τῷ Β, καὶ τοῦ ἐκ πάντων συγκειμένου κέντρον τοῦ βάρεος ἐσσεῖται τὸ Δ· ἐσσεῖται οὖν τὸ μὲν Α ἐπικείμενον κατὰ τὸ Ε, τὸ δὲ Β κατὰ τὸ Δ· ἐσσεῖται δὴ μεγέθεα ὑσα ἀλλάλως ἐτ’ εὐθείας κείμενα, ὅν τὰ κέντρα τοῦ βάρεος ὑσα ἀπ’ ἀλλάλων διέστακεν, συγκείμενα[1] ἀρτια τῷ πλήθει· δῆλον οὖν, ὃτι τοῦ ἐκ πάντων συγκειμένου μεγέθεος κέντρον ἐστὶ τοῦ βάρεος ἀ διχοτομία τὰς εὐθείας τὰς ἐχούσας τὰ κέντρα τῶν μέσων μεγεθέων. ἐπεὶ δ’ ἴσαι ἐντὶ 212
then $\Delta H : N = A : Z$. [Eucl. v., Def. 5]
And $KH : \Delta H = B : A$; [Eucl. v. 7, coroll.]
therefore, ex aequo,
$$KH : N = B : Z;$$ [Eucl. v. 22]
therefore $Z$ is the same part of $B$ as $N$ is of $KH$. Now $A$ was proved to be a multiple of $Z$; therefore $Z$ is a common measure of $A$, $B$. Therefore, if $\Delta H$ is divided into segments equal to $N$ and $A$ into segments equal to $Z$, the segments in $\Delta H$ equal in magnitude to $N$ will be equal in number to the segments of $A$ equal to $Z$. It follows that, if there be placed on each of the segments in $\Delta H$ a magnitude equal to $Z$, having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to $A$, and the centre of gravity of the figure compounded of them all will be $E$; for they are even in number, and the numbers on either side of $E$ will be equal because $\Lambda E = HE$. [Prop. 5, coroll. 2.]

Similarly it may be proved that, if a magnitude equal to $Z$ be placed on each of the segments [equal to $N$] in $KH$, having its centre of gravity at the middle of the segment, the sum of the magnitudes will be equal to $B$, and the centre of gravity of the figure compounded of them all will be $\Delta$ [Prop. 5, coroll. 2]. Therefore $A$ may be regarded as placed at $E$, and $B$ at $\Delta$. But they will be a set of magnitudes lying on a straight line, equal one to another, with their centres of gravity at equal intervals, and even in number; it is therefore clear that the centre of gravity of the magnitude compounded of them all is the point of bisection of the line containing the centres [of gravity] of the middle magnitudes [from Prop. 5, coroll. 2].

---

1 συγκείμενα om. Heiberg.
Greek Mathematics

ά μὲν ΔΕ τά ΓΔ, ἀ δὲ ΕΓ τά ΔΚ, καὶ ολα ἄρα ά ΛΓ ἵσα τά ΓΚ. ὥστε τοῦ ἐκ πάντων μεγέθεος κέντρον τοῦ βάρεος τὸ Γ σαμεῖον. τοῦ μὲν ἄρα Α κευμένου κατὰ τὸ Ε, τοῦ δὲ Β κατὰ τὸ Δ, ἰσορροπησοῦντι κατὰ τὸ Γ.

ζ'

Καὶ τοῖνυν, εἲ καὶ ἀσύμμετρα ἐσωτε τὰ μεγέθεα, ομοίως ἰσορροπησοῦντι ἀπὸ μακέων ἀντιπεπονθό-
tως τὸν αὐτὸν λόγον ἐχόντων τοῖς μεγέθεσιν. Ἑστῶ ἀσύμμετρα μεγέθεα τὰ ΑΒ, Γ, μάκεα
de τὰ ΔΕ, ΕΖ, ἐχέτω δὲ τὸ ΑΒ ποτὶ τὸ Γ τὸν αὐτὸν λόγον, δὴ καὶ τὸ ΕΔ ποτὶ τὸ ΕΖ μάκος-
λέγω, ὅτι τοῦ εἶ ἀμφιτέρων τῶν ΑΒ, Γ κέντρον
tοῦ βάρεος ἔστι τὸ Ε.

Εἰ γὰρ μή ἰσορροπήσει τὸ ΑΒ τεθέν ἐπὶ τῶν Ζ
tῶ Γ τεθέντι ἐπί τῶ Δ, ἵτου μεῖζον ἔστι τὸ ΑΒ

\[\text{Diagram of two rectangles: } \Gamma, AB, \text{ and } \text{loipon} \text{ to } \Delta \text{ symme} \text{tron} \text{ 214}\]
ARCHIMEDES

And since \( \Delta E = \Gamma \Delta \) and \( \Gamma \Delta = \Delta K \), therefore \( \Delta \Gamma = \Gamma K \); so that the centre of gravity of the magnitude compounded of them all is the point \( \Gamma \). Therefore if \( A \) is placed at \( E \) and \( B \) at \( \Delta \), they will balance about \( \Gamma \).

Prop. 7

And now, if the magnitudes be incommensurable, they will likewise balance at distances reciprocally proportional to the magnitudes.

Let \((A + B)\), \( \Gamma \) be incommensurable magnitudes,\(^a\) and let \( \Delta E, EZ \) be distances, and let

\[(A + B) : \Gamma = E\Delta : EZ ;\]

I say that the centre of gravity of the magnitude composed of both \((A + B)\), \( \Gamma \) is \( E \).

For if \((A + B)\) placed at \( Z \) do not balance \( \Gamma \) placed at \( \Delta \), either \((A + B)\) is too much greater than \( \Gamma \) to balance or less. Let it [first] be too much greater, and let there be subtracted from \((A + B)\) a magnitude less than the excess by which \((A + B)\) is too much greater than \( \Gamma \) to balance, so that the remainder \( A \) is

\(^a\) As becomes clear later in the proof, the first magnitude is regarded as made up of two parts—\( A \), which is commensurate with \( \Gamma \) and \( B \), which is not commensurate; if \((A + B)\) is too big for equilibrium with \( \Gamma \), then \( B \) is so chosen that, when it is taken away, the remainder \( A \) is still too big for equilibrium with \( \Gamma \). Similarly if \((A + B)\) is too small for equilibrium.

---

1 τὁ \( \Gamma \) om. Eutocius.
2 τὸ om. Eutocius.
εἶμεν τῷ Γ. ἐπεὶ οὖν σύμμετρά ἐστιν τὰ Α, Γ μεγέθεαι, καὶ ἐλάσσονα λόγου ἔχει τὸ Α ποτὶ τὸ Γ ἢ ἀ ΔΕ ποτὶ ΕΖ, οὐκ ἰσορροπησοῦντι τὰ Α, Γ ἀπὸ τῶν ΔΕ, ΕΖ μακέων, τεθέντος τοῦ μὲν Α ἐπὶ τῷ Ζ, τοῦ δὲ Γ ἐπὶ τῷ Δ. διὰ ταύτα δ', οὔτε εἰ τὸ Γ μεῖζόν ἐστιν ἥ ὡστε ἰσορροπεῖν τῷ ΑΒ.

(iii.) Centre of Gravity of a Parallelogram

_Ibid._, Props. 9 et 10, Archim. ed. Heiberg ii. 140. 16–144. 4

θ'

Παντὸς παραλληλόγραμμον τὸ κέντρον τοῦ βάρεός ἐστιν ἐπὶ τὰς εὐθείας τὰς ἐπιζευγνυώσας τὰς διχοτομίας τὰν κατ' ἐναντίον τοῦ παραλληλόγραμμον πλευράν.

*Εστὶν παραλληλόγραμμον τὸ ΑΒΓΔ, ἐπὶ δὲ τὰν διχοτομίαν τὰν ΑΒ, ΓΔ ἀ ΕΖ· φαμὶ δὴ, ὅτι τοῦ ΑΒΓΔ παραλληλόγραμμον τὸ κέντρον τοῦ βάρεος ἐσσεῖται ἐπὶ τὰς ΕΖ.

Μὴ γάρ, ἄλλ', εἰ δυνατόν, ἐστὶν τὸ Θ, καὶ ἄχθων παρὰ τὰν ΑΒ ἀ ΘΙ. τὰς [δὲ]¹ δὴ ΕΒ διχοτομομένας αἰεὶ ἐσσεῖται ποικὰ κἀ καταλεῖπομένα ἑλάσσων

¹ δὲ om. Heiberg.

* The proof is incomplete and obscure; it may be thus completed.

Since

A : Γ < ΔΕ : EZ,

Δ will be depressed, which is impossible, since there has been taken away from (A + B) a magnitude less than the _deduco-

216
ARCHIMEDES

commensurate with $\Gamma$. Then, since $A$, $\Gamma$ are commensurable magnitudes, and

$$A : \Gamma < \Delta E : EZ,$$

$A, \Gamma$ will not balance at the distances $\Delta E, EZ, A$ being placed at $Z$ and $\Gamma$ at $\Delta$. By the same reasoning, they will not do so if $\Gamma$ is greater than the magnitude necessary to balance $(A + B)$.

(iii.) Centre of Gravity of a Parallelogram


ii. 140. 16–144. 4

Prop. 9

The centre of gravity of any parallelogram is on the straight line joining the points of bisection of opposite sides of the parallelogram.

Let $AB\Gamma\Delta$ be a parallelogram, and let $EZ$ be the straight line joining the mid-points of $AB, \Gamma\Delta$; then I say that the centre of gravity of the parallelogram $AB\Gamma\Delta$ will be on $EZ$.

For if it be not, let it, if possible, be $\Theta$, and let $\Theta I$ be drawn parallel to $AB$. Now if $EB$ be bisected, and the half be bisected, and so on continually, there will be left some line less than $I\Theta$; [let $EK$ be less than

The centres of gravity of a triangle and a trapezium are also found by Archimedes in the first book; the second book is wholly devoted to finding the centres of gravity of a parabolic segment and of a portion of it cut off by a parallel to the base.

217
τὰς ἘΖ· καὶ διηρήσθω ἐκατέρα τῶν ΑΕ, ΕΒ εἰς τὰς τὰ ἘΚ ἴσας, καὶ ἀπὸ τῶν κατὰ τὰς διαφέρεις

σαμεῖων ἄχωσαν παρὰ τὰν ΕΖ· διαιρεθήσεται δὴ τὸ ὅλον παραλληλόγραμμον εἰς παραλληλόγραμμα τὰ ἴσα καὶ ὁμοία τῷ ΚΖ. τῶν οὖν παραλληλογράμμων τῶν ἴσων καὶ ὁμοίων τῷ ΚΖ ἐφαρμοζομένων ἐπ’ ἀλλαλα καὶ τὰ κέντρα τοῦ βάρεως αὐτῶν ἐπ’ ἀλλαλα πεσοῦνται. ἐσοφόται δὴ μεγέθεα τῶν, παραλληλόγραμμα ἴσα τῷ ΚΖ, ἄρτια τῷ πλῆθει, καὶ τὰ κέντρα τοῦ βάρεως αὐτῶν ἐπ’ εὐθείας κείμενα, καὶ τὰ μέσα ἴσα, καὶ πάντα τὰ ἐφ’ ἐκατέρα τῶν μέσων αὐτὰ τε ἴσα ἐντε καὶ αἱ μεταξὺ τῶν κέντρων εὐθείαι ἴσαι· τοῦ ἐκ πάντων αὐτῶν ἃρα συγκεκμένου μεγέθεος τὸ κέντρον ἐσσεῖται τοῦ βάρεως ἐπὶ τὰς εὐθείας τὰς ἐπιζευγνωσάς τὰ κέντρα τοῦ βάρεως τῶν μέσων χωρίων. οὐκ ἔστι δὲ· τὸ γὰρ Θ ἐκτὸς ἐστὶ τῶν μέσων παραλληλογράμμων. φανερὸν οὖν, ὅτι ἐπ’ τὰς ΕΖ εὐθείας τὸ κέντρον ἐστὶ τοῦ βάρεως τοῦ ΑΒΓΔ παραλληλογράμμου.

Πάντως παραλληλογράμμου τὸ κέντρον τοῦ βάρεως ἐστὶ τὸ σαμεῖον, καθ’ δ’ αἱ διαμέτροι συμπίπτοντι.
ARCHIMEDES

Prop. 10

The centre of gravity of any parallelogram is the point in which the diagonals meet.
GREEK MATHEMATICS

"Εστιν παραλληλόγραμμον τὸ ἉΒΓΔ καὶ ἐν αὐτῷ ἀ ΕΖ δίχα τέμνουσα τὰς ἉΒ, ΓΔ, ἀ δὲ ΚΛ τὰς ΑΓ, ΒΔ. ἔστιν δὴ τοῦ ἉΒΓΔ παραλληλο-γράμμον τὸ κέντρον τοῦ βάρεος ἐπὶ τὰς ΕΖ. δέδεικται γὰρ τοῦτο. διὰ ταύτα δὲ καὶ ἐπὶ τὰς ΚΛ· τὸ Θ ἀρα σαμεῖον κέντρον τοῦ βάρεος. κατὰ δὲ τὸ Θ αἱ διαμέτροι τοῦ παραλληλογράμμου συμπέπτοντε· ὥστε δέδεικται τὸ προτεθέν.

(j) Mechanical Method in Geometry

ii. 426. 3-430. 22

'Αρχιμήδης Ἐρατοσθένει εὖ πράττειν . . .
'Ορὼν δὲ σὲ, καθάπερ λέγω, σπουδαίον καὶ φιλοσοφίας προεστῶτα ἀξιολόγως καὶ τὴν ἐν τοῖσ

* According to Heath (H.G.M. ii. 21), Wallis has observed that Archimedes might seem, "as it were of set purpose to have covered up the traces of his investigation, as if he had grudged posterity the secret of his method of inquiry, while he wished to extort from them assent to his results." A comparison of the Method with other treatises now reveals to us how Archimedes found the areas and volumes of certain figures. His method was to balance elements of the figure against elements of another figure whose mensuration was 220
ARCHIMEDES

For let $AB\Gamma\Delta$ be a parallelogram, and in it let $EZ$ bisect $AB$, $\Gamma\Delta$ and let $KA$ bisect $\Gamma\Gamma$, $BA$; now the centre of gravity of the parallelogram $AB\Gamma\Delta$ is on $EZ$—for this has been proved. By the same reasoning it lies on $KA$; therefore the point $\Theta$ is the centre of gravity. And the diagonals of the parallelogram meet at $\Theta$; so that the proposition has been proved.

(j) Mechanical Method in Geometry


ii. 426. 3–430. 22

Archimedes to Eratosthenes greeting . . .

Moreover, seeing in you, as I say, a zealous student and a man of considerable eminence in philosophy, known. This gave him the result, and then he proved it by rigorous geometrical methods based on the principle of reductio ad absurdum.

The case of the parabola is particularly instructive. In the Method, Prop. 1, Archimedes conceives a segment of a parabola as made up of straight lines, and by his mechanical method he proves that the segment is four-thirds of the triangle having the same base and equal height. In his Quadrature of a Parabola, Prop. 14, he conceives the parabola as made up of a large number of trapezia, and by mechanical methods again reaches the same result. This is more satisfactory, but still not completely rigorous, so in Prop. 24 he proves the theorem without any help from mechanics by reductio ad absurdum.

The Method had to be classed among the lost works of Archimedes until 1906, when it was discovered at Constantinople by Heiberg in the ms. which he has termed C. Unfortunately the ms. is often difficult to decipher, and students of the text should consult Heiberg's edition. Moreover, the diagrams have to be supplied as they are undecipherable in the ms.


221
GREEK MATHEMATICS

μαθήμασιν κατὰ τὸ ὑποπττον θεωρίαν τετμηκότα ἐδοκίμασα γράφας σοι καὶ εἰς τὸ αὐτὸ βιβλίον ἐξορίσαι τρόπον τινὸς ἱδιότητα, καθ’ ὦν σοι παρεχόμενον ἔσται λαμβάνειν ἀφορμὰς εἰς τὸ δύνασθαι τινα τῶν ἐν τοῖς μαθήμασι θεωρεῖν διὰ τῶν μηχανικῶν. τούτῳ δὲ πέπεισμαι χρῆσιμον εἶναι οὐδὲν ἱσσον καὶ εἰς τὴν ἀπόδειξιν αὐτῶν τῶν θεωρημάτων. καὶ γάρ τινα τῶν πρότερον μοι φανέρων μηχανικῶς ὑποτελεῖν γεωμετρικῶς ἀπεδείχθη διὰ τὸ χωρίς ἀποδείξεως εἶναι τὴν διὰ τούτου τοῦ τρόπου θεωρίαν. ἐτοιμότερον γάρ ἐστι προλαβόντα διὰ τοῦ τρόπου γνώσιν τινα τῶν ζητημάτων πορίσασθαι τὴν ἀπόδειξιν μᾶλλον ἡ μηδενὸς ἐγνωσμένου ζητεῖν. . . . γράφομεν οὖν πρῶτον τὸ καὶ πρῶτον φανὲρ διὰ τῶν μηχανικῶν, ὅτι πάντες τρίγωνοι κόνων τομῆς ἐπίτριτῶν ἐστιν τριγώνου τοῦ βάσιν ἔχοντος τὴν αὐτὴν καὶ ὑψός ἵσον.

Ibid., Prop. 1, Archim. ed. Heiberg ii. 434. 14–438. 21

"Εστὶν τρίγωνο τὸ ΑΒΓ περιεχόμενον ύπὸ εὐθείας τῆς ΑΓ καὶ ὀρθογωνίου κόνων τομῆς τῆς ΑΒΓ, καὶ τετμήσθω διάχα ἡ ΑΓ τῷ Δ, καὶ παρὰ τὴν διάμετρον ἵκθω ἡ ΔΒΕ, καὶ ἐπεξεύχθωσαν αἱ ΑΒ, ΒΓ.

Λέγω, ὅτι ἐπίτριτῶν ἐστὶν τὸ ΑΒΓ τριγώνον τοῦ ΑΒΓ τριγώνου.

"Ηχθωσαν ἀπὸ τῶν Α, Γ σημείων ἡ μὲν ΑΖ παρὰ τὴν ΔΒΕ, ἡ δὲ ΓΖ ἐπισαῦσεται τῆς τομῆς, καὶ ἐκβεβλήσθω ἡ ΓΒ ἐπὶ τὸ Κ, καὶ κείσθω τῇ ΓΚ ἵση ἡ ΚΘ. νοείσθω ξυγὸς ὁ ΓΘ καὶ μέσον 222
who gives due honour to mathematical inquiries when they arise, I have thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, with which furnished you will be able to make a beginning in the investigation by mechanics of some of the problems in mathematics. I am persuaded that this method is no less useful even for the proof of the theorems themselves. For some things first became clear to me by mechanics, though they had later to be proved geometrically owing to the fact that investigation by this method does not amount to actual proof; but it is, of course, easier to provide the proof when some knowledge of the things sought has been acquired by this method rather than to seek it with no prior knowledge. . . . At the outset therefore I will write out the very first theorem that became clear to me through mechanics, that any segment of a section of a right-angled cone is four-thirds of the triangle having the same base and equal height.

_Ibid._, Prop. 1, Archim. ed. Heiberg ii. 434. 14-438. 21

Let $AB\Gamma$ be a segment bounded by the straight line $A\Gamma$ and the section $AB\Gamma$ of a right-angled cone, and let $A\Gamma$ be bisected at $\Delta$, and let $\Delta BE$ be drawn parallel to the axis, and let $AB$, $B\Gamma$ be joined.

I say that the segment $AB\Gamma$ is four-thirds of the triangle $AB\Gamma$.

From the points $A$, $\Gamma$ let $AZ$ be drawn parallel to $\Delta BE$, and let $\Gamma Z$ be drawn to touch the section, and let $\Gamma B$ be produced to $K$, and let $K\Theta$ be placed equal to $\Gamma K$. Let $\Gamma \Theta$ be imagined to be a balance
GREEK MATHEMATICS

αὐτοῦ τὸ Κ καὶ τῇ ΕΔ παράλληλος τυχόδια ἡ ΜΕ.

'Επει οὖν παραβολή ἐστὶν ἡ ΓΒΑ, καὶ ἐφάπτεται

ἡ ΓΖ, καὶ τεταγμένως ἡ ΓΔ, ἵση ἐστὶν ἡ EB τῇ ΒΔ. τούτῳ γὰρ ἐν τοῖς στοιχείοις δείκνυται· διὰ δὴ τούτῳ, καὶ διότι παράλληλοί εἰσιν αἱ ΖΑ, ΜΕ τῇ ΕΔ, ἵση ἐστὶν καὶ ἡ μὲν ΜΝ τῇ ΝΕ, ἡ δὲ ΖΚ τῇ ΚΑ. καὶ ἐπεὶ ἐστὶν, ὡς ἡ ΓΑ πρὸς ΑΕ, οὕτως ἡ ΜΕ πρὸς ΞΟ [τούτῳ γὰρ ἐν λήμματι δείκνυται], ως δὲ ἡ ΓΑ πρὸς ΑΕ, οὕτως ἡ ΓΚ πρὸς ΚΝ, καὶ ἵση ἐστὶν ἡ ΓΚ τῇ ΚΘ, ως ἀρα ἡ ΘΚ πρὸς KN, οὕτως ἡ ΜΕ πρὸς ΞΟ. καὶ ἐπεὶ τὸ Ν σημεῖον κέντρον τοῦ βάρους τῆς ΜΕ εὐθείας ἐστίν, ἐπείπερ ἵση ἐστὶν ἡ ΜΝ τῇ ΝΕ, εἰν ἀρα τῇ ΞΟ ἴσην θῶμεν τὴν ΘΗ καὶ κέντρον τοῦ βάρους αὐτῆς τὸ Θ, ὅπως ἵση ἡ ἡ ΘΟ τῇ ΘΗ, ἵσον ὄρθως ἡ ΘΗ τῇ ΜΕ αὐτοῦ μενοῦση διὰ τὸ ἀντιπεπονθότως τετμηθαι
with mid-point $K$, and let $ME$ be drawn parallel to $EA$.

Then since $ΓBA$ is a parabola, and $ΓZ$ touches it, and $ΓΔ$ is a semi-ordinate, $EB=BΔ$—for this is proved in the elements; for this reason, and because $ZA, ME$ are parallel to $EA, MN=NE$ and $ZK=KA$ [Eucl. vi. 4, v. 9]. And since

$$ΓA : AΞ = ME : ΞO,$$  \[Quad. parab. 5, Eucl. v. 18\]

and

$$ΓA : AΞ = ΓK : KN,$$  \[Eucl. vi. 2, v. 18\]

while

$$ΓK = KΘ,$$

therefore

$$ΘK : KN = ME : ΞO.$$  

And since the point $N$ is the centre of gravity of the straight line $ME$, inasmuch as $MN=NE$ [Lemma 4], if we place $TH=ΞO$, with $Θ$ for its centre of gravity, so that $TΘ=ΘH$ [Lemma 4], then $TΘH$ will balance $ME$ in its present position, because $ΘN$ is cut

-- Archimedes would have said "section of a right-angled cone"—δομονον κωνον τομα.

The reference will be to the Elements of Conics by Euclid and Aristaeus for which v. vol. i. pp. 486-491 and infra, p. 280 n. a; cf. similar expressions in On Conoids and Spheroids, Prop. 3 and Quadrature of a Parabola, Prop. 3; the theorem is Quadrature of a Parabola, Prop. 2.

---

1 τοῦτο . . . δείκνυται om. Heiberg. It is probably an interpolator’s reference to a marginal lemma.
GREEK MATHEMATICS

τὴν ΘΝ τοῖς ΘΗ, ΜΕ βάρεσιν, καὶ ὥς τὴν ΘΚ προς ΚΝ, οὕτως τὴν ΜΕ προς τὴν ΗΤ. ὡστε τοῦ ἐξ ἀμφοτέρων βάρους κέντρον ἐστὶν τοῦ βάρους τὸ Κ. ὁμοίως δὲ καὶ ὅσα ἄν ἀχθώσιν ἐν τῷ ΖΑΓ τριγώνῳ παράλληλοι τῇ ΕΔ, ἰσορροπήσουσιν αὐτοῦ μένουσα ταῖς ἀπολαμβανομέναις ἀπ’ αὐτῶν ὑπὸ τῆς τομῆς μετενεχθείσαις ἐπὶ τὸ Θ, ὡστε εἶναι τοῦ ἐξ ἀμφοτέρων κέντρον τοῦ βάρους τὸ Κ. καὶ ἐπεὶ ἐκ μὲν τῶν ἐν τῷ ΓΖΑ τριγώνῳ τὸ ΓΖΑ τρίγωνον συνέστηκεν, ἐκ δὲ τῶν ἐν τῇ τομῇ ὁμοίως τῇ ΞΟ λαμβανομένων συνέστηκε τὸ ΑΒΓ τμῆμα, ἰσορροπήσει ἄρα τὸ ΖΑΓ τρίγωνων αὐτοῦ μένον τῷ τμῆματι τῆς τομῆς τεθέντι περὶ κέντρον τοῦ βάρους τὸ Θ κατὰ τὸ Κ σημείον, ὡστε τοῦ ἐξ ἀμφοτέρων κέντρον εἶναι τοῦ βάρους τὸ Κ. τεθένθω δὴ ἡ ΓΚ τῶν Χ, ὡστε τριπλασίαν εἶναι τὴν ΓΚ τῆς ΚΧ. ἐσταὶ ἄρα τὸ Χ σημείον κέντρον βάρους τοῦ ΑΖΓ τριγώνου. δεδεικτα γὰρ ἐν τοῖς ἰσορροπικοῖς. ἐπεὶ οὖν ἰσόρροπον τὸ ΖΑΓ τρίγωνον αὐτοῦ μένον τῷ ΒΑΓ τμῆματι κατὰ τὸ Κ τεθέντι περὶ τὸ Θ κέντρον τοῦ βάρους, καὶ ἐστὶν τοῦ ΖΑΓ τριγώνου κέντρον βάρους τὸ Χ, ἐστὶν ἄρα, ὡς τὸ ΑΖΓ τρίγωνον πρὸς τὸ ΑΒΓ τμῆμα κείμενον περὶ τὸ Θ κέντρον, οὕτως ἡ ΘΚ πρὸς ΧΚ. τριπλασία δὲ ἐστιν ἡ ΘΚ τῆς ΚΧ. τριπλάσιον ἄρα καὶ τὸ ΑΖΓ τρίγωνον τοῦ ΑΒΓ τμῆματος. ἐστὶ δὲ καὶ τὸ ΖΑΓ τρίγωνον τετραπλάσιον τοῦ ΑΒΓ τριγώνου διὰ τὸ ἔσοχν εἶναι τὴν μὲν ΖΚ τῇ ΚΑ, τὴν δὲ ΑΔ τῇ ΔΓ. ἐπίτροπον ἄρα ἐστὶν τὸ ΑΒΓ τμῆμα τοῦ ΑΒΓ τριγώνου. [τοῦτο οὖν φανερὸν ἐστὶν].

1 τοῦτο . . . ἐστιν om. Heiberg.

226
ARCHIMEDES

In the inverse proportion of the weights TH, ΜΞ, and

\[ \Theta K : KN = MΞ : HT; \]

therefore the centre of gravity of both [TH, MΞ] taken together is K. In the same way, as often as parallels to EΔ are drawn in the triangle ZΑΓ, these parallels, remaining in the same position, will balance the parts cut off from them by the section and transferred to Θ, so that the centre of gravity of both together is K. And since the triangle ΓΖΑ is composed of the [straight lines drawn] in ΓΖΑ, and the segment ΑΒΓ is composed of the lines in the section formed in the same way as ΞΟ, therefore the triangle ZΑΓ in its present position will be balanced about K by the segment of the section placed with Θ for its centre of gravity, so that the centre of gravity of both combined is K. Now let ΓΚ be cut at X so that ΓΚ = 3ΚΧ; then the point X will be the centre of gravity of the triangle AZΓ; for this has been proved in the books On Equilibriums.a Then since the triangle ZΑΓ in its present position is balanced about K by the segment BΑΓ placed so as to have Θ for its centre of gravity, and since the centre of gravity of the triangle ZΑΓ is X, therefore the ratio of the triangle AZΓ to the segment ABΓ placed about Θ as its centre [of gravity] is equal to ΘK : XK. But ΘK = 3ΚΧ; therefore

triangle AZΓ = 3 . segment ABΓ.

And triangle ZΑΓ = 4 . triangle ABΓ,

because ZK = KA and AΔ = ΔΓ;

therefore segment ABΓ = \( \frac{4}{3} \) triangle ABΓ.

* Cf. De Plan. Equil. i. 15.
GREEK MATHEMATICS

'Τοῦτο δὴ διὰ μὲν τῶν νῦν εἰρημένων οὐκ ἀποδεικταὶ, ἐμφασιν δὲ τινα πεποίηκε τὸ συμπέρασμα ἄληθὲς εἶναι· διόπερ ἥμεις ὁρῶντες μὲν οὐκ ἀποδειγμένον, ὑπονοοῦντες δὲ τὸ συμπέρασμα ἄληθὲς εἶναι, τάξομεν τὴν γεωμετρουμένην ἀπόδειξιν ἐξευρόντες αὐτοὶ τὴν ἐκδοθείσαν πρότερον.\

Archim. Quadr. Parab., Praef., Archim. ed. Heiberg ii, 262. 2-266. 4

'Ἀρχιμήδης Δοσιθέω εὖ πράττεω.
Ἀκούσας Κόνωνα μὲν τετελευτηκέναι, ὅτι ἦν οὐδὲν ἐπιλείπων ἀμῖν ἐν φιλίᾳ, τὸν δὲ Κόνωνος γνώριμον γεγενήθαι καὶ γεωμετρίας οἰκείων εἴμεν τοῦ μὲν τετελευτηκότος εἰνεκεν ἐλυπήθημεν ὡς καὶ φίλου τοῦ ἀνδρός γεναμένου καὶ ἐν τοῖς μαθημάτεσιν θαυμαστὸν τινος, ἐπροχειριζόμεθα δὲ ἀποστείλαι τοις γράφαντες, ὡς Κόνωνι γράφειν ἐγνωκότες ἡμεῖς, γεωμετρικῶν θεωρημάτων, δὲ πρότερον μὲν οὐκ ἦν τεθεωρημένον, νῦν δὲ υφ' ἀμῖν τεθεώρηται, πρότερον μὲν διὰ μηχανικῶν εὑρεθέν, ἔπειτα δὲ καὶ διὰ τῶν γεωμετρικῶν ἐπιδειχθέν. τῶν μὲν οὖν πρότερον περὶ γεωμετρίαν πραγματευθέντων ἐπεξείρησάν τινες γράφειν ὡς δυνατὸν ἐνοι κύκλῳ τῷ δοθέντι καὶ κύκλου τριήμετρη τῷ δοθέντι χωρίον εὑρεῖν εὐθύγραμμον ἵσον, καὶ μετὰ ταῦτα τὸ περιεχόμενον χωρίον ὑπὸ τε τὰς

1 τοῦτο . . . πρότερον. In the ms. the whole paragraph from τοῦτο to πρότερον comes at the beginning of Prop. 2; it is more appropriate at the end of Prop. 1.

228
This, indeed, has not been actually demonstrated by the arguments now used, but they have given some indication that the conclusion is true; seeing, therefore, that the theorem is not demonstrated, but suspecting that the conclusion is true, we shall have recourse to the geometrical proof which I myself discovered and have already published.

Archimedes, Quadrature of a Parabola, Preface, Archim. ed. Heiberg ii. 262. 2-266. 4

Archimedes to Dositheus greeting.

On hearing that Conon, who fulfilled in the highest degree the obligations of friendship, was dead, but that you were an acquaintance of Conon and also versed in geometry, while I grieved for the death of a friend and an excellent mathematician, I set myself the task of communicating to you, as I had determined to communicate to Conon, a certain geometrical theorem, which had not been investigated before, but has now been investigated by me, and which I first discovered by means of mechanics and later proved by means of geometry. Now some of those who in former times engaged in mathematics tried to find a rectilineal area equal to a given circle and to a given segment of a circle, and afterwards they tried to square the area bounded by the section

a I have followed Heath’s rendering of τάξομεν, which seems more probable than Heiberg’s “suo loco proponemus,” though it is a difficult meaning to extract from τάξομεν.

b Presumably Quadr. Parab. 24, the second of the proofs now to be given. The theorem has not been demonstrated, of course, because the triangle and the segment may not be supposed to be composed of straight lines.

c This seems to indicate that Archimedes had not at this time written his own book On the Measurement of a Circle. For attempts to square the circle, v. vol. i. pp. 305-347.
GREEK MATHEMATICS

όλου τοῦ κόμον τομᾶς καὶ εὐθείας τετραγωνίζειν ἐπειρώντο λαμβάνοντες οὐκ εὐπαρακώρητα λήμματα, διόπερ αὐτοῖς ὑπὸ τῶν πλεῖστων οὐκ εὑρισκόμενα ταῦτα κατεγνώσθεν. τὸ δὲ ὑπ’ εὐθείας τε καὶ ὀρθογωνίου κόμον τομᾶς τμῆμα περιεχόμενον οὐδένα τῶν προτέρων ἐγχειρήσαντα τετραγωνίζειν ἐπιστάμεθα, ὃ δὴ νῦν ύφ’ ἀμῶν εὑρίσκει δεῖκνυται γάρ, ὅτι πᾶν τμῆμα περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κόμον τομᾶς ἐπιτρεπτὸν ἔστι τοῦ τριγώνου τοῦ βάσιν ἐχόντος τὰν αὐτὰν καὶ ύψος ἵσον τῷ τμῆματι λαμβανομένου τοῦτο τοῦ λήμματος ἐς τὰν ἀποδείξεων αὐτοῦ τῶν ἀνίσων χωρίων τῶν ὑπερδικάν, ὃ ὑπερέχει τὸ μεῖζον τοῦ ἐλάσσονος, δυνατὸν εἶμεν αὐτὰν ἐαιτῇ συντιθεμέναν παντὸς ὑπερέχειν τοῦ προτεθέντος πεπερασμένου χωρίου. Κέχρηνται δὲ καὶ οἱ πρότερον γεωμέτραι τῶδε τῷ λήμματι τούς τε γάρ κύκλους διπλασίονα λόγον ἔχειν ἔποι ἀλλάλους τῶν διαμετρῶν ἀποδεδείχασιν αὐτῷ τούτῳ τῷ λήμματι χρωμένοι, καὶ τὰς σφαίρας ὁτι τριπλασίονα λόγον ἔχοντι ποτ’ ἀλλάλας τῶν διαμετρῶν, ἐτι δὲ καὶ ὅτι πᾶσα πυραμίς τρίτον μέρος ἔστι τοῦ πρίσματος τοῦ τάν αὐτὰν βάσιν ἐχόντος τά πυραμίδι καὶ ύψος ἵσον καὶ διὸτι πᾶσ κώνος τρίτον μέρος ἔστι τοῦ κυλίνδρου τοῦ τάν αὐτὰν βάσιν ἐχόντος τά κώνῳ καὶ ύψος ἵσον, ὅμοιον τῷ προειρημένῳ λήμμα τι λαμβάνοντες ἐγραφον. Συμβαίνει δὲ τῶν προειρημένων ὑπερημάτων ἐκαστὸν μηδενὸς ἠσον τῶν ἀνευ τοῦτον τοῦ λήμματος ἀποδειγμάτων πεποιηκέναι· ἀρκεῖ δὲ ἐς τὰν ὁμοίαν πίστιν τοῦτος ἀναγιγμένων τῶν ύφ’ ἀμῶν ἐκδομένων. ἀναγράψαντες οὖν αὐτοῦ τὰς ἀποδείξεις ἀποστέλλομες 230
of the whole cone and a straight line, assuming lemmas far from obvious, so that it was recognized by most people that the problem had not been solved. But I do not know that any of my predecessors has attempted to square the area bounded by a straight line and a section of a right-angled cone, the solution of which problem I have now discovered; for it is shown that any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle which has the same base and height equal to the segment, and for the proof this lemma is assumed: given [two] unequal areas, the excess by which the greater exceeds the less can, by being added to itself, be made to exceed any given finite area. Earlier geometers have also used this lemma: for, by using this same lemma, they proved that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and also that any pyramid is a third part of the prism having the same base as the pyramid and equal height; and, further, by assuming a lemma similar to that aforesaid, they proved that any cone is a third part of the cylinder having the same base as the cone and equal height. In the event, each of the aforesaid theorems has been accepted, no less than those proved without this lemma; and it will satisfy me if the theorems now published by me obtain the same degree of acceptance. I have therefore written out the proofs, and now send them, first

\[ a \]

A "section of the whole cone" is probably a section cutting right through it, i.e., an ellipse, but the expression is odd.

\[ b \]

For this lemma, v. supra, p. 46 n. a.
GREEK MATHEMATICS

πρῶτον μὲν, ὡς διὰ τῶν μηχανικῶν ἔθεωρῆθη, μετὰ ταῦτα δὲ καὶ, ὡς διὰ τῶν γεωμετρομένων ἀποδείκνυται. προγράφεται δὲ καὶ στοιχεῖα κω- νικὰ χρείαιν ἔχοντα ἐς τὰς ἀπὸδείξειν. ἔρρωσο.


'Εστω τμῆμα τὸ ΒΘΓ περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς. ἐστώ δὴ πρῶτον

ά ΒΓ ποτ' ὀρθᾶς τὰ διαμέτρω, καὶ ἀχθων ἀπὸ μὲν τοῦ Β σαμείων ἀ ΒΔ παρὰ τὰν διάμετρον, ἀπὸ δὲ τοῦ Γ ἀ ΓΔ ἐπιφανοῦσα τὰς τοῦ κώνου τομᾶς κατὰ τὸ Γ. ἐσσεῖται δὴ τὸ ΒΓΔ τρίγωνον ὀρθο- γώνιον. διηγήσω στὸ ἐς ίσα τμήματα ὀποσαοῦν τὰ ΒΕ, ΕΖ, ΖΗ, ΗΓ, καὶ ἀπὸ τῶν τομῶν ἀχθωσαν παρὰ τὰν διάμετρον αἱ ΕΣ, ΖΤ, ΗΥ, ΙΕ, ἀπὸ δὲ τῶν σαμείων, καθ' ἀ τέμνοντι

232
ARCHIMEDES

as they were investigated by means of mechanics, and also as they may be proved by means of geometry. By way of preface are included the elements of conics which are needed in the demonstration. Farewell.


Let $\theta \Omega \Gamma$ be a segment bounded by a straight line and a section of a right-angled cone. First let $B \Gamma$ be at right angles to the axis, and from $B$ let $B \Delta$ be drawn parallel to the axis, and from $\Gamma$ let $\Gamma \Delta$ be drawn touching the section of the cone at $\Gamma$; then the triangle $B \Gamma \Delta$ will be right-angled [Eucl. i. 29]. Let $B \Gamma$ be divided into any number of equal segments $BE$, $EZ$, $ZH$, $HI$, $I \Gamma$, and from the points of section let $E \Sigma$, $ZT$, $HY$, $I \Xi$ be drawn parallel to the axis, and from the points in which these cut the
GREEK MATHEMATICS

αὐταὶ τὰν τοῦ κόσμου τομάν, ἐπεζεύχθωσαν ἐπὶ τὸ Γ καὶ ἐκβεβλήσθωσαν. φαμὶ δὴ τὸ τριγώνον τὸ ΒΔΓ τῶν μὲν τραπέζιων τῶν ΚΕ, ΛΖ, ΜΗ, ΝΙ καὶ τοῦ Ε Ion τριγώνου ἐλασσον εἰμὲν ἡ τριπλάσιον, τῶν δὲ τραπεζίων τῶν ΖΦ, ΗΘ, ΙΠ καὶ τοῦ IOΓ τριγώνου μείζον [ἔστων] ἡ τριπλάσιον.

"Διάχω γὰρ εὐθεῖα ἀ ΑΒΓ, καὶ ἀπολελάφθων ἀ ΑΒ ἵσα τὰ ΒΓ, καὶ νοεῖσθω ζύγιον τὸ ΑΓ· μέσον δὲ αὐτοῦ ἔσσείται τὸ Β· καὶ κρεμάσθω ἐκ τοῦ Β, κρεμάσθω δὲ καὶ τὸ ΒΔΓ ἐκ τοῦ ζύγου κατὰ τὰ Β, Γ, ἐκ δὲ τοῦ θατέρου μέρεος τοῦ ζύγου κρεμάσθω τὰ Ρ, Χ, Ψ, Ω, Δ χωρία κατὰ τὸ Α, καὶ ἰσορροπεῖτω τὸ μὲν Ρ χωρίον τῷ ΔΕ τραπεζίῳ οὔτως ἔχοντι, τὸ δὲ Χ τῷ ΖΣ τραπεζίῳ, τὸ δὲ Ψ τῷ θΗ, τὸ δὲ Ω τῷ ΥΙ, τὸ δὲ Δ τῷ ΕΙΓ τριγώνῳ· ἰσορροπήσει δὴ καὶ τὸ ὀλὸν τῷ ὅλῳ ὡστε τριπλάσιον ἄν εὑρῆ τὸ ΒΔΓ τριγώνον τοῦ ρξυωδ χωρίου. καὶ ἐπεὶ ἔστων τμῆμα τὸ ΒΓΘ, ὁ περιεχεῖνται ὑπὸ τε εὐθείας καὶ ὀρθογωνίου κόσμου τομᾶς, καὶ ἀπὸ μὲν τοῦ Β παρὰ τὰν διάμετρον ἀκταί ὁ ΒΔ, ἀπὸ δὲ τοῦ Γ ἀ ΓΔ ἐτυφαίουσα τὰς τοῦ κόσμου τομᾶς κατὰ τὸ Γ, ἀκταί δὲ τοῖς καὶ ἄλλα παρὰ τὰν διάμετρον ἀ ΣΕ, τὸν αὐτὸν ἔχει λόγον ἀ ΒΓ ποτὶ τὰν ΒΕ, δὲν ἀ ΣΕ ποτὶ τὰν ΕΦ· ὡστε καὶ ἀ ΒΑ ποτὶ τὰν ΒΕ τὸν αὐτὸν ἔχει λόγον, δὲν τὸ ΔΕ τραπεζίου ποτὶ τὸ ΚΕ. ὁμοίως δὲ δεικθήσεται ἀ ΑΒ ποτὶ τὰν ΒΖ τὸν αὐτὸν ἔχουσα λόγον, δὲν τὸ ΣΖ τραπεζίου ποτὶ τὸ ΛΖ, ποτὶ δὲ τὰν ΒΗ, δὲν τὸ θΗ ποτὶ τὸ ΜΗ, ποτὶ δὲ τὰν ΒΙ, δὲν τὸ ΥΙ ποτὶ τὸ ΝΙ. ἐπεὶ οὖν ἔστι τραπεζίου τὸ ΔΕ τὰς

1 ἔστω ομ. Heiberg.
section of the cone let straight lines be drawn to \( \Gamma \) and produced. Then I say that the triangle \( \triangle B\Delta \Gamma \) is less than three times the trapezia \( KE, \Delta Z, MH, NI \) and the triangle \( \Xi \Omega \Gamma \), but greater than three times the trapezia \( Z\Phi, H\Theta, \Pi \Pi \) and the triangle \( \Delta \Omega \Gamma \).

For let the straight line \( AB \) be drawn, and let \( AB \) be cut off equal to \( B\Gamma \), and let \( \Delta \) be imagined to be a balance; its middle point will be \( B \); let it be suspended from \( B \), and let the triangle \( B\Delta \Gamma \) be suspended from the balance at \( B \), \( \Gamma \), and from the other part of the balance let the areas \( P, X, \Psi, \Omega, \Delta \) be suspended at \( A \), and let the area \( P \) balance the trapezium \( \Delta E \) in this position, let \( X \) balance the trapezium \( Z\Sigma \), let \( \Psi \) balance \( TH \), let \( \Omega \) balance \( YI \), and let \( \Delta \) balance the triangle \( \Xi \Omega \Gamma \); then the whole will balance the whole; so that the triangle \( B\Delta \Gamma \) will be three times the area \( P + X + \Psi + \Omega + \Delta \) [Prop. 6]. And since \( B\Gamma \Theta \) is a segment bounded by a straight line and a section of a right-angled cone, and \( B\Delta \) has been drawn from \( B \) parallel to the axis, and \( \Gamma \Delta \) has been drawn from \( \Gamma \) touching the section of a cone at \( \Gamma \), and another straight line \( \Sigma \) has been drawn parallel to the axis,

\[
B\Gamma : BE = \Sigma \Sigma : E\Phi;
\]

[Prop. 5]

therefore \( BA : BE = \) trapezium \( \Delta \Sigma : \) trapezium \( KE \).

Similarly it may be proved that

\[
AB : BZ = \Sigma Z : \Delta Z,
\]

\[
AB : BH = TH : MH,
\]

\[
AB : BI = YI : NI.
\]

Therefore, since \( \Delta \Sigma \) is a trapezium with right angles

* For \( BA = B\Gamma \) and \( \Delta \Sigma : KE = \Sigma \Sigma : E\Phi \).
GREEK MATHEMATICS

μὲν ποτὶ τοῖς Β, Ε σαμεῖοις γυνίαις ὅρθὰς ἔχον, τὰς δὲ πλευρὰς ἐπὶ τὸ Γ νευόμενας, ἰσορροπεῖ δὲ τὶ χωρίον αὐτῷ τὸ Ρ κρεμάμενον ἐκ τοῦ ξυγοῦ κατὰ τὸ Α οὕτως ἔχοντος τοῦ τραπεζίου, ὡς νῦν κεῖται, καὶ ἔστων, ὡς ἂ ΒΑ ποτὶ τὰν ΒΕ, οὕτως τὸ ΔΕ τραπέζιον ποτὶ τὸ ΚΕ, μεῖζον ἡμα ἔστω τὸ ΚΕ χωρίον τοῦ Ρ χωρίου. δεδεικται γὰρ τοῦτο. πάλιν δὲ καὶ τὸ ΖΣ τραπέζιον τὰς μὲν ποτὶ τοῖς Ζ, Ε γυνίαις ὅρθὰς ἔχον, τὰν δὲ ΣΤ νευόμεναν ἐπὶ τὸ Γ, ἰσορροπεῖ δὲ αὐτῷ χωρίον τὸ Χ ἐκ τοῦ ξυγοῦ κρεμάμενον κατὰ τὸ Α οὕτως ἔχοντος τοῦ τραπεζίου, ὡς νῦν κεῖται, καὶ ἔστων, ὡς μὲν ἂ ΑΒ ποτὶ τὰν BE, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΖΦ, ὡς δὲ ἂ ΑΒ ποτὶ τὰν ΒΖ, οὕτως τὸ ΖΣ τραπέζιον ποτὶ τὸ ΔΖ. εἰ ὡς οὖν καὶ τὸ χωρίον τοῦ μὲν ΔΖ τραπεζίου ἔλασσον, τοῦ δὲ ΖΦ μεῖζον δεδεικται γὰρ καὶ τοῦτο. διὰ τὰ αὐτὰ δὴ καὶ τὸ Ψ χωρίον τοῦ μὲν ΜΗ τραπεζίου ἔλασσον, τοῦ δὲ ΘΗ μεῖζον, καὶ τὸ Ω χωρίον τοῦ μὲν ΝΟΙΗ τραπεζίου ἔλασσον, τοῦ δὲ ΠΙ μεῖζον, ὡμοίως δὲ καὶ τὸ Δ χωρίον τοῦ μὲν ΕΙΓ τρίγωνον ἔλασσον, τοῦ δὲ ΓΙΟ μεῖζον. ἐπεὶ οὖν τὸ μὲν ΚΕ τραπεζίου μεῖζον ἐστὶ τοῦ Ρ χωρίου, τὸ δὲ ΔΖ τοῦ Χ, τὸ δὲ MH τοῦ Ψ, τὸ δὲ NI τοῦ Ω, τὸ δὲ ΕΙΓ τρίγωνον τοῦ Δ, φανερὸν, ὅτι καὶ πάντα τὰ εἰρήμενα χωρία μεῖζον ἐστὶ τοῦ ΡΧΨΩΔ χωρίου. ἐστὶν δὲ τὸ ΡΧΨΩΔ τρίτον μέρος τοῦ ΒΓΔ τρίγωνον. δῆλον ἡμα, διὸ τὸ ΒΓΔ τρίγωνον ἔλασσον ἐστὶ καὶ τριπλάσιον τῶν ΚΕ, ΔΖ, MH, NI τραπεζίων καὶ τοῦ ΕΙΓ τριγώνου. πάλιν, ἐπεὶ τὸ μὲν ΖΦ τραπεζίου ἔλασσον ἐστὶ τοῦ Χ χωρίου, τὸ δὲ ΘΗ τοῦ Ψ, τὸ δὲ ΠΙ τοῦ Ω, τὸ δὲ ΙΟΓ τρίγωνον τοῦ Δ, φανερὸν, ὅτι καὶ πάντα 236
at the points B, E and with sides converging on Γ, and it balances the area P suspended from the balance at A, if the trapezium be in its present position, while

\[ BA : BE = \Delta E : KE, \]

therefore
\[ KE > P; \]
for this has been proved [Prop. 10]. Again, since \( Z\Sigma \) is a trapezium with right angles at the points Z, E and with \( \Sigma T \) converging on Γ, and it balances the area X suspended from the balance at A, if the trapezium be in its present position, while

\[ AB : BE = Z\Sigma : Z\Phi, \]
\[ AB : BZ = Z\Sigma : \Delta Z, \]
therefore
\[ \Delta Z > X > Z\Phi; \]
for this also has been proved [Prop. 12]. By the same reasoning

\[ MH > \Psi > \Theta H, \]
and
\[ NO\Theta H > \Omega > \Pi I, \]
and similarly
\[ \Xi I\Gamma > \Delta > \Gamma I O. \]
Then, since \( KE > P, \Delta Z > X, MH > \Psi, NI > \Omega, \Xi I\Gamma > \Delta, \)
it is clear that the sum of the aforesaid areas is greater than the area \( P + X + \Psi + \Omega + \Delta \). But

\[ P + X + \Psi + \Omega + \Delta = \frac{1}{3} \beta \Gamma \Delta; \quad [\text{Prop. 6}] \]
it is therefore plain that

\[ \beta \Gamma \Delta < 3(KE + \Delta Z + MH + NI + \Xi I\Gamma). \]

Again, since \( Z\Phi < X, \Theta H < \Psi, \Pi I < \Omega, I\Omega \Gamma < \Delta, \) it is
GREEK MATHEMATICS

tὰ εἰρημένα ἐλάσσονά ἐστὶ τοῦ ∆ΩΨΧ χωρίου
φανερῶν οὖν, ὅτι καὶ τὸ ΒΔΓ τρίγωνον μεῖζόν
ἐστὶν ἡ τριπλάσιον τῶν ΦΖ, ΘΗ, ΠΠ τραπεζίων
καὶ τοῦ ΙΓΟ τριγώνου, ἐλασσὸν δὲ ἡ τριπλάσιον
τῶν προγεγραμμένων.

Ibid., Prop. 24, Archim. ed. Heiberg ii. 312. 2–314. 27

Πάν τμᾶμα τὸ περιεχόμενον ὑπὸ εὐθείας καὶ
ὄρθογωνίου κώνου τομᾶς ἐπίτριτόν ἐστι τριγώνου
tοῦ τὰν αὐτὰν βάσιν ἔχοντος αὐτῷ καὶ ύψος ἴσον.
"Εστώ γὰρ τὸ ΑΔΒΕΓ τμᾶμα περιεχόμενον ὑπὸ
eὐθείας καὶ ὀρθογωνίου κώνου τομᾶς, τὸ δὲ ΑΒΓ
τρίγωνον ἐστὶν τὰν αὐτὰν βάσιν ἔχον τῷ τμάματι

καὶ ύψος ἴσον, τοῦ δὲ ΑΒΓ τριγώνου ἐστὶν ἐπί-
τριτόν τὸ Κ χωρίου. δεικτέον, ὅτι ἴσον ἐστὶ τῷ
ΑΔΒΕΓ τμάματι.

Εἰ γὰρ μὴ ἐστὶν ἴσον, ἦτοι μεῖζὸν ἐστὶν ἡ ἐλασσον.
ἐστὶν πρότερον, εἰ δυνατὸν, μεῖζον τὸ ΑΔΒΕΓ
τμᾶμα τοῦ Κ χωρίου. ἐνέγραψα δὴ τὰ ΑΔΒ,
ΒΕΓ τρίγωνα, ὡσ εἰρηταὶ, ἐνέγραψα δὲ καὶ εἰς τὰ
περιλειπόμενα τμάματα ἄλλα τρίγωνα τὰν αὐτὰν
238
clear that the sum of the aforesaid areas is greater than the area $\Delta + \Omega + \Psi + X$; it is therefore manifest that

$$B\Delta \Gamma > 3(\Phi Z + \Theta H + \Pi I + \Gamma O),$$

but is less than thrice the aforementioned areas.$^b$

Ibid., Prop. 24, Archim. ed. Heiberg ii. 312. 2–314. 27

Any segment bounded by a straight line and a section of a right-angled cone is four-thirds of the triangle having the same base and equal height.

For let $A\Delta \Gamma \delta$ be a segment bounded by a straight line and a section of a right-angled cone, and let $A\Gamma \delta$ be a triangle having the same base as the segment and equal height, and let the area $K$ be four-thirds of the triangle $A\Gamma \delta$. It is required to prove that it is equal to the segment $A\Delta \Gamma \delta$.

For if it is not equal, it is either greater or less. Let the segment $A\Delta \Gamma \delta$ first be, if possible, greater than the area $K$. Now I have inscribed the triangles $A\Delta B$, $B\Gamma \delta$, as aforesaid,$^c$ and I have inscribed in the remaining segments other triangles having the same bases.

$^a$ For $B\Delta \Gamma = 3(P + X + \Psi + \Omega + \Delta) > 3(\Delta + \Omega + \Psi + X)$.

$^b$ In Prop. 15 Archimedes shows that the same theorem holds good even if $\Gamma \delta$ is not at right angles to the axis. It is then proved in Prop. 16, by the method of exhaustion, that the segment is equal to one-third of the triangle $\Gamma \delta \Delta$. This is done by showing, on the basis of the "Axiom of Archimedes," that by taking enough parts the difference between the circumscribed and the inscribed figures can be made as small as we please. It is equivalent to integration. From this it is easily proved that the segment is equal to four-thirds of a triangle with the same base and equal height (Prop. 17).

$^c$ In earlier propositions Archimedes has used the same procedure as he now describes. $\Delta, E$ are the points in which the diameter through the mid-points of $A\Gamma$, $E\Gamma$ meet the curve.
βάσιν ἐχοντα τοῖς τμαμάτεσσων καὶ ψος τὸ αὐτό, καὶ ἄεὶ εἰς τὰ ὑστερον γινόμενα τμάματα ἐγγράφω δύο τρίγωνα τὰν αὐτὰν βάσιν ἐχοντα τοῖς τμαμάτεσσων καὶ ψος τὸ αὐτὸ· ἐσσοῦνται δὴ τὰ καταλειπόμενα τμάματα ἐλάσσονα τὰς ὑπεροχὰς, ξ ὑπερέχει τὸ ΛΔΒΕΓ τμᾶμα τοῦ Κ χωρίου. ὡστε τὸ ἐγγραφόμενον πολύγωνον μεῖζον ἔσσείταν τοῦ Κ· ὑπερ ἀδύνατον. ἐπεὶ γάρ ἔστιν ἐξῆς κείμενα χωρία ἐν τῷ τετραπλασίων λόγοι, πρῶτον μὲν τὸ ΑΒΓ τρίγωνον τετραπλάσιον τῶν ΛΔΒ, ΒΕΓ τριγώνων, ἐπειτα δὲ αὐτὰ ταῦτα τετραπλάσια τῶν εἰς τὰ ἑπόμενα τμάματα ἐγγραφέντων καὶ ἄεὶ σύμων, δὴν, ὡς σύμπαντα τὰ χωρία ἐλάσσονα ἔστιν ἡ ἐπίτριτα τοῦ μεγίστου, τὸ δὲ Κ ἐπίτριττον ἐστὶ τοῦ μεγίστου χωρίου. οὐκ ἀρα ἔστιν μεῖζον τὸ ΛΔΒΕΓ τμᾶμα τοῦ Κ χωρίου.

"Εστω δὲ, εἰ δυνατόν, ἔλασσον. κεῖσθω δὴ τὸ μὲν ΑΒΓ τρίγωνον ἵσον τῷ Ζ, τοῦ δὲ Ζ τέταρτον τὸ Η, καὶ ὁμοίως τοῦ Η τὸ Θ, καὶ ἄεὶ ἐξῆς τιθέσθω, ἐως καὶ γένηται τὸ ἐσχατὸν ἔλασσον τὰς

---

*a* This was proved geometrically in Prop. 23, and is proved generally in Eucl. ix. 35. It is equivalent to the summation

\[
1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1 - \frac{1}{2}} = \frac{\frac{1}{1 - \left(\frac{1}{2}\right)^{n+1}}}{1 - \frac{1}{2}}.
\]

240
ARCHIMEDES

... base as the segments and equal height, and so on continually I inscribe in the resulting segments two triangles having the same base as the segments and equal height; then there will be left [at some time] segments less than the excess by which the segment $\Delta\triangle ABC$ exceeds the area $K$ [Prop. 20, coroll.]. Therefore the inscribed polygon will be greater than $K$; which is impossible. For since the areas successively formed are each four times as great as the next, the triangle $AB'C'$ being four times the triangles $\Delta\triangle ABC$, $\Delta\triangle B'C'$ [Prop. 21], then these last triangles four times the triangles inscribed in the succeeding segments, and so on continually, it is clear that the sum of all the areas is less than four-thirds of the greatest [Prop. 23], and $K$ is equal to four-thirds of the greatest area. Therefore the segment $\Delta\triangle ABC$ is not greater than the area $K$.

Now let it be, if possible, less. Then let

$Z = AB\Gamma$, $H = \frac{1}{4}Z$, $\Theta = \frac{1}{4}H$,

and so on continually, until the last [area] is less than
GREEK MATHEMATICS

υπεροχάς, ἃ υπερέχει τὸ Κ χωρίον τοῦ τμάματος, καὶ ἕστω ἔλασσον τὸ Ι. ἕστω δὴ τὰ Ζ, Ἡ, Θ, I χωρία καὶ τὸ τρίτον τοῦ I ἐπίτριτα τοῦ Ζ. ἕστων δὲ καὶ τὸ K τοῦ Ζ ἐπίτριτον. ἵσον ἢρα τὸ K τοῖς Z, Ἡ, Θ, I καὶ τῷ τρίτῳ μέρει τοῦ Ι. ἐπεὶ οὖν τὸ Κ χωρίον τῶν μὲν Z, Ἡ, Θ, I χωρίων υπερέχει ἐλάσσονι τοῦ I, τοῦ δὲ τμάματος μεῖζον τοῦ I, δῆλον, ὡς μείζονα ἐντὶ τὰ Ζ, Ἡ, Θ, I χωρία τοῦ τμάματος. ὅπερ ἄδυνατον. ἐδείχθη γάρ, ὅτι, ἐὰν ἥ ὀποσαοῦν χωρία ἐξῆς κείμενα ἐν τετραπλασίοις λόγω, τὸ δὲ μέγιστον ἵσον ἥ τῷ εἰς τὸ τμάμα ἐγγραφομένω τριγώνῳ, τὰ σύμπαντα χωρία ἐλάσσονα ἐσσεῖται τοῦ τμάματος. οὐκ ἢρα τὸ ΑΔΒΕΓ τμάμα ἐλασσόν έστι τοῦ Κ χωρίου. ἐδείχθη δὲ, ὅτι οὐδὲ μείζον. ἵσον ἢρα ἐστὶν τῷ K. τὸ δὲ K χωρίον ἐπίτριτον ἐστὶ τοῦ τριγώνου τοῦ ΑΒΓ. καὶ τὸ ΑΔΒΕΓ ἢρα τμάμα ἐπίτριτον ἐστὶ τοῦ ΑΒΓ τριγώνου.

(k) Hydrostatics

(i.) Postulates

ii. 318. 2-8

Ὑποκείσθω τὸ υγρὸν φύσιν ἔχον τοιαύταν, ὡστε τῶν μερέων αὐτοῦ τῶν ἐξ ἵσου κείμενων καὶ συν-

* The Greek text of the book On Floating Bodies, the earliest extant treatise on hydrostatics, first became available in 1906 when Heiberg discovered at Constantinople the ms. which he terms C. Unfortunately many of the readings are doubtful, and those who are interested in the text should consult the Teubner edition. Still more unfortunately, it is incomplete; but, as the whole treatise was translated into Latin in 1269 by William of Moerbeke from a Greek ms. 242
ARCHIMEDES

the excess by which the area K exceeds the segment [Eucl. x. 1], and let I be [the area] less [than this excess]. Now

\[ Z + H + \Theta + I + \frac{1}{3}I = \frac{4}{3}Z. \]  

[Prop. 23]

But

\[ K = \frac{4}{3}Z; \]

therefore

\[ K = Z + H + \Theta + I + \frac{1}{3}I. \]

Therefore since the area K exceeds the areas Z, H, \Theta, I by an excess less than I, and exceeds the segment by an excess greater than I, it is clear that the areas Z, H, \Theta, I are greater than the segment; which is impossible; for it was proved that, if there be any number of areas in succession such that each is four times the next, and the greatest be equal to the triangle inscribed in the segment, then the sum of the areas will be less than the segment [Prop. 22]. Therefore the segment \( \Delta \Gamma \) is not less than the area K. And it was proved not to be greater; therefore it is equal to K. But the area K is four-thirds of the triangle \( \Delta \Gamma \); and therefore the segment \( \Delta \Gamma \) is four-thirds of the triangle \( \Delta \Gamma \).

(k) HYDROSTATICS

(i.) Postulates

Archimedes, On Floating Bodies a 1., Archim. ed.

Heiberg ii. 318. 2-8

Let the nature of a fluid be assumed to be such that, of its parts which lie evenly and are continuous, since lost, it is possible to supply the missing parts in Latin, as is done for part of Prop. 2. From a comparison with the Greek, where it survives, William’s translation is seen to be so literal as to be virtually equivalent to the original. In each case Heiberg’s figures are taken from William’s translation, as they are almost unrecognizable in C; for convenience in reading the Greek, the figures are given the appropriate Greek letters in this edition.
GREEK MATHEMATICS

eyen e'onton exwthi'sthai to h' sou thlibo' menon
upò tou mállo mou thlibo'menon, kai ekastou de tòv
mep'éwv autou thlibo'sthai tò uperánw autou ýgrò
kata káthetou éonti, ei' ka' mú' to ýgrò ò kathery-
me'von en tini kai upò állo mou tivos thlibo'menon.

Ibid. i., Archim. ed. Heiberg ii. 336. 14-16

'Ýpokesi'sthw, tòv en tò ýgrò ánvo fero'menon
ekastou anaferes'hai kata tàv káthetov tòv dia
tòv kéntrou tov báreos autou ágménav.

(ii.) Surface of Fluid at Rest

Ibid. i., Prop. 2, Archim. ed. Heiberg ii. 319. 7-320. 30

Omnis humidì consistentis ita, ut maneât inmotum,
superficies habebit figuram sperae habentis centrum
idem cum terra.

Intelligatur enim humidì consistens ita, ut
maneât non motum, et secetur ipsius superficies
plano per centrum terrae, sit autem terrae centrum
K, superficiei autem sectio linea ABGD. Dico itaque,

lineam ABGD circuli esse periferiam, centrum autem
ipsius K.

Si enim non est, rectae a K ad lineam ABGD

244
ARCHIMEDES

that which is under the lesser pressure is driven along by that under the greater pressure, and each of its parts is under pressure from the fluid which is perpendicularly above it, except when the fluid is enclosed in something and is under pressure from something else.

Ibid. i., Archim. ed. Heiberg ii. 336. 14-16

Let it be assumed that, of bodies which are borne upwards in a fluid, each is borne upwards along the perpendicular drawn through its centre of gravity.

(ii.) Surface of Fluid at Rest

Ibid. i., Prop. 2, Archim. ed. Heiberg ii. 319. 7-320. 30

The surface of any fluid at rest is the surface of a sphere having the same centre as the earth.

For let there be conceived a fluid at rest, and let its surface be cut by a plane through the centre of the earth, and let the centre of the earth be K, and let the section of the surface be the curve ABΓΔ. Then I say that the curve ABΓΔ is an arc of a circle whose centre is K.

For if it is not, straight lines drawn from K to the

* These are the only assumptions, other than the assumptions of Euclidean geometry, made in this book by Archimedes; if the object of mathematics be to base the conclusions on the fewest and most "self-evident" axioms, Archimedes' treatise On Floating Bodies must indeed be ranked highly.

* The earlier part of this proposition has to be given from William of Moerbeke's translation. The diagram is here given with the appropriate Greek letters.
occurrentes non erunt aequales. Sumatur itaque aliqua recta, quae est quarundam quidem a K occurrentium ad lineam ABGD maior, quarundam autem minor, et centro quidem K, distantia autem sumptae lineae circulus describatur; cadet igitur periferia circuli habens hoc quidem extra lineam ABGD, hoc autem intra, quoniam quae ex centro quarundam quidem a K occurrentium ad lineam ABGD est maior, quarundam autem minor. Sit igitur descripti circuli periferia quae ZBH, et a B ad K recta ducatur, et copulentur quae ZK, KEL aequales facientes angulos, describatur autem et centro K periferia quaedam quae XOP in plano et in humido; partes itaque humidi quae secundum XOP periferiam ex aequo sunt positae et continuae inuicem. Et pre- muntur quae quidem secundum XO periferiam humido quod secundum ZB locum, quae autem secundum periferiam OP humido quod secundum BE locum; inaequaliter igitur premuntur partes humidi quae secundum periferiam XO ei quae [ἡ]¹ kata τὰν ΟΠ· ὡστε ἐξωθησονται τὰ ἕσσον θλιβόμενα ὑπὸ τῶν μᾶλλον θλιβομένων· οὐ μένει ἁρα τὸ ύγρόν. ὑπέκειτο δὲ καθεστακὸς εἰμεν ὡστε μένειν ἀκίνητον· ἀναγκαῖον ἁρα τὰν ΑΒΓΔ γραμ- μάν κύκλου περιφέρειαν εἰμεν καὶ κέντρον αὐτάς τὸ Κ. ὁμοίως δὴ δειχθῆσεται καὶ, ὡστε καὶ ἄλλως ἃ ἐπιφάνεια τοῦ ύγροῦ ἐπιπέδω τμαθῇ διὰ τοῦ κέντρου τὰς γᾶς, ὅτι ἃ τομά ἐσσεῖται κύκλου περιφέρεια, καὶ κέντρον αὐτάς ἐσσεῖται, δ καὶ τὰς γᾶς ἐστι κέντρον. δῆλον οὖν, ὅτι ἃ ἐπιφάνεια τοῦ ύγροῦ καθεστακότος ἀκινήτου σφαιράς ἔχει τὸ σχῆμα τὸ αὐτὸ κέντρον ἐχούσας τὰ γᾶ, ἐπειδὴ

¹ ἡ om. Heiberg.
curve $ABG\Delta$ will not be equal. Let there be taken, therefore, any straight line which is greater than some of the straight lines drawn from $K$ to the curve $ABG\Delta$, but less than others, and with centre $K$ and radius equal to the straight line so taken let a circle be described; the circumference of the circle will fall partly outside the curve $ABG\Delta$, partly inside, inasmuch as its radii are greater than some of the straight lines drawn from $K$ to the curve $ABG\Delta$, but less than others. Let the arc of the circle so described be $ZBH$, and from $B$ let a straight line be drawn to $K$, and let $ZK$, $KE\Lambda$ be drawn making equal angles [with $KB$], and with centre $K$ let there be described, in the plane and in the fluid, an arc $\Xi\Omega\Pi$; then the parts of the fluid along $\Xi\Omega\Pi$ lie evenly and are continuous [v. supra, p. 243]. And the parts along the arc $\Xi\Omega$ are under pressure from the portion of the fluid between it and $ZB$, while the parts along the arc $\Omega\Pi$ are under pressure from the portion of the fluid between it and $BE$; therefore the parts of the fluid along $\Xi\Omega$ and the parts of the fluid along $\Omega\Pi$ are under unequal pressures; so that the parts under the lesser pressure are thrust along by the parts under the greater pressure [v. supra, p. 245]; therefore the fluid will not remain at rest. But it was postulated that the fluid would remain unmoved; therefore the curve $ABG\Delta$ must be an arc of a circle with centre $K$. Similarly it may be shown that, in whatever other manner the surface be cut by a plane through the centre of the earth, the section is an arc of a circle and its centre will also be the centre of the earth. It is therefore clear that the surface of the fluid remaining at rest has the form of a sphere with the same centre as the earth, since it is such
(iii.) Solid immersed in a Fluid


Τὰ βαρύτερα τοῦ ὕγρου ἀφεθέντα εἰς τὸ ὕγρον οἰσείται κάτω, ἐστὶν ἂν καταβάντι, καὶ ἐσοσίναι κουφότερα ἐν τῷ ὕγρῳ τοσοῦτον, ὥσον ἐξει τὸ βάρος τοῦ ὕγρου τοῦ ταλικοῦτον ὄγκον ἐχοντος, ἀλλὰς ἐστὶν ὅ τοῦ στερεοῦ μεγέθεος ὄγκος.

"Οτι μὲν οὖν οἰσείται ἔσ τὸ κάτω, ἐστὶ ἂν καταβάντι, δηλον τα γάρ ὑποκάτω αὐτοῦ μέρεα τοῦ ὕγρου θλιβησοῦνται μᾶλλον τῶν ἐσ ἵσου αὐτοὺς κεμένων μερέων, ἐπειδή βαρύτερον ὑπόκειται τὸ στερεὸν μέγεθος τοῦ ὕγρον. ὅτι δὲ κουφότερα ἐσοσίναι, ὡς εἰρηται, δειχθῆσεται.

"Εστω τι μέγεθος τὸ Α, ὅ ἐστι βαρύτερον τοῦ ὕγρου, βάρος δὲ ἐστω τοῦ μὲν ἐν ὧ Α μεγέθεος τὸ ΒΓ, τοῦ δὲ ὕγρου του ἱσον ὄγκον ἐχοντος τῷ Α το Β. δεικτέων, ὅτι τὸ Α μέγεθος ἐν τῷ ὕγρῳ ἐών βάρος ἐξει ἱσον τῷ Γ.

Λελάφθω γάρ τι μέγεθος τὸ ἐν ὧ τὸ Δ κουφότερον τοῦ ὕγρου τοῦ ἱσον ὄγκον ἐχοντος αὐτοῦ, ἐστω δὲ τοῦ μὲν ἐν ὧ τὸ Δ μεγέθεος βάρος ἱσον τῷ Β βάρει, τοῦ δὲ ὕγροι τοῦ ἱσον ὄγκον ἐχοντος τῷ Δ μεγέθει τὸ βάρος ἐστω ἱσον τῷ ΒΓ βάρει.

* Or, as we should say, “lighter by the weight of fluid displaced.”

248
that, when it is cut [by a plane] always passing through the same point, the section is an arc of a circle having for centre the point through which it is cut by the plane [Prop. 1].

(iii.) Solid immersed in a Fluid


Solids heavier than a fluid will, if placed in the fluid, sink to the bottom, and they will be lighter [if weighed] in the fluid by the weight of a volume of the fluid equal to the volume of the solid.\(^a\)

That they will sink to the bottom is manifest; for the parts of the fluid under them are under greater pressure than the parts lying evenly with them, since it is postulated that the solid is heavier than water; that they will be lighter, as aforesaid will be [thus] proved.

Let \(A\) be any magnitude heavier than the fluid, let the weight of the magnitude \(A\) be \(B + \Gamma\), and let the weight of fluid having the same volume as \(A\) be \(B\). It is required to prove that in the fluid the magnitude \(A\) will have a weight equal to \(\Gamma\).

For let there be taken any magnitude \(\Delta\) lighter than the same volume of the fluid such that the weight of the magnitude \(\Delta\) is equal to the weight \(B\), while the weight of the fluid having the same volume as the magnitude \(\Delta\) is equal to the weight \(B + \Gamma\).
This proposition suggests a method, alternative to that given by Vitruvius (*v. supra*, pp. 36-39, especially p. 38 n. a), whereby Archimedes may have discovered the proportions of gold and silver in King Hiero's crown.

Let $w$ be the weight of the crown, and let $w_1$ and $w_2$ be the weights of gold and silver in it respectively, so that $w = w_1 + w_2$.

Take a weight $w$ of gold and weigh it in a fluid, and let the loss of weight be $P_1$. Then the loss of weight when a weight $w_1$ of gold is weighed in the fluid, and consequently the weight of fluid displaced, will be $\frac{w_1}{w} \cdot P_1$. 

250
Then if we combine the magnitudes $A$, $\Delta$, the combined magnitude will be equal to the weight of the same volume of the fluid; for the weight of the combined magnitudes is equal to the weight $(B + \Gamma) + B$, while the weight of the fluid having the same volume as both the magnitudes is equal to the same weight. Therefore, if the [combined] magnitudes are placed in the fluid, they will balance the fluid, and will move neither upwards nor downwards [Prop. 3]; for this reason the magnitude $A$ will move downwards, and will be subject to the same force as that by which the magnitude $\Delta$ is thrust upwards, and since $\Delta$ is lighter than the fluid it will be thrust upwards by a force equal to the weight $\Gamma$; for it has been proved that when solid magnitudes lighter than the fluid are forcibly immersed in the fluid, they will be thrust upwards by a force equal to the difference in weight between the magnitude and an equal volume of the fluid [Prop. 6]. But the fluid having the same volume as $\Delta$ is heavier than the magnitude $\Delta$ by the weight $\Gamma$; it is therefore plain that the magnitude $A$ will be borne upwards by a force equal to $\Gamma$.

Now take a weight $w$ of silver and weigh it in the fluid, and let the loss of weight be $P_3$. Then the loss of weight when a weight $w_2$ of silver is weighed in the fluid, and consequently the weight of fluid displaced, will be $\frac{w_2}{w} \cdot P_3$.

Finally, weigh the crown itself in the fluid, and let the loss of weight, and consequently the weight of fluid displaced, be $P$.

It follows that $\frac{w_1}{w} \cdot P_1 + \frac{w_2}{w} \cdot P_3 = P$,

whence $\frac{w_1}{w} = \frac{P_2 - P}{P - P_1}$.
(iv.) Stability of a Paraboloid of Revolution


Τὸ ὀρθὸν τμῆμα τοῦ ὀρθογωνίου κωνοειδέος, ὅταν τὸν ἄξονα ἔχῃ μὴ μείζονα ἢ ἡμόλιον τὰς μέχρι τοῦ ἄξονος, πάντα λόγον ἔχου ποτὲ τὸ ύγρὸν τῷ βάρει, ἀφεθὲν εἰς τὸ υγρὸν οὕτως, ἃςτε τὰν βάσιν αὐτοῦ μὴ ἀπτεσθαί τοῦ υγροῦ, τεθὲν κεκλιμένον οὐ μενεὶ κεκλιμένον, ἀλλὰ ἀποκαταστασεῖται ὀρθὸν. ὀρθὸν δὲ λέγω καθεστακέναι τὸ τουώτο τμῆμα, ὅπόταν τὸ ἀποτετμάκος αὐτὸ ἐπίπεδον παρὰ τὰν ἐπιφάνειαν ἢ τοῦ υγροῦ.

"Εστω τμῆμα ὀρθογωνίου κωνοειδέος, οἷον εὑρηταί, καὶ κείσθω κεκλιμένον. δεικτέων, ὅτι οὐ μενεὶ, ἀλλ' ἀποκαταστασεῖται ὀρθὸν.

Τμαθέντος δὴ αὐτοῦ ἐπιπέδῳ διὰ τοῦ ἄξονος ὀρθῷ ποτὶ τὸ ἐπίπεδον τὸ ἐπὶ τὰς ἐπιφάνειας τοῦ υγροῦ τμάματος ἐστὼ τομὰ ἢ ΑΠΟΛ ὀρθογωνίου κώνου τομά, ἄξων δὲ τοῦ τμάματος καὶ διάμετρος τὰς τομᾶς ἢ ΝΟ, τὰς δὲ τοῦ υγροῦ ἐπιφάνειας τομὰ ἢ ΙΣ. ἐπεὶ οὖν τὸ τμᾶμα οὐκ ἐστὶν ὀρθὸν, οὐκ ἂν εἰ ἐπαράλληλος ἢ ΑΛ τῇ ΙΣ. ὅτε οὐ ποιήσει ὀρθῶν γωνιὰν ἢ ΝΟ ποτὶ τὰν ΙΣ. ᾿οχθω

---

* Writing of the treatise *On Floating Bodies*, Heath (*H.G.M.* ii. 94-95) justly says: "Book ii., which investigates fully the conditions of stability of a right segment of a paraboloid of revolution floating in a fluid for different values of the specific gravity and different ratios between the axis or height of the segment and the principal parameter of the generating parabola, is a veritable *tour de force* which must be read in full to be appreciated."

* In this technical term the "axis" is the axis of the
(iv.) Stability of a Paraboloid of Revolution


If there be a right segment of a right-angled conoid, whose axis is not greater than one-and-a-half times the line drawn as far as the axis, and whose weight relative to the fluid may have any ratio, and if it be placed in the fluid in an inclined position in such a manner that its base do not touch the fluid, it will not remain inclined but will return to the upright position. I mean by returning to the upright position the figure formed when the plane cutting off the segment is parallel to the surface of the fluid.

Let there be a segment of a right-angled conoid, such as has been stated, and let it be placed in an inclined position. It is required to prove that it will not remain there but will return to the upright position.

Let the segment be cut by a plane through the axis perpendicular to the plane which forms the surface of the fluid, and let ΑΠΟΛ be the section of the segment, being a section of a right-angled cone [*De Con. et Sphaer.* 11], and let ΝΟ be the axis of the segment and the axis of the section, and let ΙΣ be the section of the surface of the liquid. Then since the segment is not upright, ΑΛ will not be parallel to ΙΣ; and therefore ΝΟ will not make a right angle

right-angled cone from which the generating parabola is derived. The *latus rectum* is “the line which is double of the line drawn as far as the axis” (ά διπλασία τᾶς μέχρι τοῦ ἁξονος); and so the condition laid down by Archimedes is that the axis of the segment of the paraboloid of revolution shall not be greater than three-quarters of the *latus rectum* or *principal parameter* of the generating parabola.
GREEK MATHEMATICS

οὖν παράλληλος ἂ ἐφαπτομένα ἂ ΚΩ τᾶς τοῦ κώνου τομᾶς κατὰ τὸ Π, καὶ ἀπὸ τοῦ Π παρὰ τὰν

ΝΟ ἀχθω ἂ ΠΦ. τέμνει δὴ ἂ ΠΦ δίχα τὰν ΙΣ. δέδεικται γὰρ ἐν τοῖς κωνικοῖς. τετμάσθω ἂ ΠΦ, ὥστε εἶμεν διπλασίαν τὰν ΠΒ τὰς ΒΦ, καὶ ἂ ΝΟ κατὰ τὸ Ρ τετμάσθω, ὥστε καὶ τὰν ΩΡ τὰς ΡΝ διπλασίαν εἶμεν· ἐσσεῖται δὴ τοῦ μείζονος ἀποτμάματος τοῦ στερεοῦ κέντρου τοῦ βάρεος τὸ Ρ, τοῦ δὲ κατὰ τὰν ΠΠΟΣ τὸ Β. δέδεικται γὰρ ἐν ταῖς Ἰσορροπίαις, ὅτι παντὸς ὀρθογώνιου κωνοειδεός τμάματος τὸ κέντρον τοῦ βάρεος ἐστὶν ἐπὶ τοῦ ἄξονος διηρημένου οὐτως, ὅστε τὸ ποτὶ τὰ κορυφὰ τοῦ ἄξονος τμῆμα διπλάσιον εἶμεν τοῦ λοιποῦ. ἀφαιρεθέντος δὴ τοῦ κατὰ τὰν ΠΠΟΣ τμάματος στερεοῦ ἀπὸ τοῦ ὅλου τοῦ λοιποῦ κέντρον ἐσσεῖται τοῦ βάρεος ἐπὶ τὰς ΒΓ εὐθείας· δέδεικται γὰρ τούτῳ ἐν τοῖς Στοιχείοις τῶν μηχανικῶν, ὅτι, εἰ καὶ μέγεθος ἀφαιρεθη μὴ τὸ αὐτὸ κέντρον ἐχον τοῦ βάρεος τῷ ὅλῳ μεγέθει, τοῦ λοιποῦ τὸ κέντρον ἐσσεῖται τοῦ βάρεος ἐπὶ τὰς εὐθείας τὰς ἐπιζευγνυόμενα τὰ κέντρα τοῦ τε ὅλου μεγέθεος καὶ τοῦ

254
ARCHIMEDES

with $I\Sigma$. Therefore let $K\Omega$ be drawn parallel [to $I\Sigma$] and touching the section of the cone at $\Pi$, and from $\Pi$ let $\Pi\Phi$ be drawn parallel to $NO$; then $\Pi\Phi$ bisects $I\Sigma$—for this is proved in the [Elements of] Conics.\(^a\)

Let $\Pi\Phi$ be cut so that $\PiB = 2B\Phi$, and let $NO$ be cut at $P$ so that $OP = 2PN$; then $P$ will be the centre of gravity of the greater segment of the solid, and $B$ that of $\Pi\Pi\Omega\Sigma$; for it is proved in the books On Equilibriums that the centre of gravity of any segment of a right-angled conoid is at the point dividing the axis in such a manner that the segment towards the vertex of the axis is double of the remainder.\(^b\)

Now if the solid segment $\Pi\Pi\Omega\Sigma$ be taken away from the whole, the centre of gravity of the remainder will lie upon the straight line $B\Gamma$; for it has been proved in the Elements of Mechanics that if any magnitude be taken away not having the same centre of gravity as the whole magnitude, the centre of gravity of the remainder will be on the straight line joining the centres [of gravity] of the whole magnitude and of the part

---

\(^a\) Presumably in the works of Aristaeus or Euclid, but it is also Quad. Parab. 1.

\(^b\) The proof is not in any extant work by Archimedes.
If the normal at II meets the axis in M, then OM is greater than "the line drawn as far as the axis" except in the case where II coincides with the vertex, which case is excluded by the conditions of this proposition. Hence OM is always greater than OP; and because the angle ΩΠΙΜ is right, the angle ΩΠΠ must be acute.
taken away, produced from the extremity which is the centre of gravity of the whole magnitude [De Plan. Aequil. i. 8]. Let BP then be produced to Γ, and let Γ be the centre of gravity of the remaining magnitude. Then, since \( NO = \frac{3}{2} \cdot OP \), and \( NO > \frac{3}{2} \cdot (\text{the line drawn as far as the axis}) \), it is clear that \( PO \) (the line drawn as far as the axis); therefore \( \Pi \Pi \) makes unequal angles with \( K\Omega \), and the angle \( \Pi \Pi \Omega \) is acute \(^a\); therefore the perpendicular drawn from \( P \) to \( \Pi \Omega \) will fall between \( \Pi, \Omega \). Let it fall as \( P\Theta \); then \( P\Theta \) is perpendicular to the cutting plane containing \( \Sigma \), which is on the surface of the fluid. Now let lines be drawn from \( B, \Gamma \parallel P\Theta \); then the portion of the magnitude outside the fluid will be subject to a downward force along the line drawn through \( \Gamma \)—for it is postulated that each weight is subject to a downward force along the perpendicular drawn through its centre of gravity \(^b\); and since the magnitude in the fluid is lighter than the fluid,\(^c\) it will be subject to an upward force along the perpendicular drawn through \( B \).\(^d\) But, since they are not subject to contrary forces along the same perpendicular, the figure will not remain at rest but the portion on the side of \( A \) will move upwards and the portion on the side of \( A \) will move downwards, and this will go on continually until it is restored to the upright position.

\(^a\) Cf. supra, p. 245; possibly a similar assumption to this effect has fallen out of the text.

\(^b\) A tacit assumption, which limits the generality of the opening statement of the proposition that the segment may have any weight relative to the fluid.

\(^c\) v. supra, p. 251.
XVIII. ERATOSTHENES
Several of Eratosthenes’ achievements have already been described—his solution of the Delian problem (vol. i. pp. 290-297), and his sieve for finding successive odd numbers (vol. i. pp. 100-103). Archimedes, as we have seen, dedicated the Method to him, and the Cattle Problem, as we have also seen, is said to have been sent through him to the Alexandrian mathematicians. It is generally supposed that Ptolemy credits him with having calculated the distance between the tropics (or twice the obliquity of the ecliptic) at 11/83rds. of a complete circle or 47° 29’ 39”, but Ptolemy’s meaning is not clear. Eratosthenes also calculated the distances of the sun and moon from the earth and the size of the sun. Fragments of an astronomical poem which he wrote under the title
Eratosthenes, son of Aglaus; others say of Ambrosius; a Cyrenean, a pupil of the philosopher Ariston of Chios, of the grammarian Lysanias of Cyrene and of the poet Callimachus; he was sent for from Athens by the third Ptolemy and stayed till the fifth. Owing to taking second place in all branches of learning, though approaching the highest excellence, he was called Beta. Others called him a Second or New Plato, and yet others Pentathlon. He was born in the 126th Olympiad and died at the age of Hermes have survived. He was the first person to attempt a scientific chronology from the siege of Troy in two separate works, and he wrote a geographical work in three books. His writings are critically discussed in Bernhardy’s Eratosthenica (Berlin, 1822).

Callimachus, the famous poet and grammarian, was also a Cyrenean. He opened a school in the suburbs of Alexandria and was appointed by Ptolemy Philadelphus chief librarian of the Alexandrian library, a post which he held till his death c. 240 B.C. Eratosthenes later held the same post.

Euergetes I (reigned 246–221 B.C.), who sent for him to be tutor to his son and successor Philopator (v. vol. i. pp. 256, 296).

Epiphanes (reigned 204–181 B.C.)
πιάδι καὶ ἐτελεύτησεν π ἐτῶν γεγονός, ἀποσχόμενος τροφῆς διὰ τὸ ἀμβλυώττειν, μαθητὴν ἐπίσημον καταλιπὼν Ἀριστοφάνην τὸν Βυζάντιον· οὐ πάλιν Ἀρίσταρχος μαθητής. μαθηταί δὲ αὐτοῦ Μνασέας καὶ Μένανδρος καὶ Ἀριστις. ἔγραψε δὲ φιλόσοφα καὶ ποιήματα καὶ ἱστορίας, Ἀστρονομίαν ἡ Καταστηρίσμοις,¹ Περὶ τῶν κατὰ φιλοσοφίαν αἴρεσεων, Περὶ ἀλυπίας, διαλόγους πολλοὺς καὶ γραμματικὰ συχνά.

(b) On Means

Papp. Coll. vii. 3, ed. Hultsch 636. 18-25

Τῶν δὲ προειρημένων τοῦ Ἀναλυμένου βιβλίων ἡ τάξις ἐστὶν τοιαύτη . . . Ἐρατοσθένους περὶ μεσοτήτων δύο.

Papp. Coll. vii. 21, ed. Hultsch 660. 18-662. 18

Τῶν τόπων καθόλου οἱ μὲν εἰσὶν ἐφεκτικοὶ, ὡς καὶ Ἀπολλώνιος πρὸ τῶν ἠδίων στοιχείων λέγει σημείου μὲν τόπον σημεῖον, γραμμῆς δὲ τόπον γραμμῆν, ἑπιφανείας δὲ ἑπιφάνειαν, στερεοῦ δὲ στερεόν, οἱ δὲ διεξόδικοί, ὡς σημείου μὲν γραμμῆν, γραμμῆς δ’ ἑπιφάνειαν, ἑπιφανείας δὲ στερεόν,

¹ Καταστηρίσμοις coni. Portus, Καταστηρίσμοις codd.

---

¹ Not, of course, Aristarchus of Samos, the mathematician, but the celebrated Samothracian grammarian.
² Mnaseas was the author of a work entitled Περὶ πλοῦς, whose three sections dealt with Europe, Asia and Africa, and a collection of oracles given at Delphi.
³ This work is extant, but is not thought to be genuine in
ERATOSTHENES

of eighty of voluntary starvation, having lost his sight; he left a distinguished pupil, Aristophanes of Byzantium; of whom in turn Aristarchus was a pupil. Among his pupils were Mnaseas, Menander and Aristis. He wrote philosophical works, poems and histories, Astronomy or Placings Among the Stars, On Philosophical Divisions, On Freedom from Pain, many dialogues and numerous grammatical works.

(b) On Means

Pappus, Collection vii. 3, ed. Hultsch 636. 18-25

The order of the aforesaid books in the Treasury of Analysis is as follows . . . the two books of Eratosthenes On Means.

Pappus, Collection vii. 21, ed. Hultsch 660. 18–662. 18

Loci in general are termed fixed, as when Apollonius at the beginning of his own Elements says the locus of a point is a point, the locus of a line is a line, the locus of a surface is a surface and the locus of a solid is a solid; or progressive, as when it is said that the locus of a point is a line, the locus of a line is a surface and the locus of a surface is a solid; or circumambient as its extant form; it contains a mythology and description of the constellations under forty-four heads. The general title Αστρονομία may be a mistake for Αστροθεσία; elsewhere it is alluded to under the title Καταλογοῦ.

The inclusion of this work in the Treasury of Analysis, along with such works as those of Euclid, Aristaeus and Apollonius, shows that it was a standard treatise. It is not otherwise mentioned, but the loci with reference to means referred to in the passage from Pappus next cited were presumably discussed in it.
The passage of which this forms the concluding sentence is attributed by Hultsch to an interpolator. To fill the lacuna before εκείνωσ he suggests ἀνόμωια εκείνωσ, following Halley’s rendering, “diversa sunt ab illis.”

Tannery conjectured that these were the loci of points such that their distances from three fixed lines provided a “médiété,” i.e., loci (straight lines and conies) which can be represented in trilinear co-ordinates by such equations as

\[2y = x + z, \quad y^2 = xz, \quad y(x + z) = 2xz, \quad x(x - y) = z(y - z), \quad x(x - y) = y(y - z)\]

these represent respectively the arithmetic, geometric and harmonic means, and the means subcontrary to the harmonic and geometric means (v. vol. i. pp. 122-125). Zeuthen has
when it is said that the locus of a point is a surface and the locus of a line is a solid. [. . . the loci described by Eratosthenes as having reference to means belong to one of the aforesaid classes, but from a peculiarity in the assumptions are unlike them.]

(c) The "Platonicus"

Theon of Smyrna, ed. Hiller 81. 17–82. 5

Eratosthenes in the Platonicus says that interval and ratio are not the same. Inasmuch as a ratio is a sort of relationship of two magnitudes one towards the other, there exists a ratio both between terms that are different and also between terms that are not different. For example, the ratio of the perceptible to the intelligible is the same as the ratio of opinion to knowledge, and the difference between the intelligible and the known is the same as the difference of opinion from the perceptible. But there can be an interval only between terms that are different, according to magnitude or quality or position or in some other way. It is thence clear that ratio is

an alternative conjecture on similar lines (Die Lehre von den Kegelschnitten im Altertum, pp. 320-321).

Theon cites this work in one other passage (ed. Hiller 2. 3-12) telling how Plato was consulted about the doubling of the cube; it has already been cited (vol. i. p. 256). Eratosthenes' own solution of the problem has already been given in vol. i. pp. 290-297, and a letter purporting to be from Eratosthenes to Ptolemy Euergetes is given in vol. i. pp. 256-261. Whether the Platonicus was a commentary on Plato or a dialogue in which Plato was an interlocutor cannot be decided.


GREEK MATHEMATICS

οτι λόγος διαστήματος ἐτερων τὸ γὰρ ἦμισυ πρὸς τὸ διπλάσιον (καὶ τὸ διπλάσιον πρὸς τὸ ἦμισυ)\(^1\) λόγον μὲν οὗ τὸν αὐτὸν ἔχει, διάστημα δὲ τὸ αὐτὸ.

(d) Measurement of the Earth

Cleom. De motu circ. i. 10. 52, ed. Ziegler 94. 23–100. 23

Καὶ ἡ μὲν τοῦ Ποσειδωνίου ἐφόδους περὶ τοῦ κατὰ τὴν γῆν μεγέθους τοιαύτη, ἡ δὲ τοῦ Ἑρατοσθένους γεωμετρικῆς ἐφόδους ἐχομένη, καὶ δο-κοῦσα τι ἄσαφέστερον ἔχειν. ποιήσει δὲ σαφῆ τὰ λεγόμενα ὑπ’ αὐτοῦ τάδε προϋποθετεμένων ἦμῶν. ὑποκείσθω ἦμὶν πρῶτον μὲν κανταῦθα, ὑπὸ τῷ αὐτῷ μεσημβρινῷ κείσθαι Συνήνην καὶ Ἀλεξάνδρειαν, καὶ δεύτερον, τὸ διάστημα τὸ μεταξὺ τῶν πόλεων πεντακισχιλίων σταδίων εἶναι, καὶ τρίτον, τὰς καταπεμπομένας ἀκτίνας ἀπὸ διαφόρων μερῶν τοῦ ἡλίου ἐπὶ διάφορα τῆς γῆς μέρη παραλλήλους εἶναι: οὕτως γὰρ ἔχειν αὐτὸς οἱ γεωμέτραι ὑποτίθενται, τέταρτον ἐκεῖνο ὑπο-κείσθω, δεικνύμενον παρὰ τοῖς γεωμέτραις, τὰς εἰς παραλλήλους ἐμπιπτούσας εὐθείας τὰς ἐναλλάξ γωνίας ἴσας ποιεῖν, πέμπτον, τὰς ἐπὶ ἰσών γωνιῶν βεβηκυίας περιφερείας ὁμοίας εἶναι, τοῦτοις τῇ τῆς αὐτῆς ἀναλογίαν καὶ τὸν αὐτὸν λόγον ἔχουν πρὸς τοὺς οἰκείους κύκλους, δεικνυμένου καὶ τούτου παρὰ τοῖς γεωμέτραις. ὅποταν γὰρ περιφερείας ἐπὶ ἰσών γωνιῶν ὡς βεβηκυίας, ἄν μία ἦτισον

\(^1\) καὶ . . . ἦμισυ add. Hiller.

* The difference between ratio and interval is explained a little more neatly by Theon himself (ed. Hiller 81. 6-9): 266
ERATOSTHENES

different from interval; for the relationship of the half to the double and of the double to the half does not furnish the same ratio, but it does furnish the same interval.\textsuperscript{a}

\textbf{(d) Measurement of the Earth}

Cleomedes,\textsuperscript{b} On the Circular Motion of the Heavenly Bodies i. 10. 52, ed. Ziegler 94. 23–100. 23

Such then is Posidonius's method of investigating the size of the earth, but Eratosthenes' method depends on a geometrical argument, and gives the impression of being more obscure. What he says will, however, become clear if the following assumptions are made. Let us suppose, in this case also, first that Syene and Alexandria lie under the same meridian circle; secondly, that the distance between the two cities is 5000 stades; and thirdly, that the rays sent down from different parts of the sun upon different parts of the earth are parallel; for the geometers proceed on this assumption. Fourthly, let us assume that, as is proved by the geometers, straight lines falling on parallel straight lines make the alternate angles equal, and fifthly, that the arcs subtended by equal angles are similar, that is, have the same proportion and the same ratio to their proper circles—this also being proved by the geometers. For whenever arcs of circles are subtended by equal angles, if any one of these is (say) one-tenth

\[\text{d} \text{i} \alpha \text{f} \text{e} \text{r} \text{e} \text{i} \text{d} \text{e} \text{d} \text{i} \alpha \text{t} \text{t} \text{m} \text{a} \text{ k} \text{a} \text{i} \text{ } \text{l} \text{o} \text{g} \text{o} \text{s}, \text{e} \text{p} \text{e} \text{i} \text{d} \text{h} \text{e} \text{d} \text{i} \alpha \text{t} \text{t} \text{m} \text{a} \text{ m} \text{e} \text{n} \text{e} \text{s} \text{t} \text{i} \text{t} \text{o} \text{ m} \text{e} \text{t} \text{a} \text{z} \text{h} \text{t} \text{o} \text{v} \text{h} \text{o} \text{m} \text{o} \text{g} \text{e} \text{n} \text{w} \text{o} \text{n} \text{t} \text{e} \text{k} \text{a} \text{i} \text{a} \text{n} \text{i} \text{o} \text{w} \text{n} \text{d} \text{r} \text{o} \text{w} \text{v}, \text{l} \text{o} \text{g} \text{o} \text{s} \text{d} \text{e} \text{a} \text{p} \text{l} \text{w} \text{o} \text{s} \text{h} \text{t} \text{o} \text{v} \text{h} \text{o} \text{m} \text{o} \text{g} \text{e} \text{n} \text{w} \text{o} \text{n} \text{d} \text{r} \text{o} \text{w} \text{n} \text{p} \text{r} \text{o} \text{s} \text{a} \text{l} \text{l} \text{h} \text{l} \text{o} \text{u} \text{s} \text{s} \text{c} \text{h} \text{e} \text{s} \text{i} \text{s}.\]

\textsuperscript{b} Cleomedes probably wrote about the middle of the first century b.c. His handbook De motu circulari corporum caelestium is largely based on Posidonius.
GREEK MATHEMATICS

αὐτῶν δέκατον ἢ μέρος τοῦ οἰκείου κύκλου, καὶ αἴ λοιπαὶ πᾶσαι δέκατα μέρη γενήσονται τῶν οἰκείων κύκλων.

Τούτων ὁ κατακρατήσας οὐκ ἂν χαλεπῶς τὴν ἔφοδον τοῦ Ἐρατοσθένους καταμάθην ἔχουσαν οὕτως. ὑπὸ τῷ αὐτῷ κείσθαι μεσημβρυώ φησὶ Συήνην καὶ Ἀλεξάνδρειαν. ἐπεὶ οὖν μέγιστοι τῶν ἐν τῷ κόσμῳ οἱ μεσημβρυνοὶ, δεὶ καὶ τοὺς ὑποκείμενους τούτους τῆς γῆς κύκλους μεγίστους εἰναι ἀναγκαῖως. ὡστε ἦλικον ἃν τὸν διὰ Συήνης καὶ Ἀλεξάνδρειας ἦκοντα κύκλον τῆς γῆς ἢ ἐφοδὸς ἀποδείξει αὐτή, τηλικοῦτος καὶ ὁ μέγιστος ἔσται τῆς γῆς κύκλος. φησὶ τοῖς, καὶ ἔχει οὕτως, τὴν Συήνην ὑπὸ τῷ θερινῷ τροπικῷ κείσθαι κύκλω. ὅποταν οὖν ἐν καρκίνῳ γενόμενον ὁ ἥλιος καὶ θερινὰς ποὺν τροπὰς ἀκριβῶς μεσουρανήσῃ, ἀσκιοὶ γίνονται οἱ τῶν ὦρολογίων γνώμονες ἀναγκαῖως, καὶ ἀκριβὴς τοῦ ἥλιου ὑπερκείμενον καὶ τοῦτο γίνεσθαι λόγος ἐπὶ σταδίους τριακοσίους τὴν διάμετρον. ἐν Ἀλεξάνδρεια δὲ τῇ αὐτῇ ὥρᾳ ἀποβάλλουσαν οἱ τῶν ὦρολογίων γνώμονες σκιάν, ὅτε πρὸς ἄρκτω μᾶλλον τῆς Συήνης ταύτης τῆς πόλεως κείμενης. ὑπὸ τῷ αὐτῷ μεσημβρυώ τοῖς καὶ μεγίστω κύκλῳ τῶν πόλεως κείμενων, ἄν περισσάκωσιν περιφέρειαν ἀπὸ τοῦ ἄκρου τῆς τοῦ γνώμονος σκιάς ἐπὶ τὴν βάσιν αὐτὴν τοῦ γνώμονος τοῦ ἐν Ἀλεξάνδρεια ὦρολογίῳ, αὐτῇ ἡ περιφέρεια τῆς γενήσεται τοῦ μεγίστου τῶν ἐν τῇ σκάφῃ κύκλων, ἐπεὶ μεγίστῳ κύκλῳ ὑποκείται ἡ τοῦ ὦρολογίου σκάφη. εἰ οὖν εἰς νοῆσαι εὐθείας διὰ τῆς γῆς ἐκβαλλομένας αἱ ἐκατέρου τῶν γνώμονων, πρὸς τῷ κέντρῳ τῆς γῆς 268
of its proper circle, all the remaining arcs will be tenth parts of their proper circles.

Anyone who has mastered these facts will have no difficulty in understanding the method of Eratosthenes, which is as follows. Syene and Alexandria, he asserts, are under the same meridian. Since meridian circles are great circles in the universe, the circles on the earth which lie under them are necessarily great circles also. Therefore, of whatever size this method shows the circle on the earth through Syene and Alexandria to be, this will be the size of the great circle on the earth. He then asserts, as is indeed the case, that Syene lies under the summer tropic. Therefore, whenever the sun, being in the Crab at the summer solstice, is exactly in the middle of the heavens, the pointers of the sundials necessarily throw no shadows, the sun being in the exact vertical line above them; and this is said to be true over a space 300 stades in diameter. But in Alexandria at the same hour the pointers of the sundials throw shadows, because this city lies farther to the north than Syene. As the two cities lie under the same meridian great circle, if we draw an arc from the extremity of the shadow of the pointer to the base of the pointer of the sundial in Alexandria, the arc will be a segment of a great circle in the bowl of the sundial, since the bowl lies under the great circle. If then we conceive straight lines produced in order from each of the pointers through the earth, they
GREEK MATHEMATICS

συμπεσοῦνται. ἔπει δὲν τὸ ἐν Συήνῃ ὦρολόγιον κατὰ κάθετον ὑπόκειται τῷ ἥλιῳ, ἂν ἐπινοῆσωμεν εὐθείαν ἀπὸ τοῦ ἥλιου ἡκουσαν ἐπὶ ἀκρον τοῦ τοῦ ὦρολογίου γνώμονα, μία γενῆσεται εὐθεία ἡ ἀπὸ τοῦ ἥλιου μέχρι τοῦ κέντρου τῆς γῆς ἡκουσα. ἐὰν οὖν ἔτέραν εὐθείαν νοῆσωμεν ἀπὸ τοῦ ἀκροῦ τῆς σκιᾶς τοῦ γνώμονος δι’ ἀκροῦ τοῦ γνώμονος ἐπὶ τοῦ ἥλιου ἀναγομένην ἀπὸ τῆς ἐν Ἀλεξανδρείᾳ σκάφης, αὐτὴ καὶ ἡ προερημένη εὐθεία παράλληλοι γενῆσονται ἀπὸ διαφόρων γε τοῦ ἥλιου μερῶν ἐπὶ διάφορα μέρη τῆς γῆς διήκουσαν. εἰς ταῦτας τοῖνυν παραλλήλους οὕσας ἐμπίπτει εὐθεία ἡ ἀπὸ τοῦ κέντρου τῆς γῆς ἐπὶ τὸν ἐν Ἀλεξανδρείᾳ γνώμονα ἡκουσα, ὥστε τὰς ἐναλλάξ γνωσίας ἴσας ποιεῖν· ὅν ἡ μὲν ἐστὶ πρὸς τῷ κέντρῳ τῆς γῆς κατὰ σύμπτωσιν τῶν εὐθείων, αἱ ἀπὸ τῶν ὦρολογίων ἡχθησαν ἐπὶ τὸ κέντρον τῆς γῆς, γνωμένην, ἡ δὲ κατὰ σύμπτωσιν ἀκρον τοῦ ἐν Ἀλεξανδρείᾳ γνώμονος καὶ τῆς ἀπ’ ἀκρον τῆς σκιᾶς αὐτοῦ ἐπὶ τὸν ἥλιον διὰ τῆς πρὸς αὐτὸν ψαύσεως ἀναχθείσης γεγενημένη. καὶ ἐπὶ μὲν ταύτης βέβηκε περιφέρεια ἡ ἀπ’ ἀκρον τῆς σκιᾶς τοῦ γνώμονος ἐπὶ τῆς βάσιν αὐτοῦ περιαχθείσα, ἐπὶ δὲ τῆς πρὸς τῷ κέντρῳ τῆς γῆς ἡ ἀπὸ Συήνης διήκουσα εἰς Ἀλεξάνδρειαν. ὁμοίως τοῖνυν αἱ περιφέρειαι εἴσων ἀλλήλαις ἐπ’ ἴσων γε γνωσὶν βιβλικά. ὅν ἄρα λόγον ἐχει ἡ ἐν τῇ σκάφῃ πρὸς τὸν οἰκεῖον κύκλου, τοῦτον ἐχει τὸν λόγον καὶ ἡ ἀπὸ Συήνης εἰς Ἀλεξάνδρειαν ἡκουσα. ἡ δὲ γε ἐν τῇ σκάφῃ πεντηκοστὸν μέρος εὐρίσκεται τοῦ οἰκείου κύκλου. δει οὖν ἀναγκαίως καὶ τὸ ἀπὸ Συήνης εἰς Ἀλεξάνδρειαν διάστημα πεντηκοστὸν εἶναι μέρος τοῦ
ERATOSTHENES

will meet at the centre of the earth. Now since the sundial at Syene is vertically under the sun, if we conceive a straight line drawn from the sun to the top of the pointer of the sundial, the line stretching from the sun to the centre of the earth will be one straight line. If now we conceive another straight line drawn upwards from the extremity of the shadow of the pointer of the sundial in Alexandria, through the top of the pointer to the sun, this straight line and the aforesaid straight line will be parallel, being straight lines drawn through from different parts of the sun to different parts of the earth. Now on these parallel straight lines there falls the straight line drawn from the centre of the earth to the pointer at Alexandria, so that it makes the alternate angles equal; one of these is formed at the centre of the earth by the intersection of the straight lines drawn from the sundials to the centre of the earth; the other is at the intersection of the top of the pointer in Alexandria and the straight line drawn from the extremity of its shadow to the sun through the point where it meets the pointer. Now this latter angle subtends the arc carried round from the extremity of the shadow of the pointer to its base, while the angle at the centre of the earth subtends the arc stretching from Syene to Alexandria. But the arcs are similar since they are subtended by equal angles. Whatever ratio, therefore, the arc in the bowl of the sundial has to its proper circle, the arc reaching from Syene to Alexandria has the same ratio. But the arc in the bowl is found to be the fiftieth part of its proper circle. Therefore the distance from Syene to Alexandria must necessarily be a fiftieth part of the great
GREEK MATHEMATICS

Heron, Dioptra 36, ed. H. Schöne 302. 10-17

Δέον δὲ ἦστω, εἰ τύχοι, τὴν μεταξὺ Ἀλεξανδρείας καὶ Ῥώμης ὅδον ἐκμετρῆσαι τὴν ἐπ’ εὐθείας, τὴν γε ἐπὶ κύκλου περιφερείας μεγίστου τοῦ ἐν τῇ γῇ, προσομολογομένου τοῦ ὅτι περίμετρος τῆς γῆς σταδίων ἦστι μὲ καὶ ἐτὶ ἶ, ὡς ὁ μᾶλιστα τῶν ἄλλων ἀκριβέστερον πεπραγματευμένος Ἑρατο-
σθένης δείκνυσιν ἐν (τῷ) ἑπταγραμμαῖον Περὶ τῆς ἀναμετρήσεως τῆς γῆς.

1 τῷ add. H. Schöne.

---

*The attached figure will help to elucidate Cleomedes. S is Syene and A Alexandria; the centre of the earth is O. The sun’s rays at the two places are represented by the broken straight lines. If \( \alpha \) be the angle made by the sun’s rays with the pointer of the sundial at Alexandria (OA produced), the angle SOA is also equal to \( \alpha \), or one-fiftieth of four right angles. The arc SA is known to be 5000 stades and it follows that the whole circumference of the earth must be 250000 stades.*
ERATOSTHENES

circle of the earth. And this distance is 5000 stades. Therefore the whole great circle is 250000 stades. Such is the method of Eratosthenes.a

Heron, *Dioptra* 36, ed. H. Schöne 302. 10-17

Let it be required, perchance, to measure the distance between Alexandria and Rome along the arc of a great circle, b on the assumption that the perimeter of the earth is 252000 stades, as Eratosthenes, who investigated this question more accurately than others, shows in the book which he wrote *On the Measurement of the Earth.* c

b Lit. "along the circumference of the greatest circle on the earth."

c Strabo (ii. 5. 7) and Theon of Smyrna (ed. Hiller 124. 10-12) also give Eratosthenes' measurement as 252000 stades against the 250000 of Cleomedes. "The reason of the discrepancy is not known; it is possible that Eratosthenes corrected 250000 to 252000 for some reason, perhaps in order to get a figure devisible by 60 and, incidentally, a round number (700) of stades for one degree. If Pliny (*N.H.* xii. 13. 53) is right in saying that Eratosthenes made 40 stades equal to the Egyptian σχοῖνος, then, taking the σχοῖνος at 12000 Royal cubits of 0.525 metres, we get 300 such cubits, or 157.5 metres, i.e., 516.73 feet, as the length of the stade. On this basis 252000 stades works out to 24662 miles, and the diameter of the earth to about 7850 miles, only 50 miles shorter than the true polar diameter, a surprisingly close approximation, however much it owes to happy accidents in the calculation" (Heath, *H.G.M.* ii. 107).

273
XIX. APOLLONIUS OF PERGA
XIX. APOLLONIUS OF PERGA

(a) The Conic Sections

(i.) Relation to Previous Works


'Απολλώνιος ὁ γεωμέτρης, ὁ φιλε ἔταΐρε Ἀνθέμε, γέγονε μὲν ἐκ Πέργης τῆς ἐν Παμφυλίᾳ ἐν χρόνοις τοῦ Εὐεργέτου Πτολεμαίου, ως ἱστορεῖ Ἡράκλειος ὁ τὸν βίον Ἀρχιμήδους γράφων, ὃς καὶ φησι τὰ κωνικὰ θεωρήματα ἐπινοῆσαι μὲν πρῶτον τὸν Ἀρχιμήδη, τὸν δὲ Ἀπολλώνιον αὐτὰ εὐρόντα ὑπὸ Ἀρχιμήδους μὴ ἐκδοθέντα ἰδιοποιήσασθαι, οὐκ ἀληθεύων κατὰ γε τὴν ἐμῆν. ὃ τε γὰρ Ἀρχιμήδης ἐν πολλοῖς φαίνεται ὡς παλαιοτέρας τῆς στοιχείωσεως τῶν κωνικῶν μεμνημένοις, καὶ ὃ Ἀπολλώνιος οὐχ ὃς ἰδίας ἐπινοίας γράφειν οὔ γὰρ ἂν ἐφη "ἐπὶ πλέον καὶ καθόλου μᾶλλον

a Scarcely anything more is known of the life of one of the greatest geometers of all time than is stated in this brief reference. From Pappus, Coll. vii., ed. Hultsch 67 (quoted in vol. i. p. 488), it is known that he spent much time at Alexandria with Euclid’s successors. Ptolemy Euergetes reigned 246–221 B.C., and as Ptolemaeus Chennus (apud Photii Bibl., cod. exc., ed. Bekker 151 b 18) mentions an astro-
XIX. APOLLONIUS OF PERGA

(a) The Conic Sections

(i.) Relation to Previous Works

Eutocius, Commentary on Apollonius's Conics, Apoll. Perg. ed. Heiberg ii. 168. 5-170. 26

Apollonius the geometer, my dear Anthemius, flourished at Perga in Pamphylia during the time of Ptolemy Euergetes, as is related in the life of Archimedes written by Heraclius, who also says that Archimedes first conceived the theorems in conics and that Apollonius, finding they had been discovered by Archimedes but not published, appropriated them for himself, but in my opinion he errs. For in many places Archimedes appears to refer to the elements of conics as an older work, and moreover Apollonius does not claim to be giving his own discoveries; otherwise he would not have described his purpose as "to investigate these properties more fully and more

nomer named Apollonius who flourished in the time of Ptolemy Philopator (221-204 B.C.), the great geometer is probably meant. This fits in with Apollonius's dedication of Books iv.-viii. of his Conics to King Attalus I (247-197 B.C.). From the preface to Book i., quoted infra (p. 281), we gather that Apollonius visited Eudemus at Pergamum, and to Eudemus he dedicated the first two books of the second edition of his work.

More probably Heraclides, v. supra, p. 18 n. a.
ξειραγάθαι ταῦτα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα." ἀλλ' ὅπερ φησίν ὁ Γέμινος ἁληθὲς ἔστιν, ὅτι οἱ παλαιοὶ κάνων ὀριζόμενοι τὴν τοῦ ὀρθογώνιον τριγώνου περιφοράν μενοῦσης μὲν τῶν περὶ τὴν ὀρθὴν εἰκότως καὶ τοὺς κόνων πάντας ὀρθοὺς ὑπελάμβανον γίνεσθαι καὶ μίαν τομὴν ἐν ἐκάστῳ, ἐν μὲν τῷ ὀρθογωνίῳ τὴν νῦν καλουμένην παραβολὴν, ἐν δὲ τῷ ἀμβλυγωνίῳ τὴν ὑπερβολὴν, ἐν δὲ τῷ δεύνων ὑπὲρβολήν καὶ ἐστὶ παρ' αὐτοῖς εὗρεῖν ὅτι ὁμοιομένας τάς τομάς. ὦστερον οὖν τῶν ἀρχαίων ἐπὶ ἐνῶς ἐκάστου εἴδους τριγώνου θεωρησάντων τάς δύο ὀρθᾶς πρότερον ἐν τῷ ἱσοπλεύρῳ καὶ πάλιν ἐν τῷ ἱσο- σκελεῖ καὶ ὑστερον ἐν τῷ σκαλνῇ ὁ μεταγενε- στεροι καθολικῶν θεωρήμα πάντως τριγῶνου αἱ ἐντὸς τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν· ὅτι καὶ ἐπὶ τῶν κόνων τομῶν τὴν μὲν γὰρ λεγομένην ὀρθογωνίου κόνων τομὴν ἐν ὀρθογωνίῳ μόνον κόνως ἐθεώροιν τεμνομένω ἐπιπέδῳ ὀρθῶ πρὸς μίαν πλευρὰν τοῦ κόνων, τὴν δὲ τοῦ ἀμβλυγωνίου κόνων τομὴν ἐν ἀμβλυγωνίῳ γυνωμένην κόνως ἀπεδείκνυσαν, τὴν δὲ τοῦ δεύ- νων ὑπὲρ δεύνων, ὅμως ἐπὶ πάντων τῶν κόνων ἀγωνίαι τὰ ἐπίπεδα ὀρθὰ πρὸς μίαν πλευρὰν τοῦ κόνων δηλοὶ δὲ καὶ αὐτὰ τὰ ἀρχαῖα ὀνόματα τῶν γραμμῶν. ὑστερον δὲ Ἀπολλώνιος ὁ Περ- γαῖος καθάλοι τι θεωρήσειν, ὅτι ἐν παντὶ κόνω καὶ ὀρθῷ καὶ σκαλνῇ πάσαι αἱ τομαί εἰσὶ κατὰ διάφορον τοῦ ἐπιπέδου πρὸς τὸν κόνων προσβολήν· ὅτι καὶ θαυμάσαντες οἴ καὶ αὐτῶν γενόμενοι διὰ τὸ θαυμάσιον τῶν ὑπ’ αὐτῶν δεδειγμένων κωνικῶν θεωρημάτων μέγας γεωμέτρητον ἐκάλουν. ταῦτα 278
APOLLONIUS OF PERGA

generally than is done in the works of others."

But what Geminus says is correct: defining a cone as the figure formed by the revolution of a right-angled triangle about one of the sides containing the right angle, the ancients naturally took all cones to be right with one section in each—in the right-angled cone the section now called the parabola, in the obtuse-angled the hyperbola, and in the acute-angled the ellipse; and in this may be found the reason for the names they gave to the sections. Just as the ancients, investigating each species of triangle separately, proved that there were two right angles first in the equilateral triangle, then in the isosceles, and finally in the scalene, whereas the more recent geometers have proved the general theorem, that in any triangle the three internal angles are equal to two right angles, so it has been with the sections of the cone; for the ancients investigated the so-called section of a right-angled cone in a right-angled cone only, cutting it by a plane perpendicular to one side of the cone, and they demonstrated the section of an obtuse-angled cone in an obtuse-angled cone and the section of an acute-angled cone in the acute-angled cone, in the cases of all the cones drawing the planes in the same way perpendicularly to one side of the cone; hence, it is clear, the ancient names of the curves. But later Apollonius of Perga proved generally that all the sections can be obtained in any cone, whether right or scalene, according to different relations of the plane to the cone. In admiration for this, and on account of the remarkable nature of the theorems in conics proved by him, his contemporaries called him the "Great

* This comes from the preface to Book i., v. infra, p. 283.
GREEK MATHEMATICS

μὲν οὖν ὁ Γέμινος ἐν τῷ ἕκτῳ φησὶ τῆς Τῶν μαθημάτων θεωρίας.

(ii.) Scope of the Work

i. 2. 2–4. 28

'Απολλώνιος Εὐδήμω χαίρειν.

Εἰ τῷ τε σώματι εὐ ἐπανάγεις καὶ τὰ ἄλλα κατὰ γνώμην ἔστι σοι, καλῶς ἀν ἔχοι, μετρίως δὲ ἔχομεν καὶ αὐτοῖ. καθ’ ὅν δὲ καιρὸν ἡμῖν μετὰ σου ἐν Περγάμῳ, ἔθεωρον σε σπεύδοντα μετασχεῖν τῶν πεπραγμένων ἡμῖν κωνικῶν· πέπομφα οὖν σοι τὸ πρῶτον βιβλίον διορθωσάμενος, τὰ δὲ λοιπά, ὅταν εὐαρεστήσωμεν, ἐξαποστελοῦμεν· οὐκ ἀμημονεῖν γὰρ οὐμαι σε παρ’ ἐμοῦ ἀκηκοότα, διότι τὴν περὶ ταῦτα ἐφοδον ἐποιησάμην ἄξιον θείες ὑπὸ Ναυκράτους τοῦ γεωμέτρου, καθ’ ὅν καιρὸν ἐσχόλαζε

* Menaechmus, as shown in vol. i. pp. 278-283, and more particularly p. 283 n. a, solved the problem of the doubling of the cube by means of the intersection of a parabola with a hyperbola, and also by means of the intersection of two parabolas. This is the earliest mention of the conic sections in Greek literature, and therefore Menaechmus (fl. 360-350 B.C.) is generally credited with their discovery; and as Eratosthenes’ epigram (vol. i. p. 296) speaks of “cutting the cone in the triads of Menaechmus,” he is given credit for discovering the ellipse as well. He may have obtained them all by the method suggested by Geminus, but Heath (H.G.M. ii. 111-116) gives cogent reasons for thinking that he may have obtained his rectangular hyperbola by a section of a right-angled cone parallel to the axis.

A passage already quoted (vol. i. pp. 486-489) from Pappus (ed. Hultsch 672. 18–678. 24) informs us that treatises on the conic sections were written by Aristaeus and Euclid. Aristaeus’ work, in five books, was entitled Solid Loci; Euclid’s
APOLLONIUS OF PERGA

Geometer.” Geminus relates these details in the sixth book of his Theory of Mathematics.\(^a\)

(ii.) Scope of the Work

Apollonius, Conics i., Preface, Apoll. Perg. ed. Heiberg

Apollonius to Eudemus\(^b\) greeting.

If you are in good health and matters are in other respects as you wish, it is well; I am pretty well too. During the time I spent with you at Pergamum, I noticed how eager you were to make acquaintance with my work in conics; I have therefore sent to you the first book, which I have revised, and I will send the remaining books when I am satisfied with them. I suppose you have not forgotten hearing me say that I took up this study at the request of Nauclrates the geometer, at the time when he came

Conics was in four books. The work of Aristaeus was obviously more original and more specialized; that of Euclid was admittedly a compilation largely based on Aristaeus. Euclid flourished about 300 B.C. As noted in vol. i. p. 495 n. a, the focus-directrix property must have been known to Euclid, and probably to Aristaeus; curiously, it does not appear in Apollonius’s treatise.

Many properties of conics are assumed in the works of Archimedes without proof and several have been encountered in this work; they were no doubt taken from the works of Aristaeus or Euclid. As the reader will notice, Archimedes’ terminology differs in several respects from that of Apollonius, apart from the fundamental difference on which Geminus laid stress.

The history of the conic sections in antiquity is admirably treated by Zeuthen, Die Lehre von den Kegelschnitten im Altertum (1886) and Heath, Apollonius of Perga, xvii-clvi.

\(^b\) Not, of course, the pupil of Aristotle who wrote the famous History of Geometry, unhappily lost.
GREEK MATHEMATICS

παρ' ἦμῖν παραγενηθεὶς εἰς Ἀλεξάνδρειαν, καὶ διότι πραγματεύοντες αὐτὰ ἐν ὀκτὼ βιβλίοις εἰς αὐτὴς μεταδεδωκαμεν αὐτὰ εἰς τὸ σπουδαίστερον διὰ τὸ πρὸς ἐκπλω ἀυτὸν εἶναι οὐ διακαθάραντες, ἀλλὰ πάντα τὰ ὑποτίθοντα ἦμῖν θέντες ὡς ἔσχατον ἐπελευσόμενοι. οἴθεν καὶ ὅποιαν νῦν λαβόντες ἀεὶ τὸ τυγχάνον διορθώσεως ἐκδίδομεν. καὶ ἔπει συμβέβηκε καὶ ἄλλους τινὰς τῶν συμμεριστῶν ἦμῖν μετεληφθέναι τὸ πρῶτον καὶ τὸ δεύτερον βιβλίον πρὶν ἡ διορθώθηναι, μὴ θαυμάσησι, εὰν περιπτήθησαν αὐτοῖς ἐτέρως ἔχουσιν.

Απὸ δὲ τῶν ὀκτὼ βιβλίων τὰ πρῶτα τέσσαρα πέπτωκεν εἰς ἀγωγὴν στοιχειώδη, περιέχει δὲ τὸ μὲν πρῶτον τὰς γενέσεις τῶν τριῶν τομῶν καὶ τῶν ἀντικειμένων καὶ τὰ ἐν αὐταῖς ἀρχικὰ συμπτώματα ἐπὶ πλέον καὶ καθόλου μᾶλλον ἐξειργαζόμενα παρὰ τὰ ὑπὸ τῶν ἄλλων γεγραμμένα, τὸ δὲ δεύτερον τὰ περὶ τὰς διαμέτρους καὶ τοὺς ἄξονας τῶν τομῶν συμβαινόντα καὶ τὰς ἀσυμπτώτους καὶ ἄλλα γενικὴν καὶ ἀναγκαίαν χρείαν παρεχόμενα πρὸς τοὺς διορισμούς. τίνας δὲ διαμέτρους καὶ τίνας ἄξονας καλῶ, εἰδήσεις ἐκ τοῦτο τοῦ βιβλίου. τὸ δὲ τρίτον πολλὰ καὶ παράδοξα θεωρήματα χρήσιμα πρὸς τε τὰς συνθέσεις τῶν στερεῶν τόπων καὶ τοὺς διορισμούς, ὡς τὰ πλεῖστα καὶ κάλλιστα ξένα, ἀ καὶ κατανοησάντες συνείδομεν μὴ συντιθέμενον ὑπὸ Εὐκλείδου τὸν ἐπὶ τρεῖς καὶ τέσσαρας γραμμὰς τόπου, ἀλλὰ μόριον τὸ τυχὸν αὐτοῦ καὶ τοῦτο ὠκεν εὐτυχῶς· οὐ γὰρ ἦν δυνατὸν ἄνευ τῶν προσευρημένων ἦμῖν τελειωθῆναι τὴν

* A necessary observation, because Archimedes had used the terms in a different sense.

282
to Alexandria and stayed with me, and that, when I had completed the investigation in eight books, I gave them to him at once, a little too hastily, because he was on the point of sailing, and so I was not able to correct them, but put down everything as it occurred to me, intending to make a revision at the end. Accordingly, as opportunity permits, I now publish on each occasion as much of the work as I have been able to correct. As certain other persons whom I have met have happened to get hold of the first and second books before they were corrected, do not be surprised if you come across them in a different form.

Of the eight books the first four form an elementary introduction. The first includes the methods of producing the three sections and the opposite branches [of the hyperbola] and their fundamental properties, which are investigated more fully and more generally than in the works of others. The second book includes the properties of the diameters and the axes of the sections as well as the asymptotes, with other things generally and necessarily used in determining limits of possibility; and what I call diameters and axes you will learn from this book. The third book includes many remarkable theorems useful for the syntheses of solid loci and for determining limits of possibility; most of these theorems, and the most elegant, are new, and it was their discovery which made me realize that Euclid had not worked out the synthesis of the locus with respect to three and four lines, but only a chance portion of it, and that not successfully; for the synthesis could not be completed without the theorems discovered by me.

For this locus, and Pappus's comments on Apollonius's claims, v. vol. i. pp. 486-489.
GREEK MATHEMATICS

σύνθεσιν. τὸ δὲ τέταρτον, ποσαχῶς αἱ τῶν κώνων
tomai ἀλλήλαις τε καὶ τῇ τοῦ κύκλου περιφερείας
συμβάλλοντος, καὶ ἄλλα ἐκ περισσοῦ, ὥς οὐδέτερον
ὑπὸ τῶν πρὸ ἡμῶν γέγραπται, κώνου τομῆ ἢ
κύκλου περιφέρεια κατὰ πόσα σημεῖα συμβάλ-
λοντος.

Τὰ δὲ λοιπὰ ἐστὶ περιουσιαστικώτερα. ἔστι
γὰρ τὸ μὲν περὶ ἐλαχίστων καὶ μεγίστων ἐπὶ
πλέον, τὸ δὲ περὶ ἰσων καὶ ὁμοίων κώνου τομῶν,
tὸ δὲ περὶ διοριστικῶν θεωρημάτων, τὸ δὲ προ-
βλημάτων καυκών διωρισμένων. οὐ μὴν ἄλλα
καὶ πάντων ἐκδοθέντων ἐξεστὶ τοῖς περιτυγχάνουσι
κρίνειν αὐτά, ὡς ἂν αὐτῶν ἐκαστὸς αἰρήται.
eυτύχει.

(iii.) Definitions

Ibid., Def., Apoll. Perg. ed. Heiberg l. 6. 2–8. 20

'Εὰν ἀπὸ τῶν σημείων πρὸς κύκλου περιφέρειαν,
δὲ οὔκ ἔστιν ἐν τῷ αὐτῷ ἐπιπέδῳ τῷ σημείῳ,
εὐθεία ἐπιζευγθείσα ἐφ' ἐκάτερα προσεκβληθῆ,
καὶ μένοντος τοῦ σημείου ἢ εὐθεία περιενεχθεῖσα
περὶ τὴν τοῦ κύκλου περιφέρειαν εἰς τὸ αὐτὸ πάλιν
ἀποκατασταθῆ, ὅθεν ἦρξατο φέρεσθαι, τὴν γρα-
ψείσαν ὑπὸ τῆς εὐθείας ἐπιφάνειαν, ἢ σύγκειται
ἐκ δύο ἐπιφάνειῶν κατὰ κορύφην ἀλλήλαις κει-
μένων, ὅν ἐκατέρα εἰς ἀπειρὸν αὐξεῖται τῆς

---

a Only the first four books survive in Greek. Books v.–vii. have survived in Arabic, but Book viii. is wholly lost. Halley (Oxford, 1710) edited the first seven books, and his edition is still the only source for Books vi. and vii. The first four books have since been edited by Heiberg (Leipzig, 1891–1893) and Book v. (up to Prop. 7) by L. Nix (Leipzig, 1889). The
The fourth book investigates how many times the sections of cones can meet one another and the circumference of a circle; in addition it contains other things, none of which have been discussed by previous writers, namely, in how many points a section of a cone or a circumference of a circle can meet [the opposite branches of hyperbolas]. The remaining books are thrown in by way of addition: one of them discusses fully minima and maxima, another deals with equal and similar sections of cones, another with theorems about the determinations of limits, and the last with determinate conic problems. When they are all published it will be possible for anyone who reads them to form his own judgement. Farewell.²

(iii.) Definitions

Ibid., Definitions, Apoll. Perg. ed. Heiberg i. 6. 2–8. 20

If a straight line be drawn from a point to the circumference of a circle, which is not in the same plane with the point, and be produced in either direction, and if, while the point remains stationary, the straight line be made to move round the circumference of the circle until it returns to the point whence it set out, I call the surface described by the straight line a conical surface; it is composed of two surfaces lying vertically opposite to each other, of which each surviving books have been put into mathematical notation by T. L. Heath, Apollonius of Perga (Cambridge, 1896) and translated into French by Paul Ver Eecke, Les Coniques d' Apollonius de Perga (Bruges, 1923).

In ancient times Eutocius edited the first four books with a commentary which still survives and is published in Heiberg's edition. Serenus and Hypatia also wrote commentaries, and Pappus a number of lemmas.
GREEK MATHEMATICS

γραφούσης εὐθείας εἰς ἀπειρον προσεκβαλλομένης, καλῶ κωνικῆς ἐπιφάνειαν, κορυφὴν δὲ αὐτῆς τὸ μεμενηκὸς σημεῖον, ἄξονα δὲ τὴν διὰ τοῦ σημείου καὶ τοῦ κέντρου τοῦ κύκλου ἀγομένην εὐθείαν.

Κώνον δὲ τὸ περιεχόμενον σχῆμα ὑπὸ τε τοῦ κύκλου καὶ τῆς μεταξὺ τῆς τε κορυφῆς καὶ τῆς τοῦ κύκλου περιφερείας κωνικῆς ἐπιφανείας, κορυφὴν δὲ τοῦ κώνου τὸ σημεῖον, ὃ καὶ τῆς ἐπιφανείας ἐστὶ κορυφή, ἄξονα δὲ τὴν ἀπὸ τῆς κορυφῆς ἐπὶ τὸ κέντρον τοῦ κύκλου ἀγομένην εὐθείαν, βάσιν δὲ τοῦ κύκλου.

Τῶν δὲ κώνων ὀρθῶν μὲν καλῶ τοὺς πρὸς ὀρθὰς ἔχοντας ταῖς βάσεις τοὺς ἄξονας, σκαληνοὺς δὲ τοὺς μὴ πρὸς ὀρθὰς ἔχοντας ταῖς βάσεις τοὺς ἄξονας.

Πάσης καμπύλης γραμμῆς, ἦτις ἐστὶν ἐν ἑνί ἐπιπέδῳ, διάμετρον μὲν καλῶ εὐθείαν, ἦτις ἡγμένη ἀπὸ τῆς καμπύλης γραμμῆς πάσας τὰς ἀγομένας ἐν τῇ γραμμῇ εὐθείᾳ, εὐθείᾳ τινὶ παραλλήλους δίχα διαμεῖται, κορυφὴν δὲ τῆς γραμμῆς τὸ πέρας τῆς εὐθείας τὸ πρὸς τῇ γραμμῇ, τεταγμένως δὲ ἐπὶ τὴν διάμετρον κατήχθαι ἐκάστην τῶν παράλληλων.

Ὅμως δὲ καὶ δύο καμπύλων γραμμῶν ἐν ἑνί ἐπιπέδῳ κειμένων διάμετρον καλῶ πλαγίαν μὲν, ἦτις εὐθεία τέμνουσα τὰς δύο γραμμὰς πάσας τὰς ἀγομένας ἐν ἑκατέρα τῶν γραμμῶν παρὰ τυχὸ εὐθείαις δίχα τέμνει, κορυφὰς δὲ τῶν γραμμῶν θ᾽ πρὸς ταῖς γραμμαῖς πέρατα τῆς διαμέτρου, ὀρθὰν δὲ, ἦτις κειμένη μεταξὺ τῶν δύο γραμμῶν πάσας τὰς ἀγομένας παραλλήλους εὐθείας εὐθείᾳ τινὶ καὶ ἀπολαμβανομένας μεταξὺ τῶν γραμμῶν δίχα

286
extends to infinity when the straight line which describes them is produced to infinity; I call the fixed point the *vertex*, and the straight line drawn through this point and the centre of the circle I call the *axis*.

The figure bounded by the circle and the conical surface between the vertex and the circumference of the circle I term a *cone*, and by the *vertex of the cone* I mean the point which is the vertex of the surface, and by the *axis* I mean the straight line drawn from the vertex to the centre of the circle, and by the *base* I mean the circle.

Of cones, I term those *right* which have their axes at right angles to their bases, and *scalene* those which have their axes not at right angles to their bases.

In any plane curve I mean by a *diameter* a straight line drawn from the curve which bisects all straight lines drawn in the curve parallel to a given straight line, and by the *vertex of the curve* I mean the extremity of the straight line on the curve, and I describe each of the parallels as being drawn *ordinate-wise* to the diameter.

Similarly, in a pair of plane curves I mean by a *transverse diameter* a straight line which cuts the two curves and bisects all the straight lines drawn in either curve parallel to a given straight line, and by the *vertices of the curves* I mean the extremities of the diameter on the curves; and by an *erect diameter* I mean a straight line which lies between the two curves and bisects the portions cut off between the curves of all straight lines drawn parallel to a given
GREEK MATHEMATICS

tēmnei, tetagmēnous de ἐπὶ τὴν διάμετρον κατήχθαι ἐκαστὴν τῶν παραλλήλων.

Συζυγεῖσ καλῶ διαμέτρους [δῦο]¹ καμπύλης γραμμῆς καὶ δῦο καμπύλων γραμμῶν εὐθείας, ὥσ ἐκατέρα διάμετρος οὖσα τὰς τῇ ἐτέρᾳ παραλλήλους δίχα διαει. 

"Ἄξονα δὲ καλῶ καμπύλης γραμμῆς καὶ δῦο καμπύλων γραμμῶν εὐθείαν, ἦτες διάμετροι οὖσα τῆς γραμμῆς ἡ τῶν γραμμῶν πρὸς ὀρθάς τέμνει τὰς παραλλήλους.

Συζυγεῖσ καλῶ ἄξονας καμπύλης γραμμῆς καὶ δῦο καμπύλων γραμμῶν εὐθείας, αἰτίνες διάμετροι οὖσαι συζυγεῖσ πρὸς ὀρθάς τέμνουσι τὰς ἀλλήλων παραλλήλους.

(iv.) Construction of the Sections

Ibid., Props. 7-9, Apoll. Perg. ed. Heiberg i. 22. 26-36. 5

ζ'

Ἐὰν κώνος ἐπιπέδω τιμηθῇ διὰ τοῦ ἄξονος, τιμηθῇ δὲ καὶ ἐτέρῳ ἐπιπέδῳ τέμνοντι τὸ ἐπίπεδον, ἐν ὧ ἐστὶν ἡ βάσις τοῦ κώνου, κατ᾿ εὐθείαν πρὸς ὀρθᾶς οὖσαν ἦτοι τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου ἡ τῇ ἐπ᾿ εὐθείας αὐτῆ, αἱ ἀγόμεναι εὐθεῖαι ἀπὸ τῆς γεννηθείσης τομῆς ἐν τῇ τοῦ κώνου ἐπιφανείᾳ, ἦν ἐποίησε τὸ τέμνον ἐπίπεδον, παράλληλοι τῇ πρὸς ὀρθᾶς τῇ βάσει τοῦ τριγώνου εὐθεία ἐπὶ τὴν κοινὴν τομὴν πεσοῦνται τοῦ τέμ-

¹ δῦο om. Heiberg.

This proposition defines a conic section in the most general way with reference to any diameter. It is only much 288
straight line; and I describe each of the parallels as drawn *ordinate-wise to the diameter.*

By *conjugate diameters* in a curve or pair of curves I mean straight lines of which each, being a diameter, bisects parallels to the other.

By an *axis* of a curve or pair of curves I mean a straight line which, being a diameter of the curve or pair of curves, bisects the parallels at right angles.

By *conjugate axes* in a curve or pair of curves I mean straight lines which, being conjugate diameters, bisect at right angles the parallels to each other.

(iv.) *Construction of the Sections*


**Prop. 7 a**

*If a cone be cut by a plane through the axis, and if it be also cut by another plane cutting the plane containing the base of the cone in a straight line perpendicular to the base of the axial triangle,*b or to the base produced, a section will be made on the surface of the cone by the cutting plane, and straight lines drawn in it parallel to the straight line perpendicular to the base of the axial triangle will meet the common section of the cutting plane and the axial

later in the work (i. 52-58) that the principal axes are introduced as diameters at right angles to their ordinates. The proposition is an excellent example of the generality of Apollonius’s methods.

Apollonius followed rigorously the Euclidean form of proof. In consequence his general enunciations are extremely long and often can be made tolerable in an English rendering only by splitting them up; but, though Apollonius seems to have taken a malicious pleasure in their length, they are formed on a perfect logical pattern without a superfluous word.

b Lit. “the triangle through the axis.”
GREEK MATHEMATICS

νοντος ἐπιπέδου καὶ τοῦ διὰ τοῦ ἄξονος τριγώνου καὶ προσεκβαλλόμεναι ἐως τοῦ ἔτερου μέρους τῆς
tομῆς δίχα τιμηθήσονται ὑπ' αὐτῆς, καὶ ἐὰν μὲν
ὀρθὸς ἡ ὁ κόνως, ἡ ἐν τῇ βάσει εὐθεία πρὸς ὀρθᾶς
ἔσται τῇ κοινῇ τομῇ τοῦ τέμνοντος ἐπιπέδου καὶ
tοῦ διὰ τοῦ ἄξονος τριγώνου, ἐὰν δὲ σκαληνός,
οὐκ αἰεὶ πρὸς ὀρθὰς ἔσται, ἀλλ' ὅταν τὸ διὰ τοῦ
ἀξονος ἐπιπέδου πρὸς ὀρθὰς ἡ τῇ βάσει τοῦ κόνων.

'Εστω κόνως, ὁ το χροφή μὲν τὸ Α σημείον,
βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήσω ἐπιπέδω διὰ

tοῦ ἄξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τρίγωνου,
tετμήσω δὲ καὶ ἐτέρῳ ἐπιπέδῳ τέμνοντι τό
ἐπιπέδου, ἐν δὲ ἐστὶν ὁ ΒΓ κύκλος, κατ' εὐθείαν
τὴν ΔΕ ὑπὸ πρὸς ὀρθὰς οὐσαν τῇ ΒΓ ἡ τῇ ἐπ'
euθείας αὐτῆς, καὶ ποιεῖτω τομὴν ἐν τῇ ἐπιφανείᾳ
tοῦ κόνων τὴν ΔΖΕ· κοινῇ δὴ τομῇ τοῦ τέμνοντος
ἐπιπέδου καὶ τοῦ ΑΒΓ τριγώνου ἡ ΖΗ. καὶ

290
triangle and, if produced to the other part of the section, will be bisected by it; if the cone be right, the straight line in the base will be perpendicular to the common section of the cutting plane and the axial triangle; but if it be scalene, it will not in general be perpendicular, but only when the plane through the axis is perpendicular to the base of the cone.

Let there be a cone whose vertex is the point $A$ and whose base is the circle $B\Gamma$, and let it be cut by a plane through the axis, and let the section so made be the triangle $AB\Gamma$. Now let it be cut by another plane cutting the plane containing the circle $B\Gamma$ in a straight line $\Delta E$ which is either perpendicular to $B\Gamma$ or to $B\Gamma$ produced, and let the section made on the surface of the cone be $\Delta ZE$; then the common section of the cutting plane and of the triangle $AB\Gamma$

* This applies only to the first two of the figures given in the mss.
GREEK MATHEMATICS

eιλήφθω τι σημεῖον ἐπὶ τῆς ΔΞΕ τομῆς τὸ Θ, καὶ ἤχθω διὰ τοῦ Θ τῇ ΔΕ παράλληλος ἡ ΘΚ. λέγω, ὅτι ἡ ΘΚ συμβάλει τῇ ΖΗ καὶ ἐκβαλλομένη ἐως τοῦ ἐτέρου μέρους τῆς ΔΞΕ τομῆς δίχα τμηθῆσεται ὑπὸ τῆς ΖΗ εὐθείας.

'Ἐπεὶ γὰρ κώνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, τέτμηται ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιεῖ τομὴν τὸ ΑΒΓ τρίγωνον, εἰληφται δὲ τι σημεῖον ἐπὶ τῆς ἐπιφανείας, ὃ μὴ ἐστὶν ἐπὶ πλευράς του ΑΒΓ τριγώνου, τὸ Θ, καὶ ἐστὶ κάθετος ἡ ΔΗ ἐπὶ τὴν ΒΓ, ἡ ἀρα διὰ τοῦ Θ τῇ ΔΗ παράλληλος ἁγομένη, τούτεστι ἡ ΘΚ, συμβαλλεῖ τῷ ΑΒΓ τριγώνῳ καὶ προσεκβαλλομένη ἐως τοῦ ἐτέρου μέρους τῆς ἐπιφανείας δίχα τμηθῆσεται ὑπὸ τοῦ τριγώνου. ἔπει οὖν ἡ διὰ τοῦ Θ τῇ ΔΕ παράλληλος ἁγομένη συμβάλλει τῷ ΑΒΓ τριγώνῳ καὶ ἐστὶν ἐν τῷ διὰ τῆς ΔΞΕ τομῆς ἐπιπέδῳ, ἐπὶ τὴν κοινὴν ἀρα τομῆν πεσεῖται τοῦ τείμνουτος ἐπιπέδου καὶ τοῦ ΑΒΓ τριγώνου. κοινὴ δὲ τομῆ ἐστὶ τῶν ἐπιπέδων ἡ ΖΗ. ἡ ἀρα διὰ τοῦ Θ τῇ ΔΕ παράλληλος ἁγομένη πεσεῖται ἐπὶ τὴν ΖΗ. καὶ προσεκβαλλομένη ἐως τοῦ ἐτέρου μέρους τῆς ΔΞΕ τομῆς δίχα τμηθῆσεται ὑπὸ τῆς ΖΗ εὐθείας.

Ἡτοι δὴ ὁ κώνος ὀρθὸς ἐστὶν, ἡ τοῦ διὰ τοῦ ἄξονος τρίγωνον τὸ ΑΒΓ ὀρθὸν ἐστὶ πρὸς τὸν ΒΓ κύκλον, ἡ οὐδέτερον.

'Εστὶν πρὸτερον ὁ κώνος ὀρθὸς· εἰὴ ἂν οὖν καὶ τὸ ΑΒΓ τρίγωνον ὀρθὸν πρὸς τὸν ΒΓ κύκλον. ἐπεὶ οὖν ἐπὶ πέδου τὸ ΑΒΓ πρὸς ἐπὶ πέδου τὸ ΒΓ ὀρθὸν ἐστὶ, καὶ τῇ κοινῇ αὐτῶν τομῇ τῇ ΒΓ ἐν ἐνὶ τῶν ἐπὶ πέδου τῷ ΒΓ πρὸς ὀρθὰς ἢκται ἡ ΔΕ, 292
is ZH. Let any point $\Theta$ be taken on $\triangle ZE$, and through $\Theta$ let $\Theta K$ be drawn parallel to $\Delta E$. I say that $\Theta K$ intersects ZH and, if produced to the other part of the section $\triangle ZE$, it will be bisected by the straight line ZH.

For since the cone, whose vertex is the point $A$ and base the circle $BG$, is cut by a plane through the axis and the section so made is the triangle $ABG$, and there has been taken any point $\Theta$ on the surface, not being on a side of the triangle $ABG$, and $\Delta H$ is perpendicular to $BG$, therefore the straight line drawn through $\Theta$ parallel to $\Delta H$, that is $\Theta K$, will meet the triangle $ABG$ and, if produced to the other part of the surface, will be bisected by the triangle [Prop. 6]. Therefore, since the straight line drawn through $\Theta$ parallel to $\Delta E$ meets the triangle $ABG$ and is in the plane containing the section $\triangle ZE$, it will fall upon the common section of the cutting plane and the triangle $ABG$. But the common section of those planes is ZH; therefore the straight line drawn through $\Theta$ parallel to $\Delta E$ will meet ZH; and if it be produced to the other part of the section $\triangle ZE$ it will be bisected by the straight line ZH.

Now the cone is right, or the axial triangle $ABG$ is perpendicular to the circle $BG$, or neither.

First, let the cone be right; then the triangle $ABG$ will be perpendicular to the circle $BG$ [Def. 3; Eucl. xi. 18]. Then since the plane $ABG$ is perpendicular to the plane $BG$, and $\Delta E$ is drawn in one of the planes perpendicular to their common section $BG$, therefore
GREEK MATHEMATICS

ή ΔΕ ἄρα τῷ ΑΒΓ τριγῶνῳ ἐστὶ πρὸς ὀρθὰς. καὶ πρὸς πᾶσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὕσας ἐν τῷ ΑΒΓ τριγώνῳ ὀρθὴ ἐστιν. ὥστε καὶ πρὸς τὴν ΖΗ ἐστι πρὸς ὀρθὰς.

Μὴ ἔστω δὴ ὁ κώνος ὀρθὸς. εἰ μὲν οὖν τὸ διὰ τοῦ ἄξονος τρίγωνον ὀρθὸν ἐστὶ πρὸς τὸν ΒΓ κύκλον, ὡμοίως δείξομεν, ὅτι καὶ ἡ ΔΕ τῇ ΖΗ ἐστὶ πρὸς ὀρθὰς. μὴ ἔστω δὴ τὸ διὰ τοῦ ἄξονος τρίγωνον τὸ ΑΒΓ ὀρθὸν πρὸς τὸν ΒΓ κύκλον. λέγω, ὅτι οὐδὲ ἡ ΔΕ τῇ ΖΗ ἐστὶ πρὸς ὀρθὰς. εἰ γὰρ δυνατὸν, ἐστὼ· ἐστὶ δὲ καὶ τῇ ΒΓ πρὸς ὀρθὰς. ἡ ἀρα ΔΕ ἐκατέρα τῶν ΒΓ, ΖΗ ἐστὶ πρὸς ὀρθὰς. καὶ τῷ διὰ τῶν ΒΓ, ΖΗ ἐπιπέδω ἄρα πρὸς ὀρθὰς ἐστι. τὸ δὲ διὰ τῶν ΒΓ, ΗΖ ἐπιπέδου ἐστὶ τὸ ΑΒΓ. καὶ ἡ ΔΕ ἄρα τῷ ΑΒΓ τριγῶνῳ ἐστὶ πρὸς ὀρθὰς. καὶ πάντα ἄρα τὰ δι’ αὐτῆς ἐπιπέδα τῷ ΑΒΓ τριγώνῳ ἐστὶ πρὸς ὀρθὰς. ἐν δὲ τι τῶν διὰ τῆς ΔΕ ἐπιπέδων ἐστὶν ὁ ΒΓ κύκλος. ὁ ΒΓ ἄρα κύκλος πρὸς ὀρθὰς ἐστὶ τῷ ΑΒΓ τριγώνῳ. ὥστε καὶ τὸ ΑΒΓ τριγώνον ὀρθὸν ἐσται πρὸς τὸν ΒΓ κύκλον. ὥσπερ οὐχ ὑπόκειται. οὐκ ἂρα ἡ ΔΕ τῇ ΖΗ ἐστὶ πρὸς ὀρθὰς.

Πόρισμα

Ἐκ δὴ τούτου φανερὸν, ὅτι τῆς ΔΖΕ τομῆς διά-
μετρός ἐστιν ἡ ΖΗ, ἐπείπερ τὰς ἀγομένας παραλ-
λήλους εὐθείας των τῇ ΔΕ δίχα τέμνει, καὶ ὅτι
dυνατὸν ἐστὶν ὑπὸ τῆς διαμέτρου τῆς ΖΗ παραλ-
lήλους τινὰς δίχα τέμνεσθαι καὶ μὴ πρὸς ὀρθὰς.

η'

Ἐὰν κώνος ἐπιπέδῳ τμηθῇ διὰ τοῦ ἄξονος,
ΔE is perpendicular to the triangle ABΓ [Eucl. xi. Def. 4]; and therefore it is perpendicular to all the straight lines in the triangle ABΓ which meet it [Eucl. xi. Def. 3]. Therefore it is perpendicular to ZH.

Now let the cone be not right. Then, if the axial triangle is perpendicular to the circle BΓ, we may similarly show that ΔE is perpendicular to ZH. Now let the axial triangle ABΓ be not perpendicular to the circle BΓ. I say that neither is ΔE perpendicular to ZH. For if it is possible, let it be; now it is also perpendicular to BΓ; therefore ΔE is perpendicular to both BΓ, ZH. And therefore it is perpendicular to the plane through BΓ, ZH [Eucl. xi. 4]. But the plane through BΓ, HZ is ABΓ; and therefore ΔE is perpendicular to the triangle ABΓ. Therefore all the planes through it are perpendicular to the triangle ABΓ [Eucl. xi. 18]. But one of the planes through ΔE is the circle BΓ; therefore the circle BΓ is perpendicular to the triangle ABΓ. Therefore the triangle ABΓ is perpendicular to the circle BΓ; which is contrary to hypothesis. Therefore ΔE is not perpendicular to ZH.

Corollary

From this it is clear that ZH is a diameter of the section ΔZE [Def. 4], inasmuch as it bisects the straight lines drawn parallel to the given straight line ΔE, and also that parallels can be bisected by the diameter ZH without being perpendicular to it.

Prop. 8

*If a cone be cut by a plane through the axis, and it be*
GREEK MATHEMATICS

της δὲ καὶ ἑτέρω ἐπιπέδῳ τέμνοντι τὴν βάσιν
tou κώνου κατ' εὐθείαν πρὸς ὀρθὰς οὐσαν τῇ βάσει
tou διὰ τοῦ ἀξονὸς τριγώνου, ἢ δὲ διάμετρος τῆς
gυνομένης ἐν τῇ ἐπιφάνειᾳ τομῆς ἦτοι παρὰ μίαν
ἢ τῶν τοῦ τριγώνου πλευρῶν ἡ συμπληρώσας αὐτῆς
ἐκτὸς τῆς κορυφῆς τοῦ κώνου, προσεκβάλληται
dὲ ἢ τε τοῦ κώνου ἐπιφάνεια καὶ τὸ τέμνον ἐπί-
πεδον εἰς ἀπειρόν, καὶ ἡ τομὴ εἰς ἀπειρὸν αὐξηθῆ-
σεται, καὶ ἀπὸ τῆς διαμέτρου τῆς τομῆς πρὸς τὴν
κορυφὴν πάσην τῆς δοθείσης εὐθείας ἑσθεν ἀπολήμφηται
tis εὐθείας ἀγωμένη ἀπὸ τῆς τοῦ κώνου τομῆς
παρὰ τὴν ἐν τῇ βάσει τοῦ κώνου εὐθείαν.

"Εστὶν κώνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον,
βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήθη ἐπιπέδῳ
diā tou ἀξονος, kai poieitw tomyν τo ABΓ tri-
296
also cut by another plane cutting the base of the cone in a line perpendicular to the base of the axial triangle, and if the diameter of the section made on the surface be either parallel to one of the sides of the triangle or meet it beyond the vertex of the cone, and if the surface of the cone and the cutting plane be produced to infinity, the section will also increase to infinity, and a straight line can be drawn from the section of the cone parallel to the straight line in the base of the cone so as to cut off from the diameter of the section towards the vertex an intercept equal to any given straight line.

Let there be a cone whose vertex is the point A and base the circle BÎ, and let it be cut by a plane through the axis, and let the section so made be the triangle
GREEK MATHEMATICS

γωνον' τετμησθω δε και έτερω επιπεδω τεμνοντι τον ΒΓ κυκλον κατ' ευθειαν την ΔΕ προς όρθας ουσαν τη ΒΓ, και ποιειτω τομην εν τη επιφανεια την ΔΖΕ γραμμην· η δε διαμετρος της ΔΖΕ τομης η ΖΗ ήτοι παραλληλος έστω τη ΑΓ η έκβαλλο-μενη συμπιπτετω αυτη έκτος του Α σημειου. λεγω, ότι και, εαν η τε του κωνου επιφανεια και το τεμνον επιπεδον έκβαλληται εις άπειρον, και η ΔΖΕ τομη εις άπειρον αυξηθησεται.

'Εκβεβληθων γαρ η τε του κωνου επιφανεια και το τεμνον επιπεδον φανερον δη, οτι και αι ΑΒ, ΑΓ, ΖΗ συνεκβληθησονται. έπει η ΖΗ τη ΑΓ ήτοι παραλληλος έστων η έκβαλλομενη συμ- πιπτει αυτη έκτος του Α σημειου, αι ΖΗ, ΑΓ αρα έκβαλλομεναι ως έπι τα Γ, Η μερη ουδεποτε συμπεσονται. έκβεβληθωσαν ουν, και ειληφθω τι σημειον επι της ΖΗ τυχον το Θ, και δια του Θ σημειου τη μεν ΒΓ παραλληλος ήχθων η ΚΘΛ, τη δε ΔΕ παραλληλος η ΜΘΝ· το άρα δια των ΚΛ, ΜΝ επιπεδον παραλληλον έστι τω δια των ΒΓ, ΔΕ. κυκλος αρα έστι το ΚΑΜΝ επιπεδον. και έπει τα Δ, Ε, Μ, Ν σημεια εν τω τεμνοντι έστων επιπεδω, έστι δε και έν τη επιφανεια του κωνου, επι της κοινης αρα τομης έστων ηυξηται αρα η ΔΖΕ μεχρι των Μ, Ν σημειων. αυξηθεισης αρα της επιφανειας του κωνου και του τεμνοντος επιπεδου μεχρι του ΚΑΜΝ κυκλου ηυξηται και η ΔΖΕ τομη μεχρι των Μ, Ν σημειων. ομοιως δη δειξομεν, ότι και, εαν εις άπειρον έκβαλληται η τε του κωνου επιφανεια και το τεμνον επιπεδον, και η ΜΔΖΕΝ τομη εις άπειρον αυξηθησεται.

Και φανερον, ότι παση τη δοθειση ευθεια ισην 298
APOLLONIUS OF PERGA

Let it be cut by another plane cutting the circle \(BG\) in the straight line \(DE\) perpendicular to \(BG\), and let the section made on the surface be the curve \(DE\); let \(ZH\), the diameter of the section \(DE\), be either parallel to \(AG\) or let it, when produced, meet \(AG\) beyond the point \(A\). I say that if the surface of the cone and the cutting plane be produced to infinity, the section \(DE\) will also increase to infinity.

For let the surface of the cone and the cutting plane be produced; it is clear that the straight lines, \(AB\), \(AG\), \(ZH\) are simultaneously produced. Since \(ZH\) is either parallel to \(AG\) or meets it, when produced, beyond the point \(A\), therefore \(ZH\), \(AG\) when produced in the directions \(H, G\), will never meet. Let them be produced accordingly, and let there be taken any point \(\Theta\) at random upon \(ZH\), and through the point \(\Theta\) let \(K\Theta\Delta\) be drawn parallel to \(BG\), and let \(M\Theta N\) be drawn parallel to \(DE\); the plane through \(K\Delta, MN\) is therefore parallel to the plane through \(BG, DE\) [Eucl. xi. 15]. Therefore the plane \(KAMN\) is a circle [Prop. 4]. And since the points \(\Delta, E, M, N\) are in the cutting plane, and are also on the surface of the cone, they are therefore upon the common section; therefore \(DE\) has increased to \(M, N\). Therefore, when the surface of the cone and the cutting plane increase up to the circle \(KAMN\), the section \(DE\) increases up to the points \(M, N\). Similarly we may prove that, if the surface of the cone and the cutting plane be produced to infinity, the section \(MDE\) will increase to infinity.

And it is clear that there can be cut off from the
ἀπολήμεται τις ἀπὸ τῆς ΖΘ εὐθείας πρὸς τῷ Ζ σημείῳ. Εὰν γὰρ τῇ δοθείσῃ ἴσην θώμεν τὴν ΖΞ καὶ διὰ τοῦ Ξ τῇ ΔΕ παράλληλον ἀγάγωμεν, συμπεσεῖται τῇ τομῇ, ὡσπερ καὶ ἡ διὰ τοῦ Θ ἀπεδείξθη συμπέπτουσα τῇ τομῇ κατὰ τὰ Μ, Ν σημεῖα. Οὕτω γαρ εἶναι τις εὐθεία συμπέπτουσα τῇ τομῇ παράλληλος οὐσα τῇ ΔΕ ἀπολαμβάνουσα ἀπὸ τῆς ΖΗ εὐθείαν ἴσην τῇ δοθείσῃ πρὸς τῷ Ζ σημείῳ.

θ'

'Εὰν κύκλος ἐπιπέδω τιμηθῇ συμπέπτοντι μὲν ἑκάτερα πλευρά τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παρὰ τὴν βάσιν ἴσμεν ἐν τῷ ὑπεναντίῳ, ἡ τομὴ οὐκ ἔσται κύκλος.

'Εστώ κύκλος, σοῦ κορυφῆ μὲν τῷ Α σημείῳ,
APOLLONIUS OF PERGA

straight line $Z\Theta$ in the direction of the point $Z$ an intercept equal to any given straight line. For if we place $Z\Xi$ equal to the given straight line and through $\Xi$ draw a parallel to $\Delta E$, it will meet the section, just as the parallel through $\Theta$ was shown to meet the section at the points $M, N$; therefore a straight line parallel to $\Delta E$ has been drawn to meet the section so as to cut off from $ZH$ in the direction of the point $Z$ an intercept equal to the given straight line.

Prop. 9

*If a cone be cut by a plane meeting either side of the axial triangle, but neither parallel to the base nor subcontrary,* a the section will not be a circle.

Let there be a cone whose vertex is the point $A$ and base the circle $B\Gamma$, and let it be cut by a plane neither parallel to the base nor subcontrary, and let

• In the figure of this theorem, the section of the cone by the plane $\Delta E$ would be a subcontrary section ($\upsilon\pi\epsilon\alpha\nu\alpha\rho\nu\iota\alpha\tau\omicron\mu\nu\eta$) if the triangle $A\Delta E$ were similar to the triangle $AB\Gamma$, but in a contrary sense, i.e., if angle $A\Delta E=$angle $A\Gamma B$. Apollonius proves in i. 5 that subcontrary sections of the cone are circles; it was proved in i. 4 that all sections parallel to the base are circles.
GREEK MATHEMATICS

evantaís, kai poieítw toûmê̈n eν τη̈ épifaneía tîn ΔKE γramámmн. légm, óti h ΔKE γramámmн óuk èstai kúkllos.

Ei γar dúnavtn, èstw, kai sympiptéttw tò têmwn éptípedon tî básei, kai èstw twn éptípedwn kOUH toûmê̈n ò ZH, tò de kéntron tuô BΓ kúklou èstw tò Ò, kai áp' autô ò káðetos òxhò ìrì tîn ZH ò ΘH, kai èkbeblésthò dià tîs HΩ kai tuô òxos èptípedon kai poieítw toûmâs èn tî kwniké épifaneía tâs BΑ, ΑΓ èvtheías. èpèi òndu tâ Δ, E, H sîmêía èn te tv ìa tîs ΔKE èptípedw èstw, èstì dè kai èn tv ìa tîs A, B, Γ, tà àra D, E, H sîmêía èpì tîs kouhîs toûmê̈s twn èptípedwn èstwv: èvtheía àra èstw ò HED. eîlîfhw òh tì èpì tîs ΔKE graamámmh sîmêión tò K, kai dià tòu K tî ZH parállelos òxhò h KΛ èstai òh ìsì h KΜ tî tî MΛ. òh àra ΔE diàmetró̂s èstì tòu ΔKΛΕ kúklou. òxhò òh ìa dià tòu M tî BΓ parállelos ò NΜÈ. èstì dè kai èh KΛ tî ZH parállelos: òstè tò dià tòn N€, KΜ èptípedon parállelon èstì tv ìa dià tòn BΓ, ZH, toutèstw èstì básèi, kai èstì h toûmê̈ kûk- los. èstw ò NKE. kai èpèi òh ZH tî BΗ pròs òrthás èstì, kai èh KΜ ò N€ pròs òrthás èstwv: òstè tò ùpò tòv NΜÈ ìsoun èstì tòv ìpò tîs KΜ. èstì dè tò ùpò tòv ΔME ìsou òpò èstì tòv òpò tîs KΜ. kúkllos gâr ùpòkeiñetai ò ΔKEΛ graamámmh, kai diàmetró̂s autô ò h DE. tò àra ùpò tòv NΜÈ ìsou èstì tòv ùpò ΔME. èstwh àra òsw èh ΜΝ pròs MΔ, òntwsw èh EM pròs MÈ. èmouon àra èstì tò ΔMN trígwnon tòv ΞME trígwnw, kai èh ùpò ΔΝΜ γwvía ìsì èstì tî ùpò ΜΕÈ. állâ 302
the section so made on the surface be the curve $\Delta KE$. I say that the curve $\Delta KE$ will not be a circle.

For, if possible, let it be, and let the cutting plane meet the base, and let the common section of the planes be $ZH$, and let the centre of the circle $B\Gamma$ be $\Theta$, and from it let $\Theta H$ be drawn perpendicular to $ZH$, and let the plane through $H\Theta$ and the axis be produced, and let the sections made on the conical surface be the straight lines $BA$, $AT$. Then since the points $\Delta$, $E$, $H$ are in the plane through $\Delta KE$, and are also in the plane through $A$, $B$, $\Gamma$, therefore the points $\Delta$, $E$, $H$ are on the common section of the planes; therefore $HE\Delta$ is a straight line [Eucl. xi. 3]. Now let there be taken any point $K$ on the curve $\Delta KE$, and through $K$ let $KA$ be drawn parallel to $ZH$; then $KM$ will be equal to $MA$ [Prop. 7]. Therefore $AE$ is a diameter of the circle $\Delta KE\Delta$ [Prop. 7, coroll.]. Now let $NM\Xi$ be drawn through $M$ parallel to $B\Gamma$; but $KA$ is parallel to $ZH$; therefore the plane through $N\Xi$, $KM$ is parallel to the plane through $B\Gamma$, $ZH$ [Eucl. xi. 15], that is to the base, and the section will be a circle [Prop. 4]. Let it be $NK\Xi$. And since $ZH$ is perpendicular to $BH$, $KM$ is also perpendicular to $N\Xi$ [Eucl. xi. 10]; therefore $NM \cdot M\Xi = KM^2$. But $\Delta M \cdot ME = KM^2$; for the curve $\Delta KE\Delta$ is by hypothesis a circle, and $AE$ is a diameter in it. Therefore $NM \cdot M\Xi = \Delta M \cdot ME$. Therefore $MN : MA = EM : M\Xi$. Therefore the triangle $\Delta MN$ is similar to the triangle $\Xi ME$, and the angle $\Delta NM$ is equal to the angle $M\Xi$. 
GREEK MATHEMATICS

ή ὑπὸ ΔΝΜ γωνία τῇ ὑπὸ ΑΒΓ ἐστὶν ἵση παράλληλος γὰρ ἡ ΝΕ τῇ ΒΓ· καὶ ἡ ὑπὸ ΑΒΓ ἀρα ἵση ἐστὶ τῇ ὑπὸ ΜΕΞ. ὑπεναντία ἀρα ἐστὶν ἡ τομή· ὀπερ οὐχ ὑπόκειται. οὐκ ἀρα κύκλος ἐστὶν ἡ ΔΚΕ γραμμή.

(v.) Fundamental Properties

Ibid., Props. 11-14, Apoll. Perg. ed. Heiberg i. 36. 26–58. 7

η'

Ἐὰν κώνος ἐπιπέδω τμηθῇ διὰ τοῦ ἄξονος, τμηθῇ δὲ καὶ ἔτερῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ᾽ εὐθείαν πρὸς ὀρθὰς οὖσαν τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου, ἐτι δὲ ἡ διάμετρος τῆς τομῆς παράλληλος ἢ μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου, ἢτις ἂν ἀπὸ τῆς τομῆς τοῦ κώνου παράλληλος ἀχθῇ τῇ κοινῇ τομῇ τοῦ τέμνοντος ἐπιπέδου καὶ τῆς βάσεως τοῦ κώνου μέχρι τῆς διαμέτρου τῆς τομῆς, δυνάται τὸ περιεχόμενον ὑπὸ τῇ ἀπολαμβανομένης ὑπ᾽ αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς καὶ ἅλλης τινὸς εὐθείας, ἢ λόγον ἔχει πρὸς τὴν μεταξὺ τῆς τοῦ κώνου γωνίας καὶ τῆς κορυφῆς τῆς τομῆς, ὅν τὸ τετράγωνον τὸ ἀπὸ τῆς βάσεως τοῦ διὰ τοῦ ἄξονος τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν· καλείσθω δὲ ἡ τοιαύτη τομὴ παραβολή.

Ἑστὶν κώνος, οὗ τὸ Α σημεῖον κορυφῆς, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμῆσθω ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιεῖτο τομὴν τὸ ΑΒΓ τρίγωνον, τετμήσθω δὲ καὶ ἔτερῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ᾽ εὐθείαν τὴν ΔΕ πρὸς ὀρθὰς 304
But the angle $\Delta NM$ is equal to the angle $\Delta B\Gamma$; for $N\Xi$ is parallel to $B\Gamma$; and therefore the angle $\Delta B\Gamma$ is equal to the angle $M\Xi\Xi$. Therefore the section is subcontrary [Prop. 5]; which is contrary to hypothesis. Therefore the curve $\Delta K\Xi$ is not a circle.

(v.) Fundamental Properties


**Prop. 11**

*Let a cone be cut by a plane through the axis, and let it be also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and further let the diameter of the section be parallel to one side of the axial triangle; then if any straight line be drawn from the section of the cone parallel to the common section of the cutting plane and the base of the cone as far as the diameter of the section, its square will be equal to the rectangle bounded by the intercept made by it on the diameter in the direction of the vertex of the section and a certain other straight line; this straight line will bear the same ratio to the intercept between the angle of the cone and the vertex of the segment as the square on the base of the axial triangle bears to the rectangle bounded by the remaining two sides of the triangle; and let such a section be called a parabola.*

For let there be a cone whose vertex is the point $A$ and whose base is the circle $B\Gamma$, and let it be cut by a plane through the axis, and let the section so made be the triangle $A\Gamma B$, and let it be cut by another plane cutting the base of the cone in the straight line
GREEK MATHEMATICS

οὖσαν τῇ ΒΓ, καὶ ποιεῖτω τομὴν ἐν τῇ ἐπιφάνειᾳ τοῦ κώνου τῆν ΔΖΕ, ἥ δὲ διάμετρος τῆς τομῆς ἡ ΖΗ παράλληλος ἐστω μᾶζ πλευρὰ τοῦ διὰ τοῦ ἄξονος τριγώνου τῇ ΔΓ, καὶ ἀπὸ τοῦ Ζ σημείου τῇ ΖΗ εὐθείᾳ πρὸς ὑρθὰς ἡχθῷ ἡ ΖΘ, καὶ πε-
ποιήσθω, ὡς τὸ ἀπὸ ΒΓ πρὸς τὸ ὑπὸ ΒΑΓ, οὕτως ἡ ΖΘ πρὸς ΖΑ, καὶ εἰλήφθω τῷ σημείῳ ἑπὶ τῆς τομῆς τυχόν τὸ Κ, καὶ διὰ τοῦ Κ τῇ ΔΕ παρά-
ληλος ἡ ΚΑ. λέγω, ὅτι τὸ ἀπὸ τῆς ΚΑ ἴσον ἑστὶ τῷ ὑπὸ τῶν ΘΖΑ.

Ἡχθῷ γὰρ διὰ τοῦ Λ τῇ ΒΓ παράλληλος ἡ ΜΝ. ἔστι δὲ καὶ ἡ ΚΑ τῇ ΔΕ παράλληλος. τὸ ἀρὰ διὰ τῶν ΚΑ, ΜΝ ἐπίπεδον παράλληλον ἐστὶ τῷ διὰ τῶν ΒΓ, ΔΕ ἐπιπέδῳ, τούτεστι τῇ βάσει τοῦ κώνου. τὸ ἀρὰ διὰ τῶν ΚΑ, ΜΝ ἐπίπεδον κύκλος ἐστὶν, οὔ διάμετρος ἡ ΜΝ. καὶ ἐστὶ
κάθετος ἑπὶ τῆς ΜΝ ἡ ΚΑ, ἐπεὶ καὶ ἡ ΔΕ ἑπὶ τῆς ΒΓ. τὸ ἀρὰ ὑπὸ τῶν ΜΑΝ ἴσον ἑστὶ τῷ ἀπὸ τῆς ΚΑ. καὶ ἐπεῖ ἐστὶν, ὡς τὸ ἀπὸ τῆς ΒΓ πρὸς τὸ ὑπὸ τῶν ΒΑΓ, οὗτως ἡ ΘΖ πρὸς ΖΑ, τὸ δὲ

306
ΔΕ perpendicular to ΒΓ, and let the section so made on the surface of the cone be ΔΖΕ, and let ΖΗ, the diameter of the section, be parallel to ΑΓ, one side of the axial triangle, and from the point Ζ let ΖΘ be drawn perpendicular to ΖΗ, and let ΒΓ² : ΒΑ . ΑΓ = ΖΘ : ΖΑ, and let any point Κ be taken at random on the section, and through Κ let ΚΑ be drawn parallel to ΔΕ. I say that ΚΑ² = ΘΖ . ΖΑ.

For let MN be drawn through Α parallel to ΒΓ; but ΚΑ is parallel to ΔΕ; therefore the plane through

ΚΑ, MN is parallel to the plane through ΒΓ, ΔΕ [Eucl. xi. 15], that is to the base of the cone. Therefore the plane through ΚΑ, MN is a circle, whose diameter is MN [Prop. 4]. And ΚΑ is perpendicular to MN, since ΔΕ is perpendicular to ΒΓ [Eucl. xi. 10]; therefore ΜΑ . ΑΝ = ΚΑ².

And since ΒΓ² : ΒΑ . ΑΓ = ΘΖ : ΖΑ,
ἀπὸ τῆς ΒΓ πρὸς τὸ ύπὸ τῶν ΒΑΓ λόγον ἔχει τὸν συγκείμενον ἐκ τε τοῦ, ὅν ἔχει ἡ ΒΓ πρὸς ΓΑ καὶ ἡ ΒΓ πρὸς ΒΑ, ὥστε ἡ ΖΑ λόγος σύγκειται ἐκ τοῦ τῆς ΒΓ πρὸς ΓΑ καὶ τοῦ τῆς ΓΒ πρὸς ΒΑ. ἄλλῳ ὡς μὲν ἡ ΒΓ πρὸς ΓΑ, οὕτως ἡ ΜΝ πρὸς ΝΑ, τουτέστων ἡ ΜΑ πρὸς ΛΖ, ὡς δὲ ἡ ΒΓ πρὸς ΒΑ, οὕτως ἡ ΜΝ πρὸς ΜΑ, τουτέστων ἡ ΛΜ πρὸς ΜΖ, καὶ λοιπῷ ἡ ΝΛ πρὸς ΖΑ. ὁ ἀρα τῆς ΘΖ πρὸς ΖΑ λόγος σύγκειται ἐκ τοῦ τῆς ΜΑ πρὸς ΛΖ καὶ τοῦ τῆς ΝΛ πρὸς ΖΑ. ὁ δὲ συγκείμενος λόγος ἐκ τοῦ τῆς ΜΑ πρὸς ΛΖ καὶ τοῦ τῆς ΛΝ πρὸς ΖΑ ὁ τοῦ ύπὸ ΜΛΝ ἐστὶ πρὸς τὸ ύπὸ ΛΑΖ. ὡς ἀρα ἡ ΘΖ πρὸς ΖΑ, οὕτως τὸ ύπὸ ΜΛΝ πρὸς τὸ ύπὸ ΛΑΖ. ὡς δὲ ἡ ΘΖ πρὸς ΖΑ, τῆς ΖΛ κοινοῦ ύψους λαμβανομένης οὕτως τὸ ύπὸ ΘΖΛ πρὸς τὸ ύπὸ ΛΖΑ· ὡς ἀρα τὸ ύπὸ ΜΛΝ πρὸς τὸ ύπὸ ΛΑΖ, οὕτως τὸ ύπὸ ΘΖΛ πρὸς τὸ ύπὸ ΛΑΖ. ἵσον ἀρα ἐστὶ τὸ ύπὸ ΜΛΝ τῷ ύπὸ ΘΖΛ. τὸ δὲ ύπὸ ΜΛΝ ἵσον ἐστὶ τῷ ἀπὸ τῆς ΚΛ· καὶ τὸ ἀπὸ τῆς ΚΛ ἁρα ἵσον ἐστὶ τῷ ύπὸ τῶν ΘΖΛ.

Καλείσθω δὲ ὡς τοιαύτη τομὴ παραβολῆ, ἡ δὲ ΘΖ παρ’ ἣν ὑπάρχειν αἱ καταγόμεναι τεταγμένως ἐπὶ τῆν ΖΗ διάμετρον, καλείσθω δὲ καὶ ὀρθα.

ib'

'Εὰν κώνος ἐπιπέδῳ τιμηθὴ διὰ τοῦ ἄξονος, τιμηθὴ δὲ καὶ ἐτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν

---

* A parabola (parabolē) because the square on the ordinate ΚΛ is applied (parabolēit) to the parameter ΘΖ in the form 308
APOLLONIUS OF PERGA

while $\beta_2 : \beta \cdot \alpha \gamma = (\beta \gamma : \gamma \alpha)(\beta \gamma : \beta \alpha)$,
therefore $\theta \zeta : \zeta \alpha = (\beta \gamma : \gamma \alpha)(\gamma \beta : \beta \alpha)$.
But $\beta \gamma : \gamma \alpha = \mu \nu : \nu \alpha$
    $= \mu \alpha : \alpha \zeta$, [Eucl. vi. 4
and $\beta \gamma : \beta \alpha = \mu \nu : \nu \alpha$
    $= \mu \alpha : \zeta \nu$          [ibid.
    $= \mu \alpha : \zeta \nu$. [Eucl. vi. 2
Therefore $\theta \zeta : \zeta \alpha = (\mu \alpha : \alpha \zeta)(\nu \alpha : \zeta \alpha)$.
But $(\mu \alpha : \alpha \zeta)(\alpha \mu : \zeta \alpha) = \mu \alpha \cdot \alpha \mu : \alpha \zeta \cdot \zeta \alpha$.
Therefore $\theta \zeta : \zeta \alpha = \mu \alpha \cdot \alpha \mu : \alpha \zeta \cdot \zeta \alpha$.
But $\theta \zeta : \zeta \alpha = \theta \zeta \cdot \zeta \alpha : \alpha \zeta \cdot \zeta \alpha$,
by taking a common height $\zeta \alpha$;
therefore $\mu \alpha \cdot \alpha \mu : \alpha \zeta \cdot \zeta \alpha = \theta \zeta \cdot \zeta \alpha : \alpha \zeta \cdot \zeta \alpha$.
Therefore $\mu \alpha \cdot \alpha \mu = \theta \zeta \cdot \zeta \alpha$. [Eucl. v. 9
But $\mu \alpha \cdot \alpha \mu = \kappa \lambda^2$;
and therefore $\kappa \lambda^2 = \theta \zeta \cdot \zeta \alpha$.

Let such a section be called a parabola, and let $\theta \zeta$
be called the parameter of the ordinates to the diameter $\zeta \mu$, and let it also be called the erect side (latus rectum).\(^a\)

Prop. 12

Let a cone be cut by a plane through the axis, and let it be cut by another plane cutting the base of the cone in of the rectangle $\theta \zeta \cdot \zeta \alpha$, and is exactly equal to this rectangle. It was Apollonius’s most distinctive achievement to have based his treatment of the conic sections on the Pythagorean theory of the application of areas (παραβόλη τῶν χωρίων), for which v. vol. i. pp. 186-215. The explanation of the term latus rectum will become more obvious in the cases of the hyperbola and the ellipse; v. infra, p. 317 n. a.
GREEK MATHEMATICS

tού κώνου κατ' ευθείαν πρὸς ὅρθᾶς οὖσαν τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου, καὶ ἡ διάμετρος τῆς τομῆς ἐκβαλλομένη συμπίπτῇ μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου ἐκτὸς τῆς τοῦ κώνου κορυφῆς, ἓτει ἄν ἀπὸ τῆς τομῆς ἀχθῆ παράλληλος τῇ κοινῇ τομῇ τοῦ τέμνοντος ἐπιπέδου καὶ τῆς βάσεως τοῦ κώνου, ἐως τῆς διαμέτρου τῆς τομῆς δινησταὶ τι χωρίον παρακεῖμενον παρά τινα εὐθείαν, πρὸς ἣν λόγον ἔχει ἢ ἐπ' εὐθείας μὲν οὖσα τῇ διαμέτρῳ τῆς τομῆς, ὑποτείνουσα δὲ τὴν ἐκτὸς τοῦ τριγώνου γωνίαν, δὲν τὸ τετράγωνον τὸ ἀπὸ τῆς ἡγμένης ἀπὸ τῆς κορυφῆς τοῦ κώνου παρὰ τὴν διαμέτρου τῆς τομῆς ἐως τῆς βάσεως τοῦ τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν τῆς βάσεως τριγμάτων, ὅν ποιεῖ ἢ ἀχθείσα, πλάτος ἔχον τὴν ἀπολαμβανομένην ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῇ τῆς τομῆς, ὑπερβάλλον εἰδεὶ ὁμοίως τε καὶ ὁμοίως κειμένῳ τῷ περιεχομένῳ ὑπὸ τε τῆς ὑποτείνουσης τῆς ἐκτὸς γωνίαν τοῦ τριγώνου καὶ τῆς παρ' ἦν δύνανται αἱ καταγόμεναι· καλείσθω δὲ ἡ τουαύτη τομὴ ὑπερβολῆ.

'Εστὼ κώνος, οὗ κορυφή μὲν τὸ Α σημεῖον, βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμήθησθω ἐπιπέδῳ διὰ τοῦ ἄξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τριγώνον, τετμήσθω δὲ καὶ ἐτέρῳ ἐπιπέδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ' εὐθείαν τὴν ΔΕ πρὸς ὅρθᾶς οὖσαν τῇ ΒΓ βάσει τοῦ ΑΒΓ τριγώνου, καὶ ποιεῖτω τομὴν ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τὴν ΔΖΕ γραμμήν, ἢ δὲ διάμετρος τῆς τομῆς ἢ ΖΗ ἐκβαλλομένη συμπίπτετω μιᾷ πλευρᾷ τοῦ ΑΒΓ τριγώνου τῇ ΑΓ ἐκτὸς τῆς τοῦ κώνου κορυφῆς κατὰ τὸ Θ, καὶ διὰ τοῦ Α τῇ διαμέτρῳ τῆς τομῆς

310
a straight line perpendicular to the base of the axial triangle, and let the diameter of the section, when produced, meet one side of the axial triangle beyond the vertex of the cone; then if any straight line be drawn from the section of the cone parallel to the common section of the cutting plane and the base of the cone as far as the diameter of the section, its square will be equal to the area applied to a certain straight line; this line is such that the straight line subtending the external angle of the triangle, lying in the same straight line with the diameter of the section, will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle bounded by the segments of the base made by the line so drawn; the breadth of the applied figure will be the intercept made by the ordinate on the diameter in the direction of the vertex of the section; and the applied figure will exceed by a figure similar and similarly situated to the rectangle bounded by the straight line subtending the external angle of the triangle and the parameter of the ordinates; and let such a section be called a hyperbola.

Let there be a cone whose vertex is the point A and whose base is the circle $B\Gamma$, and let it be cut by a plane through the axis, and let the section so made be the triangle $AB\Gamma$, and let it be cut by another plane cutting the base of the cone in the straight line $\Delta E$ perpendicular to $B\Gamma$, the base of the triangle $AB\Gamma$, and let the section so made on the surface of the cone be the curve $\Delta ZE$, and let $ZH$, the diameter of the section, when produced, meet $A\Gamma$, one side of the triangle $AB\Gamma$, beyond the vertex of the cone at $\Theta$, and through $A$ let $AK$ be drawn parallel to $ZH$, the
GREEK MATHEMATICS

τὴ ΖΗ παράλληλος ἡχθω ἡ ΑΚ, καὶ τεμνέτω τὴν ΒΓ, καὶ ἀπὸ τοῦ Ζ τὴ ΖΗ πρὸς ὅρθασ ἡχθω ἡ

ΖΛ, καὶ πεπουήσθω, ὡς τὸ ἀπὸ ΚΑ πρὸς τὸ ὕπο ΒΚΓ, οὕτως ἡ ΖΘ πρὸς ΖΛ, καὶ εἴληφθω τι σημεῖον ἐπὶ τῆς τομῆς τυχόν τὸ Μ, καὶ διὰ τοῦ Μ τῇ ΔΕ παράλληλος ἡχθω ἡ ΜΝ, διὰ δὲ τοῦ Ν τῇ ΖΛ παράλληλος ἡ ΝΟΞ, καὶ ἐπιζευγθείσα ἡ ΘΛ ἐκβεβλήσθω ἐπὶ τὸ Ε, καὶ διὰ τῶν Λ, Ε τῇ ΖΝ παράλληλοι ἡχθωσαν αἱ ΛΟ, ΕΠ. λέγω, ὅτι ἡ ΜΝ δύναται τὸ ΖΕ, δὲ παράκειται παρὰ τῆν ΖΛ, πλάτος ἔχον τὴν ΖΝ, ὑπερβάλλον εἰδεὶ τῷ ΔΕ ὁμοῖο ὃντι τῷ ὕπο τῶν ΘΖΛ.

"Ἡχθω γὰρ διὰ τοῦ Ν τῇ ΒΓ παράλληλος ἡ ΡΝΣ· ἔστι δὲ καὶ ἡ ΝΜ τῇ ΔΕ παράλληλος· τὸ 312
diameter of the section, and let it cut $B \Gamma$, and from $Z$ let $Z \Lambda$ be drawn perpendicular to $ZH$, and let $KA^2 : BK \cdot K \Gamma = Z \Theta : Z \Lambda$, and let there be taken at random any point $M$ on the section, and through $M$ let $MN$ be drawn parallel to $\Delta E$, and through $N$ let $NO \Xi$ be drawn parallel to $Z \Lambda$, and let $\Theta \Lambda$ be joined and produced to $\Xi$, and through $\Lambda, \Xi$, let $\Lambda \Omega, \Xi \Pi$ be drawn parallel to $ZN$. I say that the square on $MN$ is equal to $Z \Xi$, which is applied to the straight line $Z \Lambda$, having $ZN$ for its breadth, and exceeding by the figure $\Lambda \Xi$ which is similar to the rectangle contained by $\Theta Z, Z \Lambda$.

For let $PN \Xi$ be drawn through $N$ parallel to $B \Gamma$; but $NM$ is parallel to $\Delta E$; therefore the plane through
ΓΕΝΙΚΗ ΜΑΘΗΜΑΤΙΚΗ

άρα διὰ τῶν ΜΝ, ΡΣ ἐπίπεδον παράλληλον ἐστὶν
tῷ διὰ τῶν ΒΓ, ΔΕ, τούτεστι τῇ βάσει τοῦ κώνου.
ἐάν ἀρα ἐκβληθῇ τὸ διὰ τῶν ΜΝ, ΡΣ ἐπίπεδον,
ἡ τομὴ κύκλος ἐσται, οὐ διάμετρος ἢ ΡΝΣ. καὶ
ἔστιν ἐπὶ αὐτὴν κάθετος ἢ ΜΝ· τὸ ἀρα ὑπὸ τῶν
ΡΝΣ ἱσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ. καὶ ἐπεὶ ἔστων,
ὡς τὸ ἀπὸ ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ, οὔτως ἢ ΖΘ
πρὸς ΖΛ, ὦ δὲ τοῦ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ
λόγος σύγκειται ἐκ τε τοῦ, διὸ ἔχει ἢ ΑΚ πρὸς ΚΓ
καὶ ἢ ΑΚ πρὸς ΚΒ, καὶ δὲ τῆς ΖΘ ἀρα πρὸς τὴν
ΖΛ λόγος σύγκειται ἐκ τοῦ, διὸ ἔχει ἢ ΑΚ πρὸς
ΚΓ καὶ ἢ ΑΚ πρὸς ΚΒ. ἀλλ' ὡς μὲν ἢ ΑΚ
πρὸς ΚΓ, οὔτως ἢ ΘΗ πρὸς ΗΓ, τούτεστιν ἢ ΘΝ
πρὸς ΝΣ, ὡς δὲ ἢ ΑΚ πρὸς ΚΒ, οὔτως ἢ ΖΗ πρὸς
ΗΒ, τούτεστιν ἢ ΖΝ πρὸς ΝΠ. ὦ ἀρα τῆς θΣ
πρὸς ΖΛ λόγος σύγκειται ἐκ τε τοῦ τῆς ΘΝ πρὸς
ΝΣ καὶ τοῦ τῆς ΖΝ πρὸς ΝΠ. ὦ δὲ συγκειμένος
λόγος ἐκ τοῦ τῆς ΘΝ πρὸς ΝΣ καὶ τοῦ τῆς ΖΝ
πρὸς ΝΠ ὦ τοῦ ὑπὸ τῶν ΘΝΖ ἐστὶ πρὸς τὸ ὑπὸ
tῶν ΣΝΠ· καὶ ὡς ἄρα τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ
ὑπὸ τῶν ΣΝΠ, οὔτως ἢ ΘΖ πρὸς ΖΛ, τούτεστιν
ἡ ΘΝ πρὸς ΝΕ. ἀλλ' ὡς ἢ ΘΝ πρὸς ΝΕ, τῆς
ΖΝ κοινοῦ ύψους λαμβανομένης οὔτως τὸ ὑπὸ
tῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΖΝΕ. καὶ ὡς ἄρα
τὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΣΝΠ, οὔτως
tὸ ὑπὸ τῶν ΘΝΖ πρὸς τὸ ὑπὸ τῶν ΣΝΠ. τὸ ἄρα
ὑπὸ ΣΝΠ ἱσον ἐστὶ τῷ ὑπὸ ΣΝΠ. τὸ δὲ
ἀπὸ ΜΝ ἱσον ἐδείχθη τῷ ὑπὸ ΣΝΠ· καὶ τὸ ἀπὸ
tῆς ΜΝ ἄρα ἱσον ἐστὶ τῷ ὑπὸ τῶν ΕΝΖ. τὸ ἄρα
ὑπὸ ΣΝΠ ἱσον ἐστὶ τῷ ὑπὸ ΣΝΠ. τὸ δὲ
ἀπὸ ΜΝ ἱσον ἐδείχθη τῷ ὑπὸ ΣΝΠ· καὶ τὸ ἀπὸ
tῆς ΜΝ ἄρα ἱσον ἐστὶ τῷ ὑπὸ τῶν ΕΝΖ. τὸ δὲ
ὑπὸ ΣΝΠ ἱσον ἐστὶ τῷ ΕΖ παράλληλόγραμμον.
APOLLONIUS OF PERGA

$MN$, $P\Sigma$ is parallel to the plane through $B\Gamma$, $\Delta E$ [Eucl. xi. 15], that is to the base of the cone. If, then, the plane through $MN$, $P\Sigma$ be produced, the section will be a circle with diameter $PN\Sigma$ [Prop. 4]. And $MN$ is perpendicular to it; therefore

$$PN \cdot N\Sigma = MN^2.$$

And since $AK^2 : BK \cdot K\Gamma = Z\Theta : Z\Lambda,$

while $AK^2 : BK \cdot K\Gamma = (AK : K\Gamma)(AK : KB),$

therefore $Z\Theta : Z\Lambda = (AK : K\Gamma)(AK : KB).$

But $AK : K\Gamma = \Theta H : H\Gamma,$

$i.e.,$ $\Theta N : N\Sigma,$ [Eucl. vi. 4

and $AK : KB = ZH : HB,$

$i.e.,$ $ZN : NP.$ [ibid.

Therefore $\Theta Z : Z\Lambda = (\Theta N : N\Sigma)(ZN : NP).$

But $(\Theta N : N\Sigma)(ZN : NP) = \Theta N \cdot NZ : \Sigma N \cdot NP;$

and therefore

$$\Theta N \cdot NZ : \Sigma N \cdot NP = \Theta Z : Z\Lambda$$

$$= \Theta N : N\Xi.$$ [ibid.

But $\Theta N : N\Xi = \Theta N \cdot NZ : ZN \cdot N\Xi,$

by taking a common height $ZN$.

And therefore

$$\Theta N \cdot NZ : \Sigma N \cdot NP = \Theta N \cdot NZ : \Xi N \cdot NZ.$$

Therefore $\Sigma N \cdot NP = \Xi N \cdot NZ.$ [Eucl. v. 9

But $MN^2 = \Sigma N \cdot NP,$

as was proved;

and therefore $MN^2 = \Xi N \cdot NZ.$

But the rectangle $\Xi N \cdot NZ$ is the parallelogram $\Xi Z.$
GREEK MATHEMATICS

ΜΝ δύναται τὸ ΕΖ, δ' παράκειται παρὰ τὴν ΖΛ, πλάτος ἔχουν τὴν ΖΝ, ὑπερβάλλον τῷ ΔΕ ὦμοιω ὀντὶ τῷ ὑπὸ τῶν ΘΖΛ. καλείσθω δὲ ἡ μὲν τοιαύτη τομὴ ὑπερβολῆ, ἡ δὲ ΔΖ παρ' ἦν δύνανται αἰ ἐπὶ τὴν ΖΗ καταγόμεναν τεταγμένως· καλείσθω δὲ ἡ αὐτὴ καὶ ὀρθία, πλαγία δὲ ἡ ΖΘ.

ῥ'

Εἰςεν κώνος ἐπιπέδω τμηθῆ διὰ τοῦ ἄξονος, τμηθῇ δὲ καὶ ἑτέρω ἐπιπέδω συμμετοντι μὲν ἑκατέρα πλευρά τοῦ διὰ τοῦ ἄξονος τριγώνου, μήτε δὲ παρὰ τὴν βάσιν τοῦ κώνου ἡγομένω μήτε ὑπεναντίως, τὸ δὲ ἐπιπέδου, ἐν ὦ ἐστὶν ἡ βάσις τοῦ κώνου, καὶ τὸ τέμνον ἐπιπέδου συμπίπτῃ κατ' ευθείαν πρὸς ὀρθὰς ὄψαν ήτοι τῇ βάσει τοῦ διὰ τοῦ ἄξονος τριγώνου ἢ τῇ ἑπ' εὐθείας αὐτῆς, ἡτίς ἂν ἀπὸ τῆς τομῆς τοῦ κώνου παράλληλος ἀχθῇ τῇ κοινῇ τομῇ τῶν ἐπιπέδων ἔως τῆς διαμέτρου τῆς τομῆς, δυνήσεται τι χωρίων παρακείμενον παρὰ τινα εὐθείαν, πρὸς ἦν λόγον ἔχει ἡ διάμετρος τῆς τομῆς, διὸ τὸ τετράγωνον τὸ ἀπὸ τῆς ἡγομένης ἀπὸ τῆς κορυφῆς τοῦ κώνου παρὰ τὴν διαμέτρου τῆς τομῆς ἔως τῆς βάσεως τοῦ τριγώνου πρὸς τὸ περιεχόμενον ὑπὸ τῶν ἀπολαμβανομένων ὑπ' αὐτῆς πρὸς ταῖς τοῦ τριγώνου εὐθείας, πλάτος ἔχον τὴν ἀπολαμβανομένην ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῇ κορυφῆ τῆς τομῆς, ἐλλεῖπον εἶδε όμοιώ τε καὶ όμοίως κειμένω τῷ περιεχομένῳ ὑπὸ τε τῆς διαμέτρου καὶ τῆς παρ' ἦν δύνανται καλείσθω δὲ ἡ τοιαύτη τομὴ ἐλλεῖψις.

"Εστω κώνος, οὗ κορυφὴ μὲν τὸ Α σημεῖον, 316
APOLLONIUS OF PERGA

Therefore the square on MN is equal to EZ, which is applied to ZΛ, having ZN for its breadth, and exceeding by ΛΞ similar to the rectangle contained by ΘZ, ZΛ. Let such a section be called a hyperbola, let AZ be called the parameter to the ordinates to ZH; and let this line be also called the erect side (latus rectum), and ZΘ the transverse side.∗

Prop. 13

Let a cone be cut by a plane through the axis, and let it be cut by another plane meeting each side of the axial triangle, being neither parallel to the base nor subcontrary, and let the plane containing the base of the cone meet the cutting plane in a straight line perpendicular either to the base of the axial triangle or to the base produced; then if a straight line be drawn from any point of the section of the cone parallel to the common section of the planes as far as the diameter of the section, its square will be equal to an area applied to a certain straight line; this line is such that the diameter of the section will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle contained by the intercepts made by it on the sides of the triangle; the breadth of the applied figure will be the intercept made by it on the diameter in the direction of the vertex of the section; and the applied figure will be deficient by a figure similar and similarly situated to the rectangle bounded by the diameter and the parameter; and let such a section be called an ellipse.

Let there be a cone, whose vertex is the point A

∗ The erect and transverse side, that is to say, of the figure (εἰδῶς) applied to the diameter. In the case of the parabola, the transverse side is infinite.
GREEK MATHEMATICS

βάσις δὲ ὁ ΒΓ κύκλος, καὶ τετμῆσθω ἐπιπέδω διὰ τοῦ ἀξονος, καὶ ποιεῖτω τομὴν τὸ ΑΒΓ τριγώνου, τετμῆσθω δὲ καὶ ἐτέρῳ ἐπιπέδῳ συμπίπτοντι μὲν ἑκατέρα πλευρὰ τοῦ διὰ τοῦ ἀξονος τριγώνου, μήτε δὲ παραλλήλῳ τῇ βάσει τοῦ κώνου μήτε ὑπεναντίως ἡγμένῳ, καὶ ποιεῖτω τομὴν ἐν τῇ ἐπιφανείᾳ τοῦ κώνου τῆν ΔΕ γραμμῆν· κοινὴ

dὲ τομὴ τοῦ τέμνοντος ἐπιπέδου καὶ τοῦ, ἐν ὧν ἐστὶν ἡ βάσις τοῦ κώνου, ἐστώ ἡ ΖΗ πρὸς ὄρθας ὀψα τῇ ΒΓ, ἡ δὲ διάμετρος τῆς τομῆς ἐστώ ἡ ΕΔ, καὶ ἀπὸ τοῦ Ε τῇ ΕΔ πρὸς ὄρθας ἡχθω ἡ ΕΘ, καὶ διὰ τοῦ Α τῇ ΕΔ παράλληλος ἡχθω ἡ ΑΚ, καὶ πεποιησθῶν ὡς τὸ ἀπὸ ΑΚ πρὸς τὸ ὑπὸ ΒΚΓ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΘ, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς τομῆς τὸ Λ, καὶ διὰ τοῦ Α τῇ ΖΗ παράλληλος ἡχθω ἡ ΛΜ. λέγω, ὅτι ἡ ΛΜ δύναται τι χωρίων, δὲ παράκειται παρὰ τὴν ΕΘ, πλάτος ἐχου ἑλλείπου εἰδεὶ ὁμοῖῳ τῷ ὑπὸ τῶν ΔΕΘ. Ἐπεξεύχθω γὰρ ἡ ΔΘ, καὶ διὰ μὲν τοῦ Μ τῇ

318
and whose base is the circle $B\Gamma$, and let it be cut by a plane through the axis, and let the section so made be the triangle $A\beta \Gamma$, and let it be cut by another plane meeting either side of the axial triangle, being drawn neither parallel to the base nor subcontrary, and let the section made on the surface of the cone be the curve $\Delta E$; let the common section of the cutting plane and of that containing the base of the cone be $ZH$, perpendicular to $B\Gamma$, and let the diameter of the section $bcE\Delta$, and from $E$ let $E\Theta$ be drawn perpendicular to $E\Delta$, and through $A$ let $AK$ be drawn parallel to $E\Delta$, and let $AK^2 : BK = \Delta E : E\Theta$, and let any point $A$ be taken on the section, and through $A$ let $AM$ be drawn parallel to $ZH$. I say that the square on $AM$ is equal to an area applied to the straight line $E\Theta$, having $EM$ for its breadth, and being deficient by a figure similar to the rectangle contained by $\Delta E$, $E\Theta$.

For let $\Delta \Theta$ be joined, and through $M$ let $MEN$ be
GREEK MATHEMATICS

ΘΕ παράλληλος ἡχθω ή ΜΕΝ, διὰ δὲ τῶν Θ, Ξ
tή EM παράλληλοι ἡχθωσαν αἰ ΘΝ, ΞΟ, καὶ διὰ
tοῦ Μ τή ΒΓ παράλληλος ἡχθω ή ΠΜΡ. ἐπεὶ
οὖν ή ΠΡ τή ΒΓ παράλληλος ἐστὶν, ἐστὶ δὲ καὶ
ἡ ΛΜ τή ΖΗ παράλληλος, τὸ ἄρα διὰ τῶν ΛΜ,
ΠΡ ἐπίπεδον παράλληλον ἐστὶ τῷ διὰ τῶν ΖΗ,
ΒΓ ἐπίπεδω, τούτεστι τῇ βάσει τοῦ κώνου. ἐὰν
ἄρα ἐκβληθῇ διὰ τῶν ΛΜ, ΠΡ ἐπίπεδον, ἡ τομὴ
kύκλος ἐσται, οὐ διάμετρος ἡ ΠΡ. καὶ ἐστὶ
kάθετος ἐπ’ αὐτήν ἡ ΛΜ. τὸ ἄρα ὑπὸ τῶν ΠΜΡ
ἰσον ἐστὶ τῷ ἀπὸ τῆς ΛΜ. καὶ ἐπεὶ ἐστὶν, ὡς τὸ
ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ τῶν ΒΚΓ, οὔτως ἡ ΕΔ
πρὸς τὴν ΕΘ, ὁ δὲ τοῦ ἀπὸ τῆς ΑΚ πρὸς τὸ ὑπὸ
tῶν ΒΚΓ λόγος σύγκειται ἐκ τοῦ, διὸ ἔχει ἡ ΑΚ
πρὸς ΚΒ, καὶ ἡ ΑΚ πρὸς ΚΓ, ἀλλ’ ὃς μὲν ἡ ΑΚ
πρὸς ΚΒ, οὔτως ἡ ΕΗ πρὸς ΗΒ, τούτεστιν ἡ
ΕΜ πρὸς ΜΠ, ὡς δὲ ἡ ΑΚ πρὸς ΚΓ, οὔτως ἡ
ΔΗ πρὸς ΗΓ, τούτεστιν ἡ ΔΜ πρὸς ΜΡ, ὁ ἄρα
τῆς ΔΕ πρὸς τὴν ΕΘ λόγος σύγκειται ἐκ τε τοῦ
τῆς ΕΜ πρὸς ΜΠ καὶ τοῦ τῆς ΔΜ πρὸς ΜΡ. ὁ
dὲ συγκείμενος λόγος ἐκ τε τοῦ, διὸ ἔχει ἡ ΕΜ
πρὸς ΜΠ, καὶ ἡ ΔΜ πρὸς ΜΡ, ὁ τοῦ ὑπὸ τῶν
ΕΜΔ ἐστὶ πρὸς τὸ ὑπὸ τῶν ΠΜΡ. ἐστὶν ἄρα ὡς
tο τὸ ὑπὸ τῶν ΕΜΔ πρὸς τὸ ὑπὸ τῶν ΠΜΡ, οὔτως
ἡ ΔΕ πρὸς τὴν ΕΘ, τούτεστιν ἡ ΔΜ πρὸς τὴν
ΜΕ. ὡς δὲ ἡ ΔΜ πρὸς ΜΕ, τῆς ΜΕ κοινοῦ
ὕψους λαμβανομένης, οὔτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ
ὑπὸ ΕΜΕ. καὶ ὡς ἄρα τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ
ΠΜΡ, οὔτως τὸ ὑπὸ ΔΜΕ πρὸς τὸ ὑπὸ ΕΜΕ.
ἰσον ἄρα ἐστὶ τὸ ὑπὸ ΠΜΡ τῷ ὑπὸ ΕΜΕ. τὸ δὲ
ὑπὸ ΠΜΡ ἢσον ἐδείχθη τῷ ἀπὸ τῆς ΛΜ. καὶ τὸ
ὑπὸ ΕΜΕ ἢσον ἐστὶν ἢσον τῷ ἀπὸ τῆς ΛΜ. ἡ ΛΜ
320
drawn parallel to $\Theta E$, and through $\Theta, \Xi$, let $\Theta N, \Xi O$ be drawn parallel to $EM$, and through $M$ let $\Pi MP$ be drawn parallel to $\Theta G$. Then since $\Pi P$ is parallel to $\Theta G$, and $\Delta M$ is parallel to $ZH$, therefore the plane through $\Delta M, \Pi P$ is parallel to the plane through $ZH, \Theta G$ [Eucl. xi. 15], that is to the base of the cone. If, therefore, the plane through $\Delta M, \Pi P$ be produced, the section will be a circle with diameter $\Pi P$ [Prop. 4]. And $\Delta M$ is perpendicular to it; therefore

$$\Pi M \cdot MP = \Delta M^2.$$ 

And since $AK^2 : BK \cdot K\Gamma = E\Delta : E\Theta$, and $AK^2 : BK \cdot K\Gamma = (AK : KB)(AK : K\Gamma)$, while $AK : KB = EH : HB$

$$= EM : M\Pi, \quad [\text{Eucl. vi. 4}$$

and $AK : K\Gamma = \Delta H : H\Gamma$

$$= \Delta M : MP, \quad [\text{ibid.}$$

therefore $\Delta E : E\Theta = (EM : M\Pi)(\Delta M : MP)$.

But $(EM : M\Pi)(\Delta M : MP) = EM \cdot MA : \Pi M \cdot MP$.

Therefore

$$EM \cdot MA : \Pi M \cdot MP = \Delta E : E\Theta$$  

$$= \Delta M : M\Xi. \quad [\text{ibid.}$$

But $\Delta M : M\Xi = \Delta M \cdot ME : \Xi M \cdot ME$,

by taking a common height $ME$.

Therefore $\Delta M \cdot ME : \Pi M \cdot MP = \Delta M \cdot ME : \Xi M \cdot ME$.

Therefore $\Pi M \cdot MP = \Xi M \cdot ME. \quad [\text{Eucl. v. 9}$

But $\Pi M \cdot MP = \Delta M^2$,

as was proved; and therefore $\Xi M \cdot ME = \Delta M^2$. 

321
ἀρα δύναται τὸ ΜΟ, ὁ παράκειται παρὰ τὴν ΘΕ, πλάτος ἔχου τὴν ΕΜ, ἐλλείπον εἶδει τῷ ΟΝ ὁμοίω ὀντὶ τῷ ὑπὸ ΔΕΘ. καλείσθω δὲ ἡ μὲν τοιαύτη τομὴ ἐλλειψις, ἡ δὲ ΕΘ παρ’ ἤν δύνανται αἱ καταγόμεναι ἐπὶ τὴν ΔΕ τεταγμένως, ἡ δὲ αὐτὴ καὶ ὀρθία, πλαγία δὲ ἡ ΕΔ.

18’

'Εὰν αἱ κατὰ κορυφὴν ἐπιφάνειαι ἐπιπέδῳ τμήθωσι μὴ διὰ τῆς κορυφῆς, ἐσται ἐν ἑκατέρα τῶν ἐπιφανειῶν τομῆ ἡ καλομένη ὑπερβολή, καὶ τῶν δύο τομῶν ἡ τε διάμετρος ἡ αὐτή ἐσται, καὶ παρ’ ἃς δύνανται αἱ ἐπὶ τὴν διάμετρον καταγόμεναι παράλληλοι τῇ ἐν τῇ βάσει τοῦ κόνου εὐθεία ἦσαι, καὶ τοῦ εἶδους ἡ πλαγία πλευρὰ κοινὴ ἡ μεταξὺ τῶν κορυφῶν τῶν τομῶν· καλείσθωσαν δὲ αἱ τοιαύται τομαὶ ἀντικείμεναι.

'Εστωσαν αἱ κατὰ κορυφὴν ἐπιφάνειαι, ὅν κορυφὴ τὸ Α σημείον, καὶ τετμήσωσαν ἐπιπέδῳ μὴ διὰ τῆς κορυφῆς, καὶ ποιεῖτω ἐν τῇ ἐπιφανείᾳ τομὰς τὰς ΔΕΖ, ὙΘΚ. λέγω, ὅτι ἑκατέρα τῶν ΔΕΖ, ὙΘΚ τομῶν ἐστὶν ἡ καλομένη ὑπερβολή.

---

\[a\] Let \( p \) be the parameter of a conic section and \( d \) the corresponding diameter, and let the diameter of the section and the tangent at its extremity be taken as axes of co-ordinates (in general oblique). Then Props. 11-13 are equivalent to the Cartesian equations.
Therefore the square on \( \Delta M \) is equal to \( \text{MO} \), which is applied to \( \Theta E \), having \( \text{EM} \) for its breadth, and being deficient by the figure \( \text{ON} \) similar to the rectangle \( \Delta E \cdot \Theta \). Let such a section be called an ellipse, let \( \Theta \) be called the parameter to the ordinates to \( \Delta E \), and let this line be called the erect side (latus rectum), and \( \text{E} \Delta \) the transverse side.\(^a\)

**Prop. 14**

*If the vertically opposite surfaces [of a double cone] be cut by a plane not through the vertex, there will be formed on each of the surfaces the section called a hyperbola, and the diameter of both sections will be the same, and the parameter to the ordinates drawn parallel to the straight line in the base of the cone will be equal, and the transverse side of the figure will be common, being the straight line between the vertices of the sections; and let such sections be called opposite.*

Let there be vertically opposite surfaces having the point \( \text{A} \) for vertex, and let them be cut by a plane not through the vertex, and let the sections so made on the surface be \( \Delta \text{EZ} \), \( \text{H} \Theta \text{K} \). I say that each of the sections \( \Delta \text{EZ} \), \( \text{H} \Theta \text{K} \) is the so-called hyperbola.

\[
y^2 = px \\
\text{(the parabola),}
\]

and

\[
y^2 = px \pm \frac{p}{d}x^2 \quad \text{(the hyperbola and ellipse respectively).}
\]

It is the essence of Apollonius’s treatment to express the fundamental properties of the conics as equations between areas, whereas Archimedes had given the fundamental properties of the central conics as proportions

\[
y^2 : (a^2 \pm x^2) = a^2 : b^2.
\]

This form is, however, equivalent to the Cartesian equations referred to axes through the centre.
"Εστώ γὰρ ὁ κύκλος, καθ’ οὗ φέρεται ἡ τὴν ἐπιφάνειαν γράφουσα εὐθεία, ὁ ΒΔΓΖ, καὶ ἤχθω ἐν τῇ κατὰ κορυφὴν ἐπιφάνεια παράλληλον αὐτῷ ἐπίπεδον τὸ ΞΗΟΚ. κοναὶ δὲ τοιαὶ τῶν ΗΘΚ, ΖΕΔ τομῶν καὶ τῶν κύκλων αἱ ΖΔ, ΗΚ. ἐσονται δὴ παράλληλοι. ἄξων δὲ ἐστώ τῆς κωνικῆς ἐπιφάνειας ἡ ΛΑΓ εὐθεία, κέντρα δὲ τῶν κύκλων τὰ Λ, Γ, καὶ ἀπὸ τοῦ Λ ἐπὶ τὴν ΖΔ κάθετος ἀχθεία έκβεβλῆσθω ἐπὶ τὰ Β, Γ σημεῖα, καὶ διὰ τῆς ΒΓ καὶ τοῦ ἄξονος ἐπίπεδον ἐκβεβλῆσθω. ποσὴς δὴ τομὰς ἐν μὲν τοῖς κύκλοις παράλληλους εὐθείας τὰς ΞΟ, ΒΓ, ἐν δὲ τῇ ἐπιφάνεια τὰς ΒΑΟ, ΓΑΞ.
For let $BA\Gamma Z$ be the circle round which revolves the straight line describing the surface, and in the vertically opposite surface let there be drawn parallel to it a plane $\Xi HOK$; the common sections of the sections $H\Theta K$, $Z\Xi \Delta$ and of the circles [Prop. 4] will be $Z\Delta$, $HK$; and they will be parallel [Eucl. xi. 16]. Let the axis of the conical surface be $\Delta \Lambda Y$, let the centres of the circles be $\Lambda$, $\gamma$, and from $\Lambda$ let a perpendicular be drawn to $Z\Delta$ and produced to the points $B$, $\Gamma$, and let the plane through $B\Gamma$ and the axis be produced; it will make in the circles the parallel straight lines $\Xi O$, $B\Gamma$, and on the surface $BAO$, $\Gamma A\Xi$;
GREEK MATHEMATICS

ἔσται δὴ καὶ ᾫ ΕΟ τῇ ΗΚ πρὸς ὅρθας, ἐπειδὴ καὶ ᾫ ΒΓ τῇ ΖΔ ἔστι πρὸς ὅρθας, καὶ ἔστιν ἐκατέρα παράλληλος. καὶ ἔπει τὸ διὰ τοῦ ἄξονος ἐπίπεδον ταῖς τομαῖς συμβάλλει κατὰ τὰ Μ, Ν σημεία ἐντὸς τῶν γραμμῶν, δὴ λοι, ἦς καὶ τὰς γραμμὰς τέμνει τὸ ἐπίπεδον. τεμνέτω κατὰ τὰ Θ, Ε. τὰ ἄρα Μ, Ε, Θ, Ν σημεία ἐν τε τῷ διὰ τοῦ ἄξονος ἔστιν ἐπίπεδῳ καὶ ἐν τῷ ἐπίπεδῳ, ἐν ὃ εἰσιν αἱ γραμμαί· εὐθεία ἄρα ἔστιν ἡ ΜΕΘΩΝ γραμμή. καὶ φανερὸν, ὅτι τά τε Ε, Θ, Α, Γ ἐπὶ εὐθείας ἔστι καὶ τά Β, Ε, Α, Θ. ἐν τέ γὰρ τῇ κωνικῇ ἐπιφανείᾳ ἐστὶ καὶ ἐν τῷ διὰ τοῦ ἄξονος ἐπίπεδῳ. ἤχωσαν δὴ ἀπὸ μὲν τῶν Θ, Ε τῇ ΘΕ πρὸς ὅρθας αἱ ΘΡ, ΕΠ, διὰ δὲ τοῦ Α τῇ ΜΕΘΝ παράλληλος ἤχωθι ἡ ΣΑΤ, καὶ πεποίησθι, ὡς μὲν τὸ ἀπὸ τῆς ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὕτως ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ ἀπὸ τῆς AT πρὸς τὸ ὑπὸ ΟΤΞ, οὕτως ἡ ΘΘ πρὸς ΘΡ. ἐπεὶ οὖν κώνως, οὐ κορυφή μὲν τὸ Α σημείου, βάσις δὲ ὁ ΒΓ κύκλος, τέτμηται ἐπίπεδῳ διὰ τοῦ ἄξονος, καὶ πεποίηκε τομὴν τὸ ἈΒΓ τρίγωνον, τέτμηται δὲ καὶ ἑτέρῳ ἐπίπεδῳ τέμνοντι τὴν βάσιν τοῦ κώνου κατ᾽ εὐθείαν τὴν ΔΜΖ πρὸς ὅρθας οὖσαν τῇ ΒΓ, καὶ πεποίηκε τομὴν ἐν τῇ ἐπιφανείᾳ τῆς ΔΕΖ, ἡ δὲ διάμετρος ἡ ΜΕ ἐκβαλλομένη συμ- πέπτωσε μιᾷ πλευρᾷ τοῦ διὰ τοῦ ἄξονος τριγώνου ἐκτὸς τῆς κορυφῆς τοῦ κώνου, καὶ διὰ τοῦ Α σημείου τῇ διαμέτρῳ τῆς τομῆς τῆς ΕΜ παράλληλος ἦκται ἡ ΑΣ, καὶ ἀπὸ τοῦ Ε τῇ ΕΜ πρὸς ὅρθας ἦκται ἡ ΕΠ, καὶ ἔστιν ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὕτως ἡ ΘΘ πρὸς ΕΠ, ἡ μὲν ΔΕΖ ἀρὰ τομή ὑπερβολή ἔστιν, ἡ δὲ ΕΠ παρ’ ἦν δύναται αἱ ἐπὶ τὴν ΕΜ καταγόμεναι τεταγμένως, πλαγία
326
now $\Xi O$ will be perpendicular to $HK$, since $B\Gamma$ is perpendicular to $Z\Delta$, and each is parallel [Eucl. xi. 10]. And since the plane through the axis meets the sections at the points $M$, $N$ within the curves, it is clear that the plane cuts the curves. Let it cut them at the points $\Theta$, $E$; then the points $M$, $E$, $\Theta$, $N$ are both in the plane through the axis and in the plane containing the curves; therefore the line $M\Theta\Omega N$ is a straight line [Eucl. xi. 8]. And it is clear that $\Xi$, $\Theta$, $A$, $\Gamma$ are on a straight line, and also $B$, $E$, $A$, $O$; for they are both on the conical surface and in the plane through the axis. Now let $\Theta P$, $E\Pi$ be drawn from $\Theta$, $E$ perpendicular to $\Theta E$, and through $A$ let $\Sigma A T$ be drawn parallel to $M\Theta\Omega N$, and let $A\Sigma^2 : B\Sigma \cdot \Sigma\Gamma = \Theta E : E\Pi$, and $A\Gamma^2 : O\Gamma \cdot T\Xi = E\Theta : \Theta P$.

Then since the cone, whose vertex is the point $A$ and whose base is the circle $B\Gamma$, is cut by a plane through the axis, and the section so made is the triangle $A\Theta\Gamma$, and it is cut by another plane cutting the base of the cone in the straight line $\Delta M\Xi$ perpendicular to $B\Gamma$, and the section so made on the surface is $\Delta E\Xi$, and the diameter $ME$ produced meets one side of the axial triangle beyond the vertex of the cone, and $A\Sigma$ is drawn through the point $A$ parallel to the diameter of the section $EM$, and $E\Pi$ is drawn from $E$ perpendicular to $EM$, and $A\Sigma^2 : B\Sigma \cdot \Sigma\Gamma = E\Theta : E\Pi$, therefore the section $\Delta E\Xi$ is a hyperbola, in which $E\Pi$ is the parameter to the ordinates to $EM$, and $\Theta E$ is the
GREEK MATHEMATICS

δὲ τοῦ εἶδους πλευρὰ ἡ ΘΕ. ὁμοίως δὲ καὶ ἡ ὩΚ ύπερβολή ἦστιν, ἂς διάμετρος μὲν ἡ ΘΝ, ἡ δὲ ΘΡ παρ’ ἦν δύναναι αἱ ἐπὶ τὴν ΘΝ καταγω-μεναι τεταγμένως, πλαγία δὲ τοῦ εἰδους πλευρὰ ἡ ΘΕ.

Λέγω, ὅτι ἵση ἦστιν ἡ ΘΡ τῇ ΕΠ. ἐπεὶ γὰρ παράλληλος ἦστιν ἡ ΒΓ τῇ ΞΟ, ἦστιν ὡς ἡ ΑΣ πρὸς ΣΓ, οὖτως ἡ ΑΤ πρὸς ΤΕ, καὶ ὡς ἡ ΑΣ πρὸς ΣΒ, οὖτως ἡ ΑΤ πρὸς ΤΟ. ἀλλ' ὁ τῆς ΑΣ πρὸς ΣΓ λόγος μετὰ τοῦ τῆς ΑΣ πρὸς ΣΒ ὁ τοῦ ἀπὸ ΑΣ ἦστι πρὸς τὸ ὑπὸ ΒΣΓ, ὁ δὲ τῆς ΑΤ πρὸς ΤΕ μετὰ τοῦ τῆς ΑΤ πρὸς ΤΟ ὁ τοῦ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὖτως τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΕΤΟ. ἦστιν ἑρα ὡς τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, οὖτως τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΕΤΟ. καὶ ἦστιν ὡς μὲν τὸ ἀπὸ ΑΣ πρὸς τὸ ὑπὸ ΒΣΓ, ἡ ΘΕ πρὸς ΕΠ, ὡς δὲ τὸ ἀπὸ ΑΤ πρὸς τὸ ὑπὸ ΕΤΟ, ἡ ΘΕ πρὸς ΘΡ. καὶ ὡς ἑρα ἡ ΘΕ πρὸς ΕΠ, ἡ ΕΘ πρὸς ΘΡ. ἵσῃ ἑρα ἦστιν ἡ ΕΠ τῇ ΘΡ.

(vi.) Transition to New Diameter

Ibid., Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17–154. 8

Ἐὰν ύπερβολής ἡ ἐλλείψεως ἡ κύκλου περι-φερείας εὐθεία ἐπιφανεύοισα συμπιπτή τῇ διαμέτρῳ, καὶ διὰ τῆς ἀφῆς καὶ τοῦ κέντρου εὐθεία ἐκβιληθῇ, ἀπὸ δὲ τῆς κορυφῆς ἀναχθεῖσα εὐθεία παρὰ τεταγ-μένως κατηγμένην συμπιπτή τῇ διὰ τῆς ἀφῆς καὶ

* Apollonius is the first person known to have recognized the opposite branches of a hyperbola as portions of the same
transverse side of the figure [Prop. 12]. Similarly $\Theta K$ is a hyperbola, in which $\Theta N$ is a diameter, $\Theta P$ is the parameter to the ordinates to $\Theta N$, and $\Theta E$ is the transverse side of the figure.

I say that $\Theta P = E\Pi$. For since $BG$ is parallel to $EO$,

$$\Sigma \Theta : \Theta B = AT : TO,$$

and

$$\Sigma \Theta : \Theta B = AT : TO.$$

But

$$(\Sigma \Theta : \Sigma B)(\Sigma \Theta : \Sigma B) = \Sigma B : \Sigma B,$$

and

$$(AT : \Theta E)(AT : \Theta E) = AT^2 : \Theta E : \Theta E.$$

Therefore

$$\Sigma B : \Theta E : \Theta E = AT^2 : \Theta E : \Theta E.$$

But

$$\Sigma B : \Theta E : \Theta E = \Theta E : E\Pi,$$

while

$$AT^2 : \Theta E : \Theta E = \Theta E : \Theta E;$$

therefore

$$\Theta E : E\Pi = \Theta E : \Theta E.$$

Therefore

$$E\Pi = \Theta E^a.$$

[Eucl. v. 9]

(vi.) Transition to New Diameter

Ibid., Prop. 50, Apoll. Perg. ed. Heiberg i. 148. 17-154. 8

Prop. 50

In a hyperbola, ellipse or circumference of a circle let a straight line be drawn to touch [the curve] and meet the diameter, and let the straight line through the point of contact and the centre be produced, and from the vertex let a straight line be drawn parallel to a straight line drawn ordinate-wise so as to meet the straight line drawn curve. It is his practice, however, where possible to discuss the single-branch hyperbola (or the hyperbola simpliciter as he would call it) together with the ellipse and circle, and to deal with the opposite branches separately. But occasionally, as in i. 30, the double-branch hyperbola and the ellipse are included in one enunciation.
The general enunciation is not easy to follow, but the particular enunciation will make it easier to understand. The 330
through the point of contact and the centre, and let the segment of the tangent between the point of contact and the line drawn ordinate-wise bear to the segment of the line drawn through the point of contact and the centre between the point of contact and the line drawn ordinate-wise the same ratio as a certain straight line bears to double the tangent; then if any straight line be drawn from the section parallel to the tangent so as to meet the straight line drawn through the point of contact and the centre, its square will be equal to a certain rectilineal area applied to the postulated straight line, having for its breadth the intercept between it and the point of contact, in the case of the hyperbola exceeding by a figure similar to the rectangle bounded by double the straight line between the centre and the point of contact and the postulated straight line, in the case of the ellipse and circle falling short.\(^a\)

In a hyperbola, ellipse or circumference of a circle, with diameter \(AB\) and centre \(\Gamma\), let \(\Delta E\) be a tangent, and let \(\Gamma E\) be joined and produced in either direction, and let \(\Gamma K\) be placed equal to \(E\Gamma\), and through \(B\) let \(BZH\) be drawn ordinate-wise, and through \(E\) let \(E\Theta\) be drawn perpendicular to \(E\Gamma\), and let \(ZE : EH = E\Theta : 2E\Delta\), and let \(\Theta K\) be joined and produced, and let any point \(\Lambda\) be taken on the section, and through it let \(\Lambda M\Xi\) be drawn parallel to \(E\Delta\) and purpose of this important proposition is to show that, if any other diameter be taken, the ordinate-property of the conic with reference to this diameter has the same form as the ordinate-property with reference to the original diameter. The theorem amounts to a transformation of co-ordinates from the original diameter and the tangent at its extremity to any diameter and the tangent at its extremity. In succeeding propositions, showing how to construct conics from certain data, Apollonius introduces the axes for the first time as special cases of diameters.
GREEK MATHEMATICS

ΒΗ ᾧ ΔΡΝ, τῇ δὲ ΕΘ ᾧ ΜΠ. λέγω, ὅτι τὸ ἀπὸ ΛΜ ἴσον ἐστὶ τῷ ὑπὸ ΕΜΠ.

"Ἡχθω γὰρ διὰ τοῦ Γ τῇ ΚΠ παράλληλος ᾧ ΓΣΟ. καὶ ἐπεὶ ἴση ἐστὶν ᾧ ΕΓ τῇ ΓΚ, ὡς δὲ ᾧ

ΕΓ πρὸς ΚΓ, ᾧ ΕΣ πρὸς ΣΘ, ἴση ἀρα καὶ ᾧ ΕΣ τῇ ΣΘ. καὶ ἐπεὶ ἐστὶν, ὡς ᾧ ΖΕ πρὸς ΕΗ, ᾧ ΘΕ πρὸς τὴν διπλασίαν τῆς ΕΔ, καὶ ἐστὶ τῆς ΕΘ ἡμίσεια ᾧ ΕΣ, ἐστὶν ἀρα, ὡς ᾧ ΖΕ πρὸς ΕΗ, ᾧ ΖΕ πρὸς ΕΔ. ὡς δὲ ᾧ ΖΕ πρὸς ΕΗ, ᾧ ΛΜ πρὸς ΜΡ. ὡς ἀρα ᾧ ΛΜ πρὸς ΜΡ, ᾧ ΖΕ πρὸς ΕΔ. καὶ ἐπεὶ τὸ ΡΝΓ τρίγωνον τοῦ ΗΒΓ τριγώνου, τούτου τοῦ ΓΔΕ, ἐπὶ μὲν τῆς ὑπερβολῆς μείζον ἐδείχθη, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου ἔλασσον τῷ ΛΝΞ, κοινῶν ἀφαιρεθέντων ἐπὶ μὲν τῆς ὑπερβολῆς τοῦ τε ΕΓΔ τριγώνου καὶ τοῦ ΝΡΜΞ τετράπλευρου, ἐπὶ δὲ τῆς ἐλλείψεως καὶ τοῦ κύκλου τοῦ ΜΞΓ τριγώνου, τὸ ΛΜΡ τριγώνου τῷ ΜΕΔΞ τετράπλευρῳ ἐστὶν ἴσον. καὶ ἐστί

332
APOLLONIUS OF PERGA

ΔPN parallel to ΒΠΙ, and let ΜΠ be drawn parallel to ΕΘ. I say that ΔΜ² = EM·ΜΠ.

For through Ι let ΓΣΟ be drawn parallel to ΚΠ. Then since

$$\text{ΕΓ} = \Gamma \text{Κ}$$

and

$$\text{ΕΓ} : \Gamma \text{Κ} = E\Sigma : \Sigma \Theta,$$  [Eucl. vi. 2]

therefore

$$E\Sigma = \Sigma \Theta.$$

And since

$$ZE : EH = \Theta E : 2\Delta,$$

and

$$E\Sigma = \frac{1}{2} \Theta \Sigma,$$

therefore

$$ZE : EH = \Sigma E : \Sigma \Delta.$$

But

$$ZE : EH = AM : MP;$$  [Eucl. vi. 4]

therefore

$$AM : MP = \Sigma E : \Sigma \Delta.$$

And since it has been proved [Prop. 43] that in the hyperbola

triangle PNT = triangle ΗΒΓ + triangle ΔΝΞ,

i.e., triangle PNT = triangle ΓΔE + triangle ΔΝΞ,\(^a\)

while in the ellipse and the circle

triangle PNT = triangle ΗΒΓ' -

triangle ΔΝΞ,

i.e., triangle PNT + triangle ΔΝΞ = triangle ΓΔE,\(^b\)

therefore by taking away the common elements—in the hyperbola the triangle ΕΓΔ and the quadrilateral ΝΠΜΞ, in the ellipse and the circle the triangle ΜΞΓ',

triangle ΛΜΡ = quadrilateral ΜΕΔΞ.

\(^a\) For this step v. Eutocius's comment on Prop. 43.

\(^b\) See Eutocius.
παράλληλος ἡ ΜΞ τῇ ΔΕ, ἡ δὲ ὑπὸ ΛΜΡ τῇ ὑπὸ ΕΜΞ ἐστὶν ἱσχ. ἱσον ἀρα ἐστὶ τὸ ὑπὸ ΛΜΡ τῷ ὑπὸ τῆς ΕΜ καὶ συναμφότερον τῆς ΕΔ, ΜΞ. καὶ ἐπεὶ ἐστὶν, ὡς ἡ ΜΓ πρὸς ΓΕ, ἡ τε ΜΞ πρὸς ΕΔ καὶ ἡ ΜΟ πρὸς ΕΣ, ὡς ἀρα ἡ ΜΟ πρὸς ΕΣ, ἡ ΜΞ πρὸς ΔΕ. καὶ συνθέντω, ὡς συναμφότερος ἡ ΜΟ, ΣΕ πρὸς ΕΣ, οὐτως συναμφότερος ἡ ΜΞ, ΕΔ πρὸς ΕΔ· ἐναλλάξ, ὡς συναμφότερος ἡ ΜΟ, ΣΕ πρὸς συναμφότερον τὴν ΕΜ, ΕΔ ἡ ΣΕ πρὸς ΕΔ. ἀλλ' ὡς μὲν συναμφότερος ἡ ΜΟ, ΕΣ πρὸς συναμφότερον τῆς ΜΞ, ΔΕ, τὸ ὑπὸ συναμφότερου τῆς ΜΟ, ΕΣ καὶ τῆς ΕΜ πρὸς τὸ ὑπὸ συναμφοτέρου τῆς ΜΞ, ΕΔ καὶ τῆς ΕΜ, ὡς δὲ ἡ ΣΕ πρὸς ΕΔ, ἡ ΖΕ πρὸς ΕΗ, τουτέστιν ἡ ΛΜ πρὸς ΜΡ, τουτέστι τὸ ἀπὸ ΛΜ πρὸς τὸ ὑπὸ ΛΜΡ· ὡς ἀρα τὸ ὑπὸ συναμφότερου τῆς ΜΟ, ΕΣ καὶ τῆς ΜΞ πρὸς τὸ ὑπὸ συναμφότερου τῆς ΜΞ, ΕΔ καὶ τῆς ΕΜ, τὸ ἀπὸ ΛΜ πρὸς τὸ ὑπὸ ΛΜΡ. καὶ ἐναλλάξ, ὡς τὸ ὑπὸ συναμφότερου τῆς ΜΟ, ΕΣ καὶ τῆς ΜΞ πρὸς τὸ ἀπὸ ΜΛ, οὕτως τὸ ὑπὸ συναμφότερου τῆς ΜΞ, ΕΔ καὶ τῆς ΜΞ πρὸς τὸ ὑπὸ ΛΜΡ. ἱσον δὲ τὸ ὑπὸ ΛΜΡ τῷ ὑπὸ τῆς ΜΞ καὶ συναμφοτέρου τῆς ΜΞ, ΕΔ· ἱσον ἀρα καὶ τὸ ἀπὸ ΛΜ τῷ ὑπὸ ΕΜ καὶ συναμφότερου τῆς ΜΟ, ΕΣ. καὶ ἐστὶν ἡ μὲν ΣΕ τῇ ΣΘ ἱση, ἡ δὲ ΣΘ τῇ ΟΠ· ἱσον ἀρα τὸ ἀπὸ ΛΜ τῷ ὑπὸ ΕΜΠ.

334
APOLLONIUS OF PERGA

But $MQ$ is parallel to $AE$ and angle $AMP = \angle EMQ$ (Eucl. i. 15);

therefore \[ \triangle AM \cdot MP = \triangle EM \cdot (EA + ME). \]

And since \[ MT : \triangle GE = MQ : \triangle EA, \]

and \[ MT : \triangle GE = MO : \triangle ES, \]

therefore \[ MO : \triangle ES = MQ : \triangle AE. \]

Componendo, \[ MO + ES : ES = MQ + EA : EA; \]

and permutando \[ MO + ES : EM + EA = ES : EA. \]

But \[ MO + ES : EM + EA = (MO + ES) : EM = (MQ + EA) : EM, \]

and \[ ES : EA = EZ : EH = \triangle AM \cdot MP \]

[Eucl. vi. 4]

\[ = \triangle AM^2 : \triangle AM \cdot MP; \]

therefore \[ (MO + ES) \cdot ME : (MQ + EA) \cdot EM = \triangle AM^2 : \triangle AM \cdot MP. \]

And permutando \[ (MO + ES) \cdot ME : MA^2 = (MQ + EA) \cdot ME : \triangle AM \cdot MP. \]

But \[ \triangle AM \cdot MP = ME \cdot (MQ + EA); \]

therefore \[ \triangle AM^2 = EM \cdot (MO + ES). \]

And \[ SE = SO, \text{ while } SO = OP \] [Eucl. i. 34];

therefore \[ \triangle AM^2 = EM \cdot MP. \]
(b) Other Works

(i.) General

Papp. Coll. vii. 3, ed. Hultsch 636. 18-23

Τῶν δὲ προειρημένων τοῦ Ἀναλυμένου βιβλίων ἡ τάξις ἐστὶν τοιαύτη. Ἐυκλείδου Δεδομένων βιβλίων ἄ., Ἀπολλωνίου Λόγου ἀποτομῆς β, Χωρίου ἀποτομῆς β, Διωρισμένης τομῆς δύο, Ἔπαφων δύο, Ἐυκλείδου Πορισμάτων τρία, Ἀπολλωνίου Νεύσεων δύο, τοῦ αὐτοῦ Τόπων ἐπιπέδων δύο, Κωνικῶν ἦ.

(ii.) On the Cutting-off of a Ratio

Ibid. vii. 5-6, ed. Hultsch 640. 4-22

Τῆς δ’ Ἀποτομῆς τοῦ λόγου βιβλίων ὄντων β πρότασις ἐστὶν μία ὑποδιηρμένη, διὸ καὶ μίαν πρότασιν οὕτως γράφω. διὰ τοῦ δοθέντος σημείου εὐθείαν γραμμὴν ἀγαγείν τέμνουσαν ἀπὸ τῶν τῆθεσι δοθείσων δύο εὐθείαν πρὸς τοῖς ἐπ’ αὐτῶν δοθεῖσι σημείοις λόγου ἐχούσας τὸν αὐτὸν τῷ δοθέντι. τὰς δὲ γραφὰς διαφόρους γενέσθαι καὶ πλῆθος λαβεῖν συμβεβηκεν ὑποδιαιρέσεως γενομένης ἑνεκα τῆς τε πρὸς ἀλλήλας θέσεως τῶν διδομένων εὐθείῶν καὶ τῶν διαφόρων πτωσεων τοῦ διδομένου σημείου καὶ διὰ τὰς ἀναλύσεις καὶ συνθέσεις αὐτῶν τε καὶ τῶν διορισμῶν. ἔχει γὰρ τὸ μὲν πρῶτον βιβλίον τῶν Λόγου ἀποτομῆς

* Unhappily the only work by Apollonius which has survived, in addition to the Conics, is On the Cutting-off of a
APOLLONIUS OF PERGA

(b) Other Works

(i.) General

Pappus, Collection vii. 3, ed. Hultsch 636. 18-23

The order of the aforesaid books in the Treasury of Analysis is as follows: the one book of Euclid’s Data, the two books of Apollonius’s On the Cutting-off of a Ratio, his two books On the Cutting-off of an Area, his two books On Determinate Section, his two books On Tangencies, the three books of Euclid’s Porisms, the two books of Apollonius’s On Vergings, the two books of the same writer On Plane Loci, his eight books of Conics.

(ii.) On the Cutting-off of a Ratio

Ibid. vii. 5-6, ed. Hultsch 640. 4-22

In the two books On the Cutting-off of a Ratio there is one enunciation which is subdivided, for which reason I state one enunciation thus: Through a given point to draw a straight line cutting off from two straight lines given in position intercepts, measured from two given points on them, which shall have a given ratio. When the subdivision is made, this leads to many different figures according to the position of the given straight lines in relation one to another and according to the different cases of the given point, and owing to the analysis and the synthesis both of these cases and of the propositions determining the limits of possibility. The first book of those On the Cutting-off of a Ratio, and that only in Arabic. Halley published a Latin translation in 1706. But the contents of the other works are indicated fairly closely by Pappus’s references.

337
GREEK MATHEMATICS

tópous ξ, πτώσεις κδ, διορισμούς δὲ έ, ὃν τρεῖς μὲν εἰσὶν μέγιστοι, δύο δὲ ἐλάχιστοι. . . . τὸ δὲ δεύτερον βιβλίον Λόγου ἀποτομῆς ἔχει τόπους ἰδιομούς δὲ τοὺς ἐκ τοῦ πρώτου ἀπάγεται γὰρ ὅλον εἰς τὸ πρῶτον.

(iii.) On the Cutting-off of an Area
Ibid. vii. 7, ed. Hultsch 640. 26-642. 5

Τῆς δ’ Ἀποτομῆς τοῦ χωρίου βιβλία μὲν ἑστὶν δύο, πρόβλημα δὲ κἂν τούτους ἐν ύποδιαιροῦμενον δίς, καὶ τούτων μία πρότασις ἑστὶν τὰ μὲν ἀλλὰ ὀμοίως ἔχουσα τῇ πρώτῃ, μόνῳ δὲ τούτῳ διαφέρουσα τῷ δὲν τὰς ἀποτεμνομένας δύο εὐθείας ἐν ἐκείνῃ μὲν λόγου ἔχουσας δοθέντα ποιεῖν, ἐν δὲ ταύτῃ χωρίον περιεχούσας δοθέν.

(iv.) On Determinate Section
Ibid. vii. 9, ed. Hultsch 642. 19-644. 16

Ἐξῆς τούτων ἀναδέδονται τῆς Διωρισμένης τομῆς βιβλία β, ὃν ὀμοίως τοῖς πρῶτοι μιᾶν πρότασιν πάρεστιν λέγειν, διεζευγμένην δὲ ταύτην.

* The Arabic text shows that Apollonius first discussed the cases in which the lines are parallel, then the cases in which the lines intersect but one of the given points is at the point of intersection; in the second book he proceeds to the general case, but shows that it can be reduced to the case where one
APOLLONIUS OF PERGA

*Ratio* contains seven loci, twenty-four cases and five determinations of the limits of possibility, of which three are maxima and two are minima. . . . The second book *On the Cutting-off of a Ratio* contains fourteen loci, sixty-three cases and the same determinations of the limits of possibility as the first; for they are all reduced to those in the first book.a

(iii.) *On the Cutting-off of an Area*


In the work *On the Cutting-off of an Area* there are two books, but in them there is only one problem, twice subdivided, and the one enunciation is similar in other respects to the preceding, differing only in this, that in the former work the intercepts on the two given lines were required to have a given ratio, in this to comprehend a given area.b

(iv.) *On Determinate Section*


Next in order after these are published the two books *On Determinate Section*, of which, as in the previous cases, it is possible to state one comprehension of the given points is at the intersection of the two lines. By this means the problem is reduced to the application of a rectangle. In all cases Apollonius works by analysis and synthesis.

b Halley attempted to restore this work in his edition of the *De sectione rationis*. As in that treatise, the general case can be reduced to the case where one of the given points is at the intersection of the two lines, and the problem is reduced to the application of a certain rectangle.
As the Greeks never grasped the conception of one point being two coincident points, it was not possible to enunciate this problem so concisely as we can do: Given four points $A, B, C, D$ on a straight line, of which $A$ may coincide with $C$ and $B$ with $D$, to find another point $P$ on the same straight line such that $AP : CP : BP : DP$ has a given value. If $AP : CP = \lambda : BP : DP$, where $A, B, C, D, \lambda$ are given, the determination of $P$ is equivalent to the solution of a quadratic equation, which the Greeks could achieve by means of the...
APOLLONIUS OF PERGA

sive enunciation thus: To cut a given infinite straight line in a point so that the intercepts between this point and given points on the line shall furnish a given ratio, the ratio being that of the square on one intercept, or the rectangle contained by two, towards the square on the remaining intercept, or the rectangle contained by the remaining intercept and a given independent straight line, or the rectangle contained by two remaining intercepts, whichever way the given points [are situated]. . . . The first book contains six problems, sixteen subdivisions and five limits of possibility, of which four are maxima and one is a minimum. . . . The second book On Determinate Section contains three problems, nine subdivisions, and three limits of possibility.a

(v.) On Tangencies

Ibid. vii. 11, ed. Hultsch 644, 23–646. 19

Next in order are the two books On Tangencies. Their enunciations are more numerous, but we may bring these also under one enunciation thus stated: Given three entities, of which any one may be a point or a straight line or a circle, to draw a circle which shall pass through each of the given points, so far as it is points which are given, or to touch each of the given lines.b In application of areas. But the fact that limits of possibility, and maxima and minima were discussed leads Heath (H.G.M. ii. 180-181) to conjecture that Apollonius investigated the series of point-pairs determined by the equation for different values of λ, and that "the treatise contained what amounts to a complete Theory of Involution." The importance of the work is shown by the large number of lemmas which Pappus collected.

b The word "lines" here covers both the straight lines and the circles.
πλήθη τῶν ἐν ταῖς ὑποθέσεσι δεδομένων ὁμοίων ἡ ἀνομοίων κατὰ μέρος διαφόρους προτάσεις ἀναγκαῖον γίνεσθαι δέκα· ἐκ τῶν τριῶν γὰρ ἀνομοίων γενῶν τριάδες διαφοροὶ ἀτακτοὶ γίνονται ἵ. ἦτοι γὰρ τὰ διδόμενα τρία σημεῖα ἢ τρεῖς εὑθεῖαι ἢ δύο σημεῖα καὶ εὐθεῖα ἢ δύο εὐθεῖαι καὶ σημείον ἢ δύο σημεῖα καὶ κύκλος ἢ δύο κύκλοι καὶ σημείον ἢ δύο εὐθεῖαι καὶ κύκλος ἢ δύο κύκλοι καὶ εὐθεία ἢ σημείον καὶ εὐθεία καὶ κύκλος ἢ τρεῖς κύκλοι. τούτων δύο μὲν τὰ πρῶτα δέδεκται ἐν τῷ δ' βιβλίῳ τῶν πρῶτων Στοιχείων, διὸ παρεῖ μὴ γράφων· τὸ μὲν γὰρ τριῶν δοθέντων σημείων μὴ ἑπὶ εὐθείας ὄντων τὸ αὐτὸ ἐστὶν τῷ περὶ τὸ δοθὲν τρίγωνον κύκλον περιγράψαι, τὸ δὲ γ' δοθεῖσθαι εὐθείαι τῇ παραλλήλῳ ὦσῶν, ἄλλα τῶν τριῶν συμπεπτούσων, τὸ αὐτὸ ἐστὶν τῷ εἰς τὸ δοθὲν τρίγωνον κύκλον ἐγγράψαι: τὸ δὲ δύο παραλλήλων ὦσῶν καὶ μᾶς ἐμπεπτούσης ὡς μέρος ὃν τῆς β' ὑποδιαιρέσεως προγράφεται ἐν τούτοις πάντων. καὶ τὰ ἐξῆς ἐν τῷ πρώτῳ βιβλίῳ τὰ δὲ λειπόμενα δύο, τὸ δύο δοθεῖσθαι εὐθείαι καὶ κύκλοι ἢ τριῶν δοθέντων κύκλων μονοὶ ἐν τῷ δευτέρῳ βιβλίῳ διὰ τὰς πρὸς ἀλλήλους θέσεις τῶν κύκλων τε καὶ εὐθείαν πλείονας ὦσιν καὶ πλειόνων διορισμῶν δεομένας.

* Eucl. iv. 5 and 4.
* The last problem, to describe a circle touching three
this problem, according to the number of like or unlike entities in the hypotheses, there are bound to be, when the problem is subdivided, ten enunciations. For the number of different ways in which three entities can be taken out of the three unlike sets is ten. For the given entities must be (1) three points or (2) three straight lines or (3) two points and a straight line or (4) two straight lines and a point or (5) two points and a circle or (6) two circles and a point or (7) two straight lines and a circle or (8) two circles and a straight line or (9) a point and a straight line and a circle or (10) three circles. Of these, the first two cases are proved in the fourth book of the first Elements, for which reason they will not be described; for to describe a circle through three points, not being in a straight line, is the same thing as to circumscribe a given triangle, and to describe a circle to touch three given straight lines, not being parallel but meeting each other, is the same thing as to inscribe a circle in a given triangle; the case where two of the lines are parallel and one meets them is a subdivision of the second problem but is here given first place. The next six problems in order are investigated in the first book, while the remaining two, the case of two given straight lines and a circle and the case of three circles, are the sole subjects of the second book on account of the manifold positions of the circles and straight lines with respect one to another and the need for numerous investigations of the limits of possibility.

given circles, has been investigated by many famous geometers, including Newton (Arithmetica Universalis, Prob. 47). The lemmas given by Pappus enable Heath (H.G.M. ii. 182-185) to restore Apollonius’s solution—a “plane” solution depending only on the straight line and circle.
On Plane Loci

Oι μὲν οὖν ἄρχαιοι εἰς τὴν τῶν ἐπιπέδων τούτων τόπων τάξιν ἀποβλέποντες ἐστοιχείωσαν ὡς ἀμελησαντες οἱ μετ' αὐτῶν προσέθηκαν ἐτέρους, ὡς οὖν ἀπείρων τὸ πλήθος οὐντων, εἰ θέλοι τις προσγράφειν οὐ τῆς τάξεως ἐκείνης ἐχόμενα. Ὁtester οὖν τὰ μὲν προσκείμενα ὑστερα, τὰ δ' ἐκ τῆς τάξεως πρότερα μιὰ περιλαβῶν προτάσει ταῦτῃ.

Ἐὰν δύο εὐθείαι ἀκράως ἤτοι ἀπὸ ἕνος δεδομένου σημείου ἡ ἀπὸ δύο καὶ ἤτοι ἐπ' εὐθείας ἡ παράλληλαι ἡ δεδομένη περιέχουσα γωνίαν καὶ ἤτοι λόγον ἔχουσα πρὸς ἄλληλαις ἡ χωρίον περιέχουσι δεδομένου, ἀπηται δὲ τὸ τῆς μᾶς πέρας ἐπιπέδου τόπου θέσει δεδομένου, ἄφεται καὶ τὸ τῆς ἐτέρας πέρας ἐπιπέδου τόπου θέσει δεδομένου ὅτε μὲν τοῦ ὀμογενοῦς ὅτε δὲ τοῦ ἐτέρου, καὶ ὅτε μὲν ὀμοίως κειμένου πρὸς τὴν εὐθείαν, ὅτε δὲ ἐναντίως. ταῦτα δὲ γίνεται παρὰ τὰς διαφορὰς τῶν ὑποκειμένων.

On Vergings

Νεὺειν λέγεται γραμμή ἐπὶ σημείον, ἐὰν ἐπεκβαλλομένη ἐπ' αὐτὸ παραγίνηται [ ... ]

1 τούτων is attributed by Hultsch to dittography.

---

a These words follow the passage (quoted supra, pp. 262-265) wherein Pappus divides loci into ἐφεκτικοὶ, διεξοδικοὶ and ἀναστροφικοὶ.

b It is not clear what straight line is meant—probably the most obvious straight line in each figure.
APOLLONIUS OF PERGA

(vi.) On Plane Loci

Ibid. vii. 23, ed. Hultsch 662. 19-664. 7

The ancients had regard to the arrangement of these plane loci with a view to instruction in the elements; heedless of this consideration, their successors have added others, as though the number could not be infinitely increased if one were to make additions from outside that arrangement. Accordingly I shall set out the additions later, giving first those in the arrangement, and including them in this single enunciation:

If two straight lines be drawn, from one given point or from two, which are in a straight line or parallel or include a given angle, and either bear a given ratio one towards the other or contain a given rectangle, then, if the locus of the extremity of one of the lines be a plane locus given in position, the locus of the extremity of the other will also be a plane locus given in position, which will sometimes be of the same kind as the former, sometimes of a different kind, and will sometimes be similarly situated with respect to the straight line, sometimes contrariwise. These different cases arise according to the differences in the suppositions.

(vii.) On Vergings

Ibid. vii. 27-28, ed. Hultsch 670. 4-672. 3

A line is said to verge to a point if, when produced, it passes through the point. [ . . . ] The general

d Pappus proceeds to give seven other enunciations from the first book and eight from the second book. These have enabled reconstructions of the work to be made by Fermat, van Schooten and Robert Simson.

d Examples of vergings have already been encountered several times; v. pp. 186-189 and vol. i. p. 244 n. a.
GREEK MATHEMATICS

προβλήματος δὲ ὁντὸς καθολικοῦ τούτου· δύο δοθεισῶν γραμμῶν θέσει θείναι μεταξὺ τούτων εὐθείαν τῶν μεγέθει δεδομένην νεύουσαν ἐπὶ δοθέν σημείον, ἐπὶ τούτου τῶν ἐπὶ μέρους διάφορα τὰ ὑποκείμενα ἐχόντων, ἄ μὲν ἤν ἐπίπεδα, ἄ δὲ στερεά, ἄ δὲ γραμμικά, τῶν δ' ἐπιπέδων ἀποκληρώσαντες τὰ πρὸς πολλὰ χρησιμώτερα ἔδειξαν τὰ προβλήματα ταῦτα·

Θέσει δεδομένων ἡμικυκλίου τε καὶ εὐθείας πρὸς ὀρθὰς τῇ βάσει ἣ δύο ἡμικυκλίων ἐπὶ εὐθείας ἐχόντων τὰς βάσεις θείναι δοθέσαν τῶν μεγέθει εὐθείαν μεταξὺ τῶν δύο γραμμῶν νεύουσαν ἐπὶ γωνίαν ἡμικυκλίου·

Καὶ ρόμβου δοθέντος καὶ ἐπεκβεβλημένης μᾶς πλευράς ἀρμόσαι ὑπὸ τὴν ἑκτὸς γωνίαν δεδομένην τῷ μεγέθει εὐθείαν νεύουσαν ἐπὶ τῆν ἀντικρυ γωνίαν·

Καὶ θέσει δοθέντος κύκλου ἑναρμόσαι εὐθείαν μεγέθει δεδομένην νεύουσαν ἐπὶ δοθέν.

Τούτων δὲ ἐν μὲν τῷ πρῶτῳ τεύχει δεδεικται τὸ ἐπὶ τοῦ ἑνὸς ἡμικυκλίου καὶ εὐθείας ἠχον πτώσεις δ καὶ τὸ ἐπὶ τοῦ κύκλου ἠχον πτώσεις δύο καὶ τὸ ἐπὶ τοῦ ρόμβου πτώσεις ἠχον β' ἐν δὲ τῷ δευτέρῳ τεύχει τὸ ἐπὶ τῶν δύο ἡμικυκλίων τῆς ὑποθέσεως πτώσεις ἠχούσης ἦ, εν δὲ ταύτῃς ὑποδιαιρέσεις πλείονες διοριστικαὶ ἐνεκα τοῦ δεδομένου μεγέθους τῆς εὐθείας.
problem is: Two straight lines being given in position, to place between them a straight line of given length so as to verge to a given point. When it is subdivided the subordinate problems are, according to differences in the suppositions, sometimes plane, sometimes solid, sometimes linear. Among the plane problems, a selection was made of those more generally useful, and these problems have been proved:

Given a semicircle and a straight line perpendicular to the base, or two semicircles with their bases in a straight line, to place a straight line of given length between the two lines and verging to an angle of the semicircle [or of one of the semicircles];

Given a rhombus with one side produced, to insert a straight line of given length in the external angle so that it verges to the opposite angle;

Given a circle, to insert a chord of given length verging to a given point.

Of these, there are proved in the first book four cases of the problem of one semicircle and a straight line, two cases of the circle, and two cases of the rhombus; in the second book there are proved ten cases of the problem in which two semicircles are assumed, and in these there are numerous subdivisions concerned with limits of possibility according to the given length of the straight line.α

• A restoration of Apollonius’s work On Vergings has been attempted by several writers, most completely by Samuel Horsley (Oxford, 1770). A lemma by Pappus enables Apollonius’s construction in the case of the rhombus to be restored with certainty; v. Heath, H.G.M. ii. 190-192.
GREEK MATHEMATICS

(viii.) On the Dodecahedron and the Icosahedron


'O autos küklos perílamibánei to te tou dúdekaedrou pevntaívnon kai to tou eikosaédrou tríyvnon twv eis têh autêh sfaíran égygrafofménon. touto de gráfeita upo mév 'Aristaiou en tv epigrafofménw Tôw ë sχhmátwv suykrisiei, upo de 'Apolloínoun en têh deutéra èkdosei tês Sumykrísews touto dúdekaedrou proòs to eikosaédron, òti èstiv òs ë touto dúdekaedrou épifánêia proòs têh touto eikosaédrou épifáneian, ouitwv kai autò touto dúdekaedrou proòs touto eikosaédron dià touto têh autêh eínav kábetaon apa touto kêtrown tês sfaíras épi touto dúdekaedrou pevntaívnon kai touto touto eikosaédrou tríyvnon.

(ix.) Principles of Mathematics

Marin. in Eucl. Dat., Eucl. ed. Heiberg vi. 234. 13-17

Δiò touton áplouóstereôn1 kai mé tìni diaphora periáрейn touto dédoménon prothéménon oî mév teýagménov, òs 'Apolloínoun en têh Peri neúsêu kai

1 áplouóstereon Heiberg, áplouóstérwv cod.
(viii.) On the Dodecahedron and the Icosahedron

Hypsicles [Euclid, Elements xiv.],* Eucl. ed. Heiberg
v. 6. 19–8. 5

The pentagon of the dodecahedron and the triangle of the icosahedron \(^b\) inscribed in the same sphere can be included in the same circle. For this is proved by Aristaeus in the work which he wrote On the Comparison of the Five Figures,\(^c\) and it is proved by Apollonius in the second edition of his work On the Comparison of the Dodecahedron and the Icosahedron that the surface of the dodecahedron bears to the surface of the icosahedron the same ratio as the volume of the dodecahedron bears to the volume of the icosahedron, by reason of there being a common perpendicular from the centre of the sphere to the pentagon of the dodecahedron and the triangle of the icosahedron.

(ix.) Principles of Mathematics

Marinus, Commentary on Euclid’s Data, Eucl. ed.
Heiberg vi. 234. 13-17

Therefore, among those who made it their aim to define the datum more simply and with a single differentia, some called it the assigned, such as Apollonius in his book On Vergings and in his

\(^a\) The so-called fourteenth book of Euclid’s Elements is really the work of Hypsicles, for whom v. infra, pp. 394-397.

\(^b\) For the regular solids v. vol. i. pp. 216-225. The face of the dodecahedron is a pentagon and the face of the icosahedron a triangle.

\(^c\) A proof is given by Hypsicles as Prop. 2 of his book. Whether the Aristaeus is the same person as the author of the Solid Loci is not known.
GREEK MATHEMATICS

ἐν τῇ Καθόλου πραγμάτεια, οἱ δὲ γνώριμοι, ὡς Διόδορος.

(x.) On the Cochlias

Procl. in Eucl. i., ed. Friedlein 105. 1-6

Τὴν περὶ τῶν κύλινδρων ἔλικα γραφομένην, ὅταν εὐθεῖας κινούμενης περὶ τὴν ἐπιφάνειαν τοῦ κύλινδρου σημεῖον ὀμοσχῶς ἐπʼ αὐτῆς κινήται. γίνεται γὰρ ἐλξ, ἢς ὀμοιομερῶς πάντα τὰ μέρη πᾶσιν ἐφαρμόζει, καθάπερ Ἀπολλώνιος ἐν τῷ Περὶ τοῦ κοχλίου γράμματι δείκνυσιν.

(xi.) On Unordered Irrationals

Procl. in Eucl. i., ed. Friedlein 74. 23-24

Τὰ Περὶ τῶν ἀτάκτων ἄλογων, ἀὸ Ἀπολλώνιος ἐπὶ πλέον ἐξειργάσατο.

Schol. i. in Eucl. Elem. x., Eucl. ed. Heiberg v. 414. 10-16

Ἐν μὲν οὖν τοῖς πρώτοις περὶ συμμέτρων καὶ ἀσυμμέτρων διαλαμβάνει πρὸς τὴν φύσιν αὐτῶν αὐτὰ ἔξετάζων, ἐν δὲ τοῖς ἔξης περὶ δητῶν καὶ ἄλογων οὐ πασῶν· τινὲς γὰρ αὐτῷ ὡς ἐνιστάμενοι ἐγκαλοῦσιν· ἀλλὰ τῶν ἀπλούστατων εἰδῶν, ὡν

* Heath (H.G.M. ii. 192-193) conjectures that this work must have dealt with the fundamental principles of mathematics, and to it he assigns various remarks on such subjects attributed to Apollonius by Proclus, and in particular his attempts to prove the axioms. The different ways in which entities are said to be given are stated in the definitions quoted from Euclid’s Data in vol. i. pp. 478-479.
APOLLONIUS OF PERGA

General Treatise, others the known, such as Diodorus.

(x.) On the Cochlias

Proclus, On Euclid i., ed. Friedlein 105. 1-6

The cylindrical helix is described when a point moves uniformly along a straight line which itself moves round the surface of a cylinder. For in this way there is generated a helix which is homoeomeric, any part being such that it will coincide with any other part, as is shown by Apollonius in his work On the Cochlias.

(xii.) On Unordered Irrationals

Proclus, On Euclid i., ed. Friedlein 74. 23-24

The theory of unordered irrationals, which Apollonius fully investigated.

Euclid, Elements x., Scholium i., ed. Heiberg v. 414. 10-16

Therefore in the first [theorems of the tenth book] he treats of symmetrical and asymmetrical magnitudes, investigating them according to their nature, and in the succeeding theorems he deals with rational and irrational quantities, but not all, which is held up against him by certain detractors; for he dealt only with the simplest kinds, by the combination of which

* Possibly Diodorus of Alexandria, for whom v. vol. i. p. 300 and p. 301 n. b.
* In Studien über Euklid, p. 170, Heiberg conjectured that this scholium was extracted from Pappus's commentary, and he has established his conjecture in Videnskabernes Selskabs Skrifter, 6 Raekke, hist.-philos. Afd. ii. p. 236 seq. (1888).
GREEK MATHEMATICS

συντιθεμένων γίνονται ἀπειροὶ ἀλογοὶ, ὅν τινας καὶ ὁ Ἀπολλώνιος ἀναγράφει.

(xii.) Measurement of a Circle


'Ιστέον δὲ, ὃτι καὶ ὁ Ἀπολλώνιος ὁ Περγαῖος ἐν τῷ Ὑκτοκίῳ ἀπεδείξεν αὐτῷ δὲ ἀριθμῶν ἑτέρων ἐπὶ τὸ σύνεγγυς μᾶλλον ἀγαγῶν. τούτῳ δὲ ἀκριβέστερον μὲν εἶναι δοκεῖ, οὐ χρήσιμον δὲ πρὸς τὸν Ἅρχιμήδους σκοπὸν ἐφαμεν γὰρ αὐτὸν σκοπὸν ἔχειν ἐν τῷ δὲ τῷ βιβλίῳ τὸ σύνεγγυς εὑρεῖν διὰ τὰς ἐν τῷ βίῳ χρείας.

(xiii.) Continued Multiplications


Τούτου δὴ προτεθεωρημένου πρόδηλον, πῶς ἐστιν τὸν δοθέντα στίχον πολλαπλασιάσαι καὶ εἰπεῖν τὸν γενόμενον ἀριθμὸν ἐκ τοῦ τῶν πρῶτον ἀριθμῶν ὑπ᾽ ἐλιθη τὸ πρῶτον τῶν γραμμάτων ἐπὶ τὸν δεύτερον ἀριθμὸν ὑπ᾽ ἐλιθη τὸ δεύτερον τῶν γραμμάτων πολλαπλασιασθῆναι καὶ τὸν γενόμενον ἐπὶ τὸν τρίτον ἀριθμὸν ὑπ᾽ ἐλιθη τὸ τρίτον γράμμα

1 The extensive interpolations are omitted.

* Pappus’s commentary on Eucl. Elem. x. was discovered in an Arabic translation by Woepcke (Mémoires présentées par divers savants à l’Académie des sciences, 1856, xiv.). It contains several references to Apollonius’s work, of which one is thus translated by Woepcke (p. 693): “Enfin, Apollonius distinguait les espèces des irrationnelles ordonnées, et 352
an infinite number of irrationals are formed, of which latter Apollonius also describes some.\textsuperscript{a}

(xii.) \textit{Measurement of a Circle}

Eutocius, \textit{Commentary on Archimedes' Measurement of a Circle}, Archim. ed. Heiberg iii. 258. 16-22

It should be noticed, however, that Apollonius of Perga proved the same thing (\textit{sc.} the ratio of the circumference of a circle to the diameter) in the \textit{Quick-deliverer} by a different calculation leading to a closer approximation. This appears to be more accurate, but it is of no use for Archimedes' purpose; for we have stated that his purpose in this book was to find an approximation suitable for the everyday needs of life.\textsuperscript{b}

(xiii.) \textit{Continued Multiplications} \textsuperscript{c}

Pappus, \textit{Collection} ii. 17-21, ed. Hultsch 18. 23-24. 20\textsuperscript{d}

This theorem having first been proved, it is clear how to multiply together a given verse and to tell the number which results when the number represented by the first letter is multiplied into the number represented by the second letter and the product is multiplied into the number represented by the third découvrit la science des quantités appelées (irrationnelles) inordonnées, dont il produisit un très-grand nombre par des méthodes exactes.”

\textsuperscript{a} We do not know what the approximation was.

\textsuperscript{b} Heiberg (Apollon. Perg. ed. Heiberg ii. 124, n. 1) suggests that these calculations were contained in the 'Ωκτόκιον', but there is no definite evidence.

\textsuperscript{c} The passages, chiefly detailed calculations, adjudged by Hultsch to be interpolations are omitted.
GREEK MATHEMATICS

καὶ κατὰ τὸ ἐξῆς περαινεσθαι μεχρὶ τοῦ διεξο- 
δεύεσθαι τὸν στίχον, διν ἑπεν Ἀπολλώνιος ἐν 
ἀρχῇ οὖτως:

'Αρτέμιδος κλείτε κράτος ἐξοχον ἐννέα κοῦραι
(τὸ δὲ κλείτε φησιν ἀντὶ τοῦ ὑπομνήσατε).

'Επεὶ οὖν γράμματά ἐστιν θη τοῦ στίχου, τάντα 
δὲ περιέχει ἀριθμοὺς δέκα τοὺς ἰ ἴ ἴ ἰ ἵ ἵ ἴ ἵ, 
ἐκαστὸς ἐλάσσων μὲν ἐστὶν χιλιάδος μετρεῖται 
δὲ ὑπὸ ἐκατοντάδος, καὶ ἀριθμοὺς ἦ ὁ τοὺς ἰ ἴ ἵ ἵ ἵ 
λ ἵ ἵ ἵ ἵ ἵ ἵ ἵ ἵ, ὡς ἐκαστὸς ἐλάσσων μὲν 
ἐστὶν ἐκατοντάδος μετρεῖται δὲ ὑπὸ δεκάδος, καὶ 
tous λοιποὺς ἦ τοὺς ἵ ἵ ἵ ἵ ἵ ἵ ἵ ἵ. ὡς 
ἐκαστὸς ἐλάσσων δεκάδος, ἐὰν ἀρα τοῖς μὲν δέκα 
ἀριθμοῖς ὑποτάξωμεν ἰσαρίθμους δέκα κατὰ τάξιν 
ἐκατοντάδος, τοῖς δὲ ἦ ὑποτάξωμεν 
δεκάδος ἦ, φανερὸν ἐκ τοῦ ἀνώτερον λογιστικοῦ 
θεωρήματος ιΒ' ὅτι δέκα ἐκατοντάδες μετὰ τῶν 
郐 δεκάδων ποιοῦσι μυριάδας ἐνναπλᾶς δέκα.

'Επεὶ δὲ καὶ πυθμένες ὠμοῦ τῶν μετρομένων 
ἀριθμῶν ὑπὸ ἐκατοντάδος καὶ τῶν μετρομένων 
ὑπὸ δεκάδος εἰσίν ὡς ὑποκείμενοι θὲ

\[\begin{align*}
\alpha & \gamma \beta \gamma \alpha \gamma \beta \zeta \delta \alpha \\
\delta \alpha & \zeta \beta \gamma \alpha \beta \zeta \zeta \zeta \epsilon \epsilon \epsilon \beta \zeta \alpha,
\end{align*}\]

* Apollonius, it is clear from Pappus, had a system of tetrads for calculations involving big numbers, the unit being the myriad or fourth power of 10. The tetrads are called μυριάδες ἀπλαῖ, μυριάδες διπλαῖ, μυριάδες τριπλαῖ, simple myriads, double myriads, triple myriads and so on, by which are meant 10000, 10000, \(10000^3\) and so on. In the text of 354.
letter and so on in order until the end of the verse which Apollonius gave in the beginning, that is

'Αρτέμιδος κλείτε κράτος ἐξοχον ἐννέα κοῦραν

(where he says κλείτε for ὑπομνήσατε, recall to mind).

Since there are thirty-eight letters in the verse, of which ten, namely ῥ ῥ ῥ ῥ χ ν ρ (= 100, 300, 200 300, 100, 300, 200, 600, 400, 100), represent numbers less than 1000 and divisible by 100, and seventeen, namely μ ι ο ι ι ι ο ε ο ο ν ν ν ν ν ν ν ν ν ν ν κ δ (= 40, 10, 70, 20, 30, 10, 20, 70, 60, 70, 70, 50, 50, 50, 20, 70, 10), represent numbers less than 100 and divisible by 10, while the remaining eleven, namely, α ε δ ε ε α ε ε ε ε ε ε α α (= 1, 5, 4, 5, 5, 1, 5, 5, 1, 1), represent numbers less than 10, then if for those ten numbers we substitute an equal number of hundreds, and if for the seventeen numbers we similarly substitute seventeen tens, it is clear from the above arithmetical theorem, the twelfth, that the ten hundreds together with the seventeen tens make 10,10000.

And since the bases of the numbers divisible by 100 and those divisible by 10 are the following twenty-seven

1, 3, 2, 3, 1, 3, 2, 6, 4, 1
4, 1, 7, 2, 3, 1, 2, 7, 6, 7, 7, 5, 5, 5, 2, 7, 1,

Pappus they are sometimes abbreviated to μ, μ, μγ and so on.

From Pappus, though the text is defective, Apollonius's procedure in multiplying together powers of 10 can be seen to be equivalent to adding the indices of the separate powers of 10, and then dividing by 4 to obtain the power of the myriad which the product contains. If the division is exact, the number is the n-myriad, say, meaning 10000n. If there is a remainder, 8, 2 or 1, the number is 1000, 100 or 10 times the n-myriad as the case may be.
GREEK MATHEMATICS

(xiv.) On the Burning Mirror

Fragmentum mathematicum Bobiense 113. 28-33, ed. Belger, Hermes, xvi., 1881, 279-280

Oi μὲν οὖν παλαιοὶ ὑπέλαβον τὴν ἐξαίφνη ποιεῖσθαι περὶ τὸ κέντρον τοῦ κατόπτρου, τούτο δὲ ψεύδος Ἀπολλώνιος μάλα δεόντως . . . (ἐν τῷ) πρὸς τοὺς κατοπτρικοὺς ἐδειξεν, καὶ περὶ τίνα δὲ τόπον ἡ ἐκπύρωσις ἔσται, διασεσάφηκεν ἐν τῷ Περὶ τοῦ πυρίου.

1 As amended by Heiberg, Zeitschrift für Mathematik und Physik, xxviii., 1883, hist. Abth. 124-125.
while there are eleven less than ten, that is the numbers

\[ 1, 5, 4, 5, 5, 1, 5, 5, 5, 1, 1, \]

if we multiply together the solid number formed by these eleven with the solid number formed by the twenty-seven the result will be the solid number

\[ 19 \cdot 10000^4 + 6036 \cdot 10000^3 + 8480 \cdot 10000^2. \]

When these numbers are multiplied into the solid number formed by the hundreds and the tens, that is with 10 \cdot 10000^9 as calculated above, the result is

\[ 196 \cdot 10000^{13} + 368 \cdot 10000^{12} + 4800 \cdot 10000^{11}. \]

(xiv.) *On the Burning Mirror*

*Fragmentum mathematicum Bobiense* 113. 28-33,\(^a\) ed. Belger, *Hermes*, xvi., 1881, 279-280

The older geometers thought that the burning took place at the centre of the mirror, but Apollonius very suitably showed this to be false . . . in his work on mirrors, and he explained clearly where the kindling takes place in his works *On the Burning Mirror*.\(^b\)

\(^a\) This fragment is attributed to Anthemius by Heiberg, but its antiquated terminology leads Heath (*H.G.M.* ii. 194) to suppose that it is much earlier.

\(^b\) Of Apollonius's other achievements, his solution of the problem of finding two mean proportionals has already been mentioned (vol. i. p. 267 n. b) and sufficiently indicated; for his astronomical work the reader is referred to Heath, *H.G.M.* ii. 195-196.
XX. LATER DEVELOPMENTS IN GEOMETRY
XX. LATER DEVELOPMENTS IN GEOMETRY

(a) Classification of Curves

Procl. in Eucl. i., ed. Friedlein 111. 1–112. 11

Διαιρεῖ δ' αὖ τὴν γραμμὴν ὁ Γέμινος¹ πρῶτον μὲν εἰς τὴν ἁσύνθετον καὶ τὴν σύνθετον—καλεῖ δὲ σύνθετον τὴν κεκλασμένην καὶ γωνίαν ποιοῦσαν—ἐπεὶτα τὴν ἁσύνθετον² εἰς τε τὴν σχηματοποιοῦσαν καὶ τὴν ἐπ' ἀπειρον ἐκβαλλομένην, σχῆμα λέγων ποιεῖν τὴν κυκλικὴν, τὴν τοῦ θυρεοῦ, τὴν κιττοειδῆ, μὴ ποιεῖν δὲ τὴν τοῦ ὀρθογωνίου κώνου τομὴν, τὴν τοῦ ἀμβλυγωνίου, τὴν κοινχειδῆ, τὴν εὐθεῖαν, πάσας τὰς τουαύτας. καὶ πάλιν κατ' ἄλλον τρόπον τῆς ἁσύνθετου γραμμῆς τὴν μὲν ἀπλὴν εἶναι, τὴν δὲ μικτὴν, καὶ τῆς ἀπλῆς τὴν μὲν σχῆμα ποιεῖν ὡς τὴν κυκλικὴν, τὴν δὲ ἀὁρίστον εἶναι ὡς τὴν εὐθείαν, τῆς δὲ μικτῆς τὴν μὲν ἐν τοῖς ἐπιπέδοις εἶναι, τὴν δὲ ἐν τοῖς στερεοῖς, καὶ τῆς ἐν ἐπιπέδοις τὴν μὲν ἐν αὐτῇ συμπίπτειν ὡς τὴν κιττοειδῆ, τὴν δ' ἐπ' ἀπειρον ἐκβάλλεσθαι, τῆς δὲ ἐν στερεοῖς

¹ Γέμινος Tittel, Γεμίνος Friedlein.
² σύνθετον codd., correxí.

* No great new developments in geometry were made by the Greeks after the death of Apollonius, probably through 360
XX. LATER DEVELOPMENTS IN GEOMETRY

(a) CLASSIFICATION OF CURVES

Proclus, On Euclid i., ed. Friedlein 111. 1–112. 11

Geminus first divides lines into the *incomposite* and the *composite*, meaning by composite the broken line forming an angle; and then he divides the incomposite into those *forming a figure* and those *extending without limit*, including among those forming a figure the circle, the ellipse and the cissoid, and among those not forming a figure the parabola, the hyperbola, the conchoid, the straight line, and all such lines. Again, in another manner he says that some incomposite lines are *simple*, others *mixed*, and among the simple are some *forming a figure*, such as the circle, and others *indeterminate*, such as the straight line, while the mixed include both *lines on planes* and *lines on solids*, and among the lines on planes are lines *meeting themselves*, such as the cissoid, and others *extending without limit*, and among lines on solids are the limits imposed by their methods, and the recorded additions to the corpus of Greek mathematics may be described as reflections upon existing work or "stock-taking." On the basis of geometry, however, the new sciences of trigonometry and mensuration were founded, as will be described, and the revival of geometry by Pappus will also be reserved for separate treatment.

361
GREEK MATHEMATICS

tην μὲν κατὰ τὰς τομὰς ἐπιστήμου στερεῶν, τὴν δὲ περὶ τὰ στερεὰ ὑφιστάσθαι. τὴν μὲν γὰρ ἐλικά τὴν περὶ σφαίραν ἡ κῶνον περὶ τὰ στερεὰ ὑφεστάναι, τὰς δὲ κωνικὰς τομὰς ἡ τὰς σπειρικὰς ἀπὸ τοιάσθε τομῆς γεννᾶσθαι τῶν στερεῶν. ἐπὶνενοῆσθαι δὲ ταύτας τὰς τομὰς τὰς μὲν ὑπὸ Μεναιχμον τὰς κωνικὰς, ὁ καὶ Ἐρατοσθένης ἱστορῶν λέγει: "μὴ δὲ Μεναιχμῖος κωνοτομεῖν τριάδας". τὰς δὲ ὑπὸ Περσεώς, ὃς καὶ τὸ ἐπίγραμμα ἐποίησεν ἐπὶ τῇ εὐρέσει—

Τρεῖς γραμμὰς ἐπὶ πέντε τομαῖς εὑρὼν ἐλικώδεις
Περσεύς τῶν δ’ ἔνεκεν δαίμονας ἰλάσατο.

αἱ μὲν δὴ τρεῖς τομαὶ τῶν κώνων εἰσὶν παραβολὴ καὶ ὑπερβολὴ καὶ ἐλλεψις, τῶν δὲ σπειρικῶν τομῶν ἡ μὲν ἔστων ἐμπεπλεγμένη, ἐοικύα τῇ τοῦ ἱπποῦ πέδη, ἡ δὲ κατὰ τὰ μέσα πλατύνεται, ἐξ ἐκατέρω δὲ ἀπολήγει μέρους, ἡ δὲ παραμῆκης οὔσα τῷ μὲν μέσῳ διαστήματι ἐλάττονι χρήσθη, εὑρόνεται δὲ ἐφ’ ἐκάτερα. τῶν δὲ ἅλλων μίζουν τὸ πλῆθος ἀπέραντον ἔστων καὶ γὰρ στερεῶν σχημάτων πλῆθός ἔστων ἀπειρον καὶ τομαὶ αὐτῶν συνίστανται πολυειδεῖς.

Ibid., ed. Friedlein 356. 8-12

Καὶ γὰρ Ἀπολλώνιος ἐφ’ ἐκάστης τῶν κωνικῶν γραμμῶν τί τὸ σύμπτωμα δείκνυσι, καὶ ὁ Νικομήδης ἐπὶ τῶν κογχοειδῶν, καὶ ὁ Ἰππίας ἐπὶ

1 ἐλικώδεις Knoche, εὑρὼν τὰς σπειρικὰς λέγων codd.
2 v. vol. i. pp. 296-297.
LATER DEVELOPMENTS IN GEOMETRY

lines conceived as formed by sections of the solids and lines formed round the solids. The helix round the sphere or cone is an example of the lines formed round solids, and the conic sections or the spiric curves are generated by various sections of solids. Of these sections, the conic sections were discovered by Menaechmus, and Eratosthenes in his account says: "Cut not the cone in the triads of Menaechmus"; and the others were discovered by Perseus, who wrote an epigram on the discovery—

Three spiric lines upon five sections finding,
Perseus thanked the gods therefor.

Now the three conic sections are the parabola, the hyperbola and the ellipse, while of the spiric sections one is interlaced, resembling the horse-fetter, another is widened out in the middle and contracts on each side, a third is elongated and is narrower in the middle, broadening out on either side. The number of the other mixed lines is unlimited; for the number of solid figures is infinite and there are many different kinds of section of them.

Ibid., ed. Friedlein 356. 8-12

For Apollonius shows for each of the conic curves what is its property, as does Nicomedes for the
GREEK MATHEMATICS

τῶν τετραγωνιζομένων, καὶ ὁ Περσεύς ἐπὶ τῶν σπειρικῶν.

Ibid., ed. Friedlein 119. 8-17

"Ὁ δὲ συμβαίνειν φαμέν κατὰ τὴν σπειρικῆν ἐπιφάνειαν. κατὰ γὰρ κύκλου νοεῖται στροφὴν ὅρθον διαμένοντος καὶ στρεφομένου περὶ τὸ αὐτὸ σημεῖον, ὃ μὴ ἐστι κέντρον τοῦ κύκλου, διὸ καὶ τριχῶς ἡ σπείρα γίνεται, ἡ γὰρ ἐπὶ τῆς περιφερείας ἐστὶ τὸ κέντρον ἡ ἕντος ἡ ἔκτος. καὶ εἰ μὲν ἐπὶ τῆς περιφερείας ἐστὶ τὸ κέντρον, γίνεται σπείρα συνεχῆς, εἰ δὲ ἕντος, ἡ ἐμπεπλεγμένη, εἰ δὲ ἔκτος, ἡ διεχής. καὶ τρεῖς αἱ σπειρικαὶ τομαὶ κατὰ τὰς τρεῖς ταύτας διαφοράς.

a Obviously the work of Perseus was on a substantial scale to be associated with these names, but nothing is known of him beyond these two references. He presumably flourished after Euclid (since the conic sections were probably well developed before the spiric sections were tackled) and before Geminus (since Proclus relies on Geminus for his knowledge of the spiric curves). He may therefore be placed between 300 and 75 B.C.

Nicomedes appears to have flourished between Eratosthenes and Apollonius. He is known only as the inventor of the conchoid, which has already been fully described (vol. i. pp. 298-309).

It is convenient to recall here that about a century later flourished Diocles, whose discovery of the cissoid has already been sufficiently noted (vol. i. pp. 270-279). He has also been referred to as the author of a brilliant solution of the problem of dividing a cone in a given ratio, which is equivalent to the solution of a cubic equation (supra, p. 162 n. a). The Dionysodorus who solved the same problem (ibid.) may have been the Dionysodorus of Caunus mentioned in the Herculaneum Roll, No. 1044 (so W. Schmidt in Bibliotheca mathematica, iv. pp. 321-325), a younger contemporary of Apollonius; he is presumably the same person as the 364.
LATER DEVELOPMENTS IN GEOMETRY

conchoid and Hippias for the quadratics and Perseus for the spiric curves.a

Ibid., ed. Friedlein 119. 8-17

We say that this is the case with the spiric surface; for it is conceived as generated by the revolution of a circle remaining perpendicular [to a given plane] and turning about a fixed point which is not its centre. Hence there are three forms of spire according as the centre is on the circumference, or within it, or without. If the centre is on the circumference, the spire generated is said to be continuous, if within interlaced, and if without open. And there are three spiric sections according to these three differences.’’

Dionysodorus mentioned by Heron, Metrica ii. 13 (cited infra, p. 481), as the author of a book On the Spire.

This last sentence is believed to be a slip, perhaps due to too hurried transcription from Geminus. At any rate, no satisfactory meaning can be obtained from the sentence as it stands. Tannery (Mémoires scientifiques ii. pp. 24-28) interprets Perseus’ epigram as meaning “three curves in addition to five sections.” He explains the passages thus: Let $a$ be the radius of the generating circle, $c$ the distance of the centre of the generating circle from the axis of revolution, $d$ the perpendicular distance of the plane of section (assumed to be parallel to the axis of revolution) from the axis of revolution. Then in the open spire, in which $c > a$, there are five different cases:

1. $c + a > d > c$. The curve is an oval.
2. $d = c$. Transition to (3).
3. $c > d > c - a$. The curve is a closed curve narrowest in the middle.
4. $d = c - a$. The curve is the hippocede (horse-fetter), which is shaped like the figure of 8 (v. vol. i. pp. 414-415 for the use of this curve by Eudoxus).
5. $c - a > d > 0$. The section consists of two symmetrical ovals.

Tannery identifies the “five sections” of Perseus with these five types of section of the open spire; the three curves...
Attempts to Prove the Parallel Postulate

(i.) General

Procl. in Eucl. i., ed. Friedlein 191. 16–193. 9

"Καί εάν εἰς δύο εὐθείας εὐθεία ἐμπλήττουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάττωνας ποιή, ἐκβαλλομένας τὰς εὐθείας ἐπ’ ἀπειρον συμπλήττειν, ἐφ’ ἂ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάττονες."

Τούτῳ καὶ παντελῶς διαγράφειν χρή τῶν αἰτημάτων: θεώρημα γὰρ ἐστι, πολλὰς μὲν ἀπορίας επιδεχόμενον, ὅς καὶ ὁ Πτολεμαῖος ἐν τοῖς βιβλίοις διαλύσας προὕθετο, πολλῶν δὲ εἰς ἀπόδειξιν δεόμενον καὶ ὀρῶν καὶ θεωρημάτων. καὶ τὸ γε ἀντιστρέφον καὶ ὁ Εὐκλείδης ὃς θεώρημα δείκνυσιν. ἦς δὲ ἂν τινες ἀπατώμενοι καὶ τούτῳ τάττειν ἐν τοῖς αἰτήμασιν ἀξιώσειαν, ὡς διὰ τὴν ἐλάττωσιν τῶν δύο ὀρθῶν αὐτόθεν τὴν πίστιν παρεχόμενον

described by Proclus are (1), (3) and (4). When the spire is continuous or closed, \( c = a \) and there are only three sections corresponding to (1), (2) and (3); (4) and (5) reduce to two equal circles touching one another. But the interlaced spire, in which \( c < a \), gives three new types of section, and in these Tannery sees his "three curves in addition to five sections." There are difficulties in the way of accepting this interpretation, but no better has been proposed.

Further passages on the spire by Heron, including a formula for its volume, are given infra, pp. 476-483.

* Eucl. i. Post. 5, for which v. vol. i. pp. 442-443, especially n. c.

Aristotle (Anal. Prior. ii. 16, 65 a 4) alludes to a petitio principii current in his day among those who "think they establish the theory of parallels"—τὰς παραλλήλους γράφειν. As Heath notes (The Thirteen Books of Euclid's Elements, 366
(b) Attempts to Prove the Parallel Postulate

(i.) General

Proclus, On Euclid i., ed. Friedlein 191. 16–193. 9

"If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles."  

This ought to be struck right out of the Postulates; for it is a theorem, and one involving many difficulties, which Ptolemy set himself to resolve in one of his books, and for its proof it needs a number of definitions as well as theorems. Euclid actually proves its converse as a theorem. Possibly some would erroneously consider it right to place this assumption among the Postulates, arguing that, as the angles are less than two right angles, there is vol. i. pp. 191-192), Philoponus's comment on this passage suggests that the petitio principii lay in a direction theory of parallels. Euclid appears to have admitted the validity of the criticism and, by assuming his famous postulate once and for all, to have countered any logical objections.

Nevertheless, as the extracts here given will show, ancient geometers were not prepared to accept the undemonstrable character of the postulate. Attempts to prove it continued to be made until recent times, and are summarized by R. Bonola, "Sulla teoria delle parallele e sulle geometrie non-euclidee " in Questioni riguardanti la geometria elementare, and by Heath, loc. cit., pp. 204-219. The chapter on the subject in W. Rouse Ball's Mathematical Essays and Recreations, pp. 307-326, may also be read with profit. Attempts to prove the postulate were abandoned only when it was shown that, by not conceding it, alternative geometries could be built.
GREEK MATHEMATICS

tῆς τῶν εὑθείων συνεύσεως καὶ συμπτώσεως. πρὸς οὖς ὁ Γεμῖνος ὤρθώς ἀπήντησε λέγων ὁτι
παρ' αὐτῶν ἐμάθομεν τῶν τῆς ἐπιστήμης ταύτης ήγεμόνων μὴ πάνυ προσέχειν τῶν νοῦν ταῖς
πιθαναίς φαντασίαις εἰς τὴν τῶν λόγων τῶν ἐν
γεωμετρίᾳ παραδοχήν. ὁμοιον γὰρ φησι καὶ
Ἀριστοτέλης ἰητορικοῦ ἀποδείξεις ἀπαιτεῖν καὶ
gεωμέτρου πιθανολογοῦντος ἀνέχεσθαι, καὶ ὁ παρὰ
tῷ Πλάτωνι Συμμίας, ὃτι "τοῖς ἐκ τῶν εἰκότων
tὰς ἀποδείξεις ποιομένοις σύνοιδα οὖν ἀλαζόσι." κανταῦθα τοῖνυν τὸ μὲν ἡλαττωμένων τῶν ὀρθῶν
συνεύει τὰς εὐθείας ἀλήθεις καὶ ἀναγκαῖον, τὸ
δὲ συνευούσας ἐπὶ πλέον ἐν τῷ ἕκβαλλεσθαι συμ-
πεσείθαι ποτε πιθανὸν, ἄλλ' οὐκ ἀναγκαῖον, εἰ
μὴ τὶς ἀποδείξειν λόγος, ὃτι ἐπὶ τῶν εὐθείων
τοῦτο ἀληθές. τὸ γὰρ εἰναὶ τινας γραμμὰς
συνιούσας μὲν ἐπὶ ἀπειρον, ἀσυμπτώτους δὲ
ὑπαρχοῦσας, καίτοι δοκοῦν ἀπίθανον εἰναι καὶ
παράδοξον, ὃμως ἀληθές ἐστι καὶ πεφώραται ἐπ'
ἀλλων εἰδῶν τῆς γραφῆς. μὴποτε οὖν τοῦτο
καὶ ἐπὶ τῶν εὐθειῶν δυνατόν, ὅπερ ἐπὶ ἐκείνων
tῶν γραμμῶν; ἔως γὰρ ἂν δὴ ἀποδείξεως αὐτὸ
kαταδησώμεθα, περισσά τὴν φαντασίαν τὰ ἐπ'
ἀλλων δεικνύμενα γραμμῶν. εἰ δὲ καὶ οἱ διαμφισ-
βητοῦντες λόγοι πρὸς τὴν σύμπτωσιν πολὺ τὸ
πληκτικὸν ἔχουν, πῶς οὐχὶ πολλῷ πλέον ἂν τὸ
πιθανὸν τοῦτο καὶ τὸ ἄλογον ἐκβάλλουμεν τῆς
ἡμετέρας παραδοχῆς;
'Αλλ' ὃτι μὲν ἀπόδειξιν χρῆ ἦτείν τοῦ προ-
κειμένου θεωρήματος δὴλον ἐκ τούτων, καὶ ὃτι

* For Geminus, v. infra, p. 370 n. c.

368
immediate reason for believing that the straight lines converge and meet. To such, Geminus \(^a\) rightly rejoined that we have learnt from the pioneers of this science not to incline our mind to mere plausible imaginings when it is a question of the arguments to be used in geometry. For Aristotle \(^b\) says it is as reasonable to demand scientific proof from a rhetorician as to accept mere plausibilities from a geometer, and Simmias is made to say by Plato \(^c\) that he "recognizes as quacks those who base their proofs on probabilities." In this case the convergence of the straight lines by reason of the lessening of the right angles is true and necessary, but the statement that, since they converge more and more as they are produced, they will some time meet is plausible but not necessary, unless some argument is produced to show that this is true in the case of straight lines. For the fact that there are certain lines which converge indefinitely but remain non-secant, although it seems improbable and paradoxical, is nevertheless true and well-established in the case of other species of lines. May not this same thing be possible in the case of straight lines as happens in the case of those other lines? For until it is established by rigid proof, the facts shown in the case of other lines may turn our minds the other way. And though the controversial arguments against the meeting of the two lines should contain much that is surprising, is that not all the more reason for expelling this merely plausible and irrational assumption from our accepted teaching?

It is clear that a proof of the theorem in question must be sought, and that it is alien to the special

\(^a\) Eth. Nic. i. 3. 4, 1094 b 25-27. \(^b\) Phaedo 92 d.
GREEK MATHEMATICS

τῆς τῶν αὐτημάτων ἦστιν ἄλλοτριον ἰδιότητος; πῶς δὲ ἀποδεικτέον αὐτῷ καὶ διὰ ποίων λόγων ἀναρτέον τὰς πρὸς αὐτὸ φερομένας ἐννοσάεις, τηνικαύτα λεκτέον, ἢν καὶ ὁ στοιχεωτής αὐτοῦ μέλλῃ ποιεῖσθαι μνήμην ὡς ἑναργεῖ προσ-χρώμενος. τότε γὰρ ἀναγκαῖον αὐτοῦ δεῖξαι τὴν ἑναργείαν οὐκ ἀναποδείκτως προφαινομένην ἀλλὰ δι᾽ ἀποδείξεων γνώριμον γνωρωμένην.

(ii.) Posidonius and Geminus

Ibid., ed. Friedlein 176. 5-10

Καὶ ὁ μὲν Εὐκλείδης τοῦτον ὁρίζεται τὸν τρόπον τὰς παραλλήλους εὐθείας, ὁ δὲ Ποσειδώνως, παράλληλοι, φησίν, εἰσιν αἱ μῆτε συνεύουσαι μῆτε ἀπονεύουσαι ἐν ἑνὶ ἐπιπέδῳ, ἀλλ’ ἵσας ἔχουσαι

a i.e., Eucl. i. 28.

b Posidonius was a Stoic and the teacher of Cicero; he was born at Apamea and taught at Rhodes, flourishing 151-135 B.C. He contributed a number of definitions to elementary geometry, as we know from Proclus, but is more famous for a geographical work On the Ocean (lost but copiously quoted by Strabo) and for an astronomical work Περὶ μετεώρων. In this he estimated the circumference of the earth (v. supra, p. 267) and he also wrote a separate work on the size of the sun.

c As with so many of the great mathematicians of antiquity, we know practically nothing about Geminus’s life, not even his date, birthplace or the correct spelling of his name. As he wrote a commentary on Posidonius’s Περὶ μετεώρων, we have an upper limit for his date, and “the view most generally accepted is that he was a Stoic philosopher, born probably in the island of Rhodes, and a pupil of Posidonius, and that he wrote about 73-67 B.C.” (Heath, H.G.M. ii. 223). Further details may be found in Manitius’s edition of the so-called Gemini elementa astronomiae.

Geminus wrote an encyclopaedic work on mathematics
character of the Postulates. But how it should be proved, and by what sort of arguments the objections made against it may be removed, must be stated at the point where the writer of the *Elements* is about to recall it and to use it as obvious. Then it will be necessary to prove that its obvious character does not appear independently of proof, but by proof is made a matter of knowledge.

(ii.) *Posidonius* b and *Geminus* c

*Ibid.*, ed. Friedlein 176. 5-10

Such is the manner in which Euclid defines parallel straight lines, but Posidonius says that parallels are lines in one plane which neither converge nor diverge which is referred to by ancient writers under various names, but that used by Eutocius (Τῶν μαθημάτων θεωρία, v. *supra*, pp. 280-281) was most probably the actual title. It is unfortunately no longer extant, but frequent references are made to it by Proclus, and long extracts are preserved in an Arabic commentary by an-Nairizī.

It is from this commentary that Geminus is known to have attempted to prove the parallel-postulate by a definition of parallels similar to that of Posidonius. The method is reproduced in Heath, *H.G.M.* ii. 228-230. It tacitly assumes “Playfair’s axiom,” that through a given point only one parallel can be drawn to a given straight line; this axiom—which was explicitly stated by Proclus in his commentary on Eucl. i. 30 (Procl. in *Eucl.* i., ed. Friedlein 374. 18-375. 3)—is, in fact, equivalent to Euclid’s Postulate 5. Saccheri noted an even more fundamental objection, that, before Geminus’s definition of parallels can be used, it has to be proved that the locus of points equidistant from a straight line is a straight line; and this cannot be done without some equivalent postulate. Nevertheless, Geminus deserves to be held in honour as the author of the first known attempt to prove the parallel-postulate, a worthy predecessor to Lobachewsky and Riemann.
πάσας τὰς καθέτους τὰς ἀγομένας ἀπὸ τῶν τῆς ἐτέρας σημείων ἐπὶ τὴν λοιπῆν.

(iii.) Ptolemy

Ibid., ed. Friedlein 362. 12–363. 18

'Ἀλλ' ὅπως μὲν ὁ Στοιχεωτής δείκνυσιν ὅτι δύο ὀρθαῖς ἵσων οὐσῶν τῶν ἐντὸς αἱ εὐθείαι παράλληλοι εἰσι, φανερὸν ἐκ τῶν γεγραμμένων. Πτολεμαῖος δὲ ἐν οἷς ἀποδείξαι προέθετο τὰς ἀπ' ἐλαττόνων ἡ δύο ὀρθῶν ἐκβαλλομένας συμπίπτειν, ἐφ' ἄ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἀλάσσονες, τούτῳ πρὸ πάντων δείκνυσ τὸ θεώρημα τὸ δυεῖν ὀρθαῖς ἵσων ὑπαρχοῦσῶν τῶν ἐντὸς παράλληλοις εἶναι τὰς εὐθείας οὕτω πως δείκνυσιν.

'Εστωσαν δύο εὐθείας αἱ ΑΒ, ΓΔ, καὶ τεμνέτω τὶς αὐτὰς εὐθεία ἡ ΕΖΗΘ, ὅπερ τὰς ὑπὸ ΒΖΗ καὶ ὑπὸ ΖΗΔ γωνίας δύο ὀρθαῖς ἴσας ποιεῖν. λέγω ὅτι παράλληλοί εἰσιν αἱ εὐθείαι, τούτων 372
LATER DEVELOPMENTS IN GEOMETRY

but the perpendiculairs drawn from points on one of
the lines to the other are all equal.

(iii.) Ptolemy *

Ibid., ed. Friedlein 362. 12–363. 18

How the writer of the Elements proves that, if the
interior angles be equal to two right angles, the
straight lines are parallel is clear from what has been
written. But Ptolemy, in the work b in which he
attempted to prove that straight lines produced from
angles less than two right angles will meet on the
side on which the angles are less than two right
angles, first proved this theorem, that if the interior
angles be equal to two right angles the lines are parallel,
and he proves it somewhat after this fashion.

Let the two straight lines be AB, ΠΔ, and let any
straight line EZHΘ cut them so as to make the angles
BZH and ZHΔ equal to two right angles. I say that
the straight lines are parallel, that is they are non-

* For the few details known about Ptolemy, v. infra,
p. 408 and n. b.

b This work is not otherwise known.

VOL. n 373
There is a Common Notion to this effect interpolated in the text of Euclid; v. vol. i. pp. 444 and 445 n. a.

The argument would have been clearer if it had been proved that the two interior angles on one side of ZH were severally equal to the two interior angles on the other side, that is BZH = GHZ and ΔHZ = AZH; whence, if ZA, HG meet at A, the triangle ZHA can be rotated about the mid-

* 374
secant. For, if it be possible, let $BZ$, $HA$, when pro-
duced, meet at $K$. Then since the straight line $HZ$
stands on $AB$, it makes the angles $AZH$, $BZH$ equal
to two right angles [Eucl. i. 13]. Similarly, since
$HZ$ stands on $GA$, it makes the angles $GHA$, $AHZ$ equa-
to two right angles [ibid.]. Therefore the four
angles $AZH$, $BZH$, $GHA$, $AHZ$ are equal to four right
angles, and of them two, $BZH$, $ZH\Delta$, are by hypo-
thesis equal to two right angles. Therefore the
remaining angles $AZH$, $GHA$ are also themselves
equal to two right angles. If then, the interior
angles being equal to two right angles, $ZB$, $HA$ meet
at $K$ when produced, $ZA$, $HA$ will also meet when
produced. For the angles $AZH$, $GHA$ are also equal
to two right angles. Therefore the straight lines will
either meet on both sides or on neither, since these
angles also are equal to two right angles. Let $ZA$,
$HA$ meet, then, at $A$. Then the straight lines $AABK$,
$GAK$ enclose a space, which is impossible. Therefore it is not possible that, if the interior angles be
equal to two right angles, the straight lines should
meet. Therefore they are parallel.

Ibid., ed. Friedlein 365. 5–367. 27

Therefore certain others already classed as a
theorem this postulate assumed by the writer of the Elements and demanded a proof. Ptolemy appears
point of $ZH$ so that $ZH$ lies where $HZ$ is in the figure, while
$ZK$, $HK$ lie along the sides $HA$, $ZA$ respectively: and therefore $HA$, $ZA$ must meet at the point where $K$ falls.

The proof is based on the assumption that two straight
lines cannot enclose a space. But Riemann devised a geo-
metry in which this assumption does not hold good, for all
straight lines having a common point have another point
common also.
αὐτὸ δεικνύναι ἐν τῷ περὶ τού τὰς ἀπ’ ἐλαττόνων ἦ δύο ὀρθῶν ἐκβαλλομένας συμπίπτειν, καὶ δείκνυσι πολλὰ προλαβῶν τῶν μέχρι τοῦ θεωρήματος ὑπὸ τοῦ Στοιχείωτος προαποδεδειγμένων. καὶ ὑποκείσθω πάντα εἶναι ἀληθῆ, ἵνα μὴ καὶ ἤμεισ ὀχλον ἐπεισάγωμεν ἄλλον, καὶ ἃς λημμάτιον τοῦτο δείκνυσθαι διὰ τῶν προερημένων. ἐν δὲ καὶ τοῦτο τῶν προεδειγμένων τὸ τάς ἀπὸ δυνώ ὀρθᾶς ἰσων ἐκβαλλομένας μηδαμῶς συμπίπτειν. λέγω τοῖνυν ὅτι καὶ τὸ ἀνάπαλα ἀληθὲς, καὶ τὸ παραλλήλων οὐσῶν τῶν εὐθείων καὶ τεχνομένων ὑπὸ μᾶς εὐθείας τὰς ἐντός καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθὰς ἰσας εἶναι. ἀνάγκη γὰρ τὴν τέμνουσαν τὰς παραλληλων ἦ δύο ὀρθᾶς ἰσας ποιεῖν τὰς ἐντός καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας ἦ δύο ὀρθῶν ἐλάσσους ἦ μείζους. ἔστωσαν οὖν παράλληλοι αἱ AB, ΓΔ, καὶ ἐμπιπτέτω εἰς αὐτὰς ἦ HZ; λέγω ὅτι οὐ ποιεῖ δύο ὀρθῶν μείζους τὰς ἐντός καὶ ἐπὶ τὰ αὐτά. εἰ γὰρ αἱ ὑπὸ AZH, HZH δύο ὀρθῶν μείζους, αἱ λοιπαὶ αἱ ὑπὸ BZH, ΔΗΖ δύο ὀρθῶν ἐλάσσους. ἀλλὰ καὶ δύο ὀρθῶν μείζους αἱ αὐταί· οὐδὲν γὰρ μᾶλλον αἱ AZ, ΓΗ παραλληλοὶ ἦ ZB, ΗΔ, ὅστε εἰ ἡ ἐμπεσοῦσα εἰς τὰς AZ, ΓΗ δύο ὀρθῶν μείζους ποιεῖ τὰς ἐντός, καὶ 876
LATER DEVELOPMENTS IN GEOMETRY

to have proved it in his book on the proposition that straight lines drawn from angles less than two right angles meet if produced, and he uses in the proof many of the propositions proved by the writer of the *Elements* before this theorem. Let all these be taken as true, in order that we may not introduce another mass of propositions, and by means of the aforesaid propositions this theorem is proved as a lemma, that straight lines drawn from two angles together equal to two right angles do not meet when produced—a—for this is common to both sets of preparatory theorems. I say then that the converse is also true, that if parallel straight lines be cut by one straight line the interior angles on the same side are equal to two right angles. For the straight line cutting the parallel straight lines must make the interior angles on the same side equal to two right angles or less or greater. Let AB, ΓΔ be parallel straight lines, and let HZ cut them; I say that it does not make the interior angles on the same side greater than two right angles. For if the angles AZH, ΓHZ are greater than two right angles, the remaining angles BZH, ΔHZ are less than two right angles. But these same angles are greater than two right angles; for AZ, ΓΗ are not more parallel than ZB, HΔ, so that if the straight line falling on AZ, ΓΗ make the interior angles greater than two right angles, the same straight line falling

---

\[377\]

*a* This is equivalent to Eucl. i. 28.

*b* This is equivalent to Eucl. i. 29.

*c* By Eucl. i. 13, for the angles AZH, BZH are together equal to two right angles and so are the angles ΓHZ, ΔHZ.
GREEK MATHEMATICS

ή εἰς τὰς ΖΒ, ΗΔ ἐμπίπτουσα δύο ὀρθῶν ποιῆσαι μεῖζους τὰς ἐντὸς. ἀλλ' αἱ αὐταὶ καὶ δύο ὀρθῶν ἐλάσσους. αἱ γὰρ τέσσαρες αἱ ὑπὸ ΑΖΗ, ΓΗΖ, ΒΖΗ, ΔΗΖ τέτρασιν ὀρθαῖς ἱσαὶ· ὀπερ ἀδύνατον. ὁμοίως δὴ δείξομεν ὅτι εἰς τὰς παραλλήλους ἐμπίπτουσα οὐ ποιεῖ δύο ὀρθῶν ἐλάσσους τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας. εἰ δὲ μήτε μεῖζους μήτε ἐλάσσους ποιεῖ τῶν δύο ὀρθῶν, λείπεται τὴν ἐμπίπτουσαν δύο ὀρθαῖς ἱσας ποιεῖν τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας.

Τούτων δὴ οὖν προδεδειγμένου τὸ προκείμενον ἀναμφισβητήτως ἀποδείκνυται. λέγω γὰρ ὅτι ἐὰν εἰς δύο εὐθείας εὐθεία ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιήσῃ, συμπέσονται αἱ εὐθείαι ἐκβαλλόμεναι, ἐφ' ἀ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες. μὴ γὰρ συμπιπτέτωσαν. ἀλλ' εἰ ἀσύμπτωτοι εἰσὶν, ἐφ' ἀ μέρη αἱ τῶν δύο ὀρθῶν ἐλάσσονες, πολλῷ μᾶλλον ἔσονται ἀσύμπτωτοι ἐπὶ θάτερα, ἐφ' ἀ τῶν δύο εἰσὶν ὀρθῶν αἱ μεῖζονες, ὡστε ἐφ' ἐκάτερα ἐν εἰνεν ἀσύμπτωτοι αἱ εὐθείαι. εἰ δὲ τούτο, παράλληλοι εἰσίν. ἀλλὰ δεδεικταί ὅτι ή εἰς τὰς παραλλήλους ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δύο ὀρθαῖς ἱσαὶ ποιῆσαι γωνίας. αἱ αὐταὶ ἁρα καὶ δύο ὀρθαῖς ἱσαὶ καὶ δύο ὀρθῶν ἐλάσσονες, ὀπερ ἀδύνατον.

Ταύτα προδεδειχθεῖσα δς Πτολεμαῖος καὶ καταν-

* See note c on p. 377.

b The fallacy lies in the assumption that “AZ, GH are not more parallel than ZB, ΗΔ,” so that the angles BZH, ΔΗΖ must also be greater than two right angles. This assump-

378
on ZB, $\Delta$ also makes the interior angles greater than two right angles; but these same angles are less than two right angles, for the four angles $\angle AZH$, $\angle HZ$, $\angle BZH$, $\angle AHZ$ are equal to four right angles; which is impossible. Similarly we may prove that a straight line falling on parallel straight lines does not make the interior angles on the same side less than two right angles. But if it make them neither greater nor less than two right angles, the only conclusion left is that the transversal makes the interior angles on the same side equal to two right angles.

With this preliminary proof, the theorem in question is proved beyond dispute. I mean that if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced, will meet on that side on which are the angles less than two right angles. For [, if possible,] let them not meet. But if they are non-secant on the side on which are the angles less than two right angles, by much more will they be non-secant on the other side, on which are the angles greater than two right angles, so that the straight lines would be non-secant on both sides. Now if this should be so, they are parallel. But it has been proved that a straight line falling on parallel straight lines makes the interior angles on the same side equal to two right angles. Therefore the same angles are both equal to and less than two right angles, which is impossible.

Having first proved these things and squarely faced

---

It is known as "Playfair's Axiom," but is, in fact, stated by Proclus in his note on Eucl. i. 31.
τήσας εἰς τὸ προκείμενον ἀκριβέστερον τι προσθείναι βούλεται καὶ δείξαι ὅτι, ἐὰν εἰς δύο εὐθείας εὐθεία ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δύο ὀρθῶν ποιῆ ἐλάσσονας, οὐ μόνον οὐκ εἰσὶν ἁσύμπτωτοι αἱ εὐθείαι, ὡς δεδεικταί, ἀλλὰ καὶ ἡ σύμπτωσις αὐτῶν κατ’ ἐκεῖνα γίνεται τὰ μέρη, ἐφ’ ἂν τῶν δύο ὀρθῶν ἐλάσσονες, οὐκ ἐφ’ ἂν αἱ μείζονες. ἔστωσαν γὰρ δύο εὐθείαι αἱ AB, ΓΔ καὶ ἐμπίπτουσα εἰς αὐτὰς ἡ EZΗΘ τοιείτω τὰς ὑπὸ AZΗ καὶ ὑπὸ ΓΗΖ δύο ὀρθῶν ἐλάσσονες.

αἱ λοιπαὶ ἀρὰ μείζους δύο ὀρθῶν. ὅτι μὲν [οὖν] οὐκ ἁσύμπτωτοι αἱ εὐθείαι δεδεικταί. εἰ δὲ συμπίπτουσιν, ἢ ἐπὶ τὰ A, Γ συμπέσουνται, ἢ ἐπὶ τὰ B, Δ. συμπίπττετωσαν ἐπὶ τὰ B, Δ κατὰ τὸ K. ἐπεὶ οὖν αἱ μὲν ὑπὸ AZΗ καὶ ΓΗΖ δύο ὀρθῶν εἰσὶν ἐλάσσονες, αἱ δὲ ὑπὸ AZΗ, BΖΗ δύο ὀρθαῖς ἱσαι, κοινῆς ἀφαίρεθείσης τῆς ὑπὸ AZΗ, 380
LATER DEVELOPMENTS IN GEOMETRY

the theorem in question, Ptolemy tries to make a more precise addition and to prove that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, not only are the straight lines not non-secant, as has been proved, but their meeting takes place on that side on which the angles are less than two right angles, and not on the side on which they are greater. For let \( AB, \Gamma \Delta \) be two straight lines and let \( EZH\Theta \) fall on them and make the angles \( AZH, \GammaHZ \) less than two right angles. Then the remaining angles are greater than two right angles [Eucl. i. 13]. Now it has been proved that the straight lines are not non-secant. If they meet, they will meet either on the side of \( A, \Gamma \) or on the side of \( B, \Delta \). Let them meet on the side of \( B, \Delta \) at \( K \). Then since the angles \( AZH, \GammaHZ \) are less than two right angles, while the angles \( AZH, BZH \) are equal to two right angles, when the common angle \( AZH \) is taken away, the angle \( \GammaHZ \) will be less

\[ \text{\textsuperscript{1} οὔ is clearly out of place.} \]
GREEK MATHEMATICS

η υπὸ ΓΗΖ ἐλάσσων ἐσταῖ τῆς ὑπὸ ΒΖΗ,

τριγώνου ἄρα τοῦ ΚΖΗ ἡ ἐκτὸς τῆς ἐντὸς καὶ

ἀπεναντίον ἐλάσσων, ὅπερ ἀδύνατον. οὐκ ἄρα

κατὰ ταῦτα συμπίπτουσιν. ἄλλα μὴν συμπίπτουσιν.

κατὰ θάτερα ἄρα ἡ σύμπτωσις αὐτῶν ἐσταῖ, καθ'

ἀι τῶν δύο ὅρθων εἶσιν ἐλάσσονες.

(iv.) Proclus

Ibid., ed. Friedlein 371. 23–373. 2

Τούτων δὴ προσποτεθέντος λέγω ὅτι, ἕαν παραλλήλων εὐθειῶν τὴν ἑτέραν τέμνει τις εὐθεία, τεμεῖ καὶ τὴν λοιπὴν.

"Εστῶσαν γὰρ παράλληλοι αἱ ΑΒ, ΓΔ, καὶ τεμνόμεν τὴν ΑΒ ἡ ΕΖΗ. λέγω ὅτι τὴν ΓΔ τεμεῖ.

Επεὶ γὰρ δύο εὐθειάι εἶσιν ἂφ' ἐνὸς σημείου

tοῦ Ζ, εἰς ἀπειρον ἐκβαλλόμεναι αἱ ΒΖ, ΖΗ,

παντὸς μεγέθους μείζονα ἔχουσι διάστασιν, ὥστε

καὶ τούτων, ὅσον ἐστὶ τὸ μεταξὺ τῶν παραλλήλων.

382
LATER DEVELOPMENTS IN GEOMETRY

than the angle BZH. Therefore the exterior angle of the triangle KZH will be less than the interior and opposite angle, which is impossible [Eucl. i. 16]. Therefore they will not meet on this side. But they do meet. Therefore their meeting will be on the other side, on which the angles are less than two right angles.

(iv.) Proclus

_Ibid., ed. Friedlein 371. 23–373. 2_

This having first been assumed, I say that, if any straight line cut one of parallel straight lines, it will cut the other also.

For let AB, ΓΔ be parallel straight lines, and let EZH cut AB. I say that it will cut ΓΔ.

For since BZ, ZH are two straight lines drawn from one point Z, they have, when produced indefinitely, a distance greater than any magnitude, so that it will also be greater than that between the parallels. 383
The method is ingenious, but Clavius detected the flaw, which lies in the initial assumption, taken from Aristotle, that two divergent straight lines will eventually be so far apart that a perpendicular drawn from a point on one to the other will be greater than any assigned distance; Clavius draws attention to the conchoid of Nicomedes (v. vol. i. pp. 298-301), which continually approaches its asymptote, and therefore continually gets farther away from the tangent at the vertex; but the perpendicular from any point on the curve to that tangent will always be less than the distance between the tangent and the asymptote.
Whenever, therefore, they are at a distance from one another greater than the distance between the parallels, ZH will cut \( \Gamma\Delta \). If, therefore, any straight line cuts one of parallels, it will cut the other also.

This having first been established, we shall prove in turn the theorem in question. For let \( AB, \Gamma\Delta \) be two straight lines, and let \( EZ \) fall on them so as to make the angles \( \text{BEZ}, \Delta ZE \) less than two right angles. I say that the straight lines will meet on that side on which are the angles less than two right angles.

For since the angles \( \text{BEZ}, \Delta ZE \) are less than two right angles, let the angle \( \text{OEB} \) be equal to the excess of the two right angles. And let \( \text{O}E \) be produced to \( K \). Then since \( EZ \) falls on \( K\Theta, \Gamma\Delta \) and makes the interior angles \( \text{OEZ}, \Delta ZE \) equal to two right angles, the straight lines \( \Theta K, \Gamma\Delta \) are parallel. And \( AB \) cuts \( K\Theta \); therefore, by what was before shown, it will also cut \( \Gamma\Delta \). Therefore \( AB, \Gamma\Delta \) will meet on that side on which are the angles less than two right angles, so that the theorem in question is proved.
(c) Isoperimetric Figures


"'Ωσαύτως δ' ὅτι, τῶν ἰσην περίμετρον ἐχόντων σχημάτων διαφόρων, ἐπειδὴ μείζονα ἐστὶν τὰ πολυγωνότερα, τῶν μὲν ἐπιτεδῶν ὁ κύκλος γίνεται μείζων, τῶν δὲ στερεῶν ἡ σφαῖρα."

Ποιησόμεθα δὴ τὴν τούτων ἀπόδειξιν ἐν ἔπιτομῇ ἐκ τῶν Ζηνοδώρου δεδειγμένων ἐν τῷ Περὶ ἰσοπερίμετρων σχημάτων.
Τῶν ἰσην περίμετρον ἐχόντων τεταγμένων εὐ-

* Ptolemy, Math. Syn. i. 3, ed. Heiberg i. pars i. 13. 16-19.

b Zenodorus, as will shortly be seen, cites a proposition by Archimedes, and therefore must be later in date than Archimedes; as he follows the style of Archimedes closely, he is generally put not much later. Zenodorus's work is not
LATER DEVELOPMENTS IN GEOMETRY

(c) ISOPERIMETRIC FIGURES

Theon of Alexandria, Commentary on Ptolemy’s Syntaxis 1. 3, ed. Rome, Studi e Testi, lxxii. (1936), 354. 19–357. 22

“... In the same way, since the greatest of the various figures having an equal perimeter is that which has most angles, the circle is the greatest among plane figures and the sphere among solid.""

We shall give the proof of these propositions in a summary taken from the proofs by Zenodorus in his book On Isoperimetric Figures.

Of all rectilinear figures having an equal perimeter—

extant, but Pappus also quotes from it extensively (Coll. v. ad init.), and so does the passage edited by Hultsch (Papp. Coll., ed. Hultsch 1138–1165) which is extracted from an introduction to Ptolemy’s Syntaxis of uncertain authorship (v. Rome, Studi e Testi, liv., 1931, pp. xiii-xvii). It is disputed which of these versions is the most faithful.
GREEK MATHEMATICS

θυγράμμων σχημάτων, λέγω δὴ ἴσοπλευρῶν τε καὶ ἴσογωνών, τὸ πολυγωνότερον μείζον ἐστὶν.

"Εστω γὰρ ἴσοπερίμετρα ἴσοπλευρά τε καὶ ἴσο-

γώνια τὰ ΑΒΓ, ΔΕΖ, πολυγωνότερον δὲ ἐστὶν

τὸ ΑΒΓ. λέγω, ὅτι μείζον ἐστὶν τὸ ΑΒΓ.

Εὐλύθρω γὰρ τὰ κέντρα τῶν περὶ τὰ ΑΒΓ,

ΔΕΖ πολύγωνα περιγραφομένων κύκλων τὰ Ἡ,

Θ, καὶ ἐπεξεύθεσαν αἱ ΗΒ, ΗΓ, ΘΕ, ΘΖ. καὶ

ἐτὶ ἀπὸ τῶν Ἡ, Θ ἐπὶ τὰς ΒΓ, ΕΖ καθετοὶ ἥχθωσαν

αἱ ΗΚ, ΘΛ. ἐπεὶ οὖν πολυγωνότερον ἐστὶν τὸ

ΑΒΓ τοῦ ΔΕΖ, πλεονάκις ἡ ΒΓ τὴν τοῦ ΑΒΓ

περίμετρον καταμετρεῖ ἦπερ ἡ ΕΖ τὴν τοῦ ΔΕΖ.

καὶ εἰσίν ἵσαι αἱ περίμετροι. μείζων ἁρὰ ἡ ΕΖ
tῆς ΒΓ. ὥστε καὶ ἡ ΕΔ τῆς ΒΚ. κείσθω τῇ

ΒΚ ἴση ἡ ΛΜ, καὶ ἐπεξεύθεω ἡ ΘΜ. καὶ ἐπεὶ

ἐστὶν ὡς ἡ ΕΖ εὐθεία πρὸς τὴν τοῦ ΔΕΖ πολυ-

γώνου περίμετρον οὐτως ἡ ὑπὸ ΕΘΖ πρὸς δ ὀρθός,

διὰ τὸ ἴσοπλευρον εἶναι τὸ πολυγώνων καὶ ἵσας

ἀπολαμβάνειν περιφερείας τοῦ περιγραφομένου
cύκλου καὶ τὰς πρὸς τὰ κέντρο γωνίας τὸν αὐτὸν

ἐχειν λόγων ταῖς περιφερείαις ἐφ' ὧν βεβήκασιν,

ὡς δὲ ἡ τοῦ ΔΕΖ περίμετρος, τουτέστων ἡ τοῦ

ΑΒΓ, πρὸς τὴν ΒΓ οὖτως αἱ δ ὀρθαὶ πρὸς τὴν

ὑπὸ ΒΗΓ, δι' ἴσου ἁρὰ ὡς ἡ ΕΖ πρὸς ΒΓ, του-

tέστων ἡ ΕΔ πρὸς ΛΜ, οὖτως καὶ ἡ ὑπὸ ΕΘΖ

gωνία πρὸς τὴν ὑπὸ ΒΗΓ, τουτέστων ἡ ὑπὸ ΕΘΛ

πρὸς τὴν ὑπὸ ΒΗΚ. καὶ ἐπεὶ ἡ ΕΔ πρὸς ΛΜ

μείζωνα λόγων ἐχεῖ ἦπερ ἡ ὑπὸ ΕΘΛ γωνία πρὸς
tὴν ὑπὸ ΜΘΛ, ὡς ἐξής δείξομεν, ὡς δὲ ἡ ΕΔ

* ΘΖ is not, in fact, joined in the ms. figures.

* This is proved in a lemma immediately following the proposition by drawing an arc of a circle with Θ as centre

388
LATER DEVELOPMENTS IN GEOMETRY

I mean equilateral and equiangular figures—the greatest is that which has most angles.

For let $AB\Gamma$, $\Delta EZ$ be equilateral and equiangular figures having equal perimeters, and let $AB\Gamma$ have the more angles. I say that $AB\Gamma$ is the greater.

For let $H$, $\Theta$ be the centres of the circles circumscribed about the polygons $AB\Gamma$, $\Delta EZ$, and let $HB$, $\Theta H$, $\Theta E$, $\Theta Z$ be joined. And from $H$, $\Theta$ let $HK$, $\Theta \Lambda$ be drawn perpendicular to $B\Gamma$, $EZ$. Then since $AB\Gamma$ has more angles than $\Delta EZ$, $B\Gamma$ is contained more often in the perimeter of $AB\Gamma$ than $EZ$ is contained in the perimeter of $\Delta EZ$. And the perimeters are equal. Therefore $EZ > B\Gamma$; and therefore $EA > BK$. Let $AM$ be placed equal to $BK$, and let $\Theta M$ be joined. Then since the straight line $EZ$ bears to the perimeter of the polygon $\Delta EZ$ the same ratio as the angle $E\Theta Z$ bears to four right angles—owing to the fact that the polygon is equilateral and the sides cut off equal arcs from the circumscribing circle, while the angles at the centre are in the same ratio as the arcs on which they stand [Eucl. iii. 26]—and the perimeter of $\Delta EZ$, that is the perimeter of $AB\Gamma$, bears to $B\Gamma$ the same ratio as four right angles bears to the angle $B\Theta \Gamma$, therefore $ex aequali$ [Eucl. v. 17]

$$EZ : B\Gamma = \text{angle } E\Theta Z : \text{angle } B\Theta \Gamma,$$

i.e.,

$$EA : AM = \text{angle } E\Theta Z : \text{angle } B\Theta \Gamma,$$

i.e.,

$$EA : AM = \text{angle } E\Theta \Lambda : \text{angle } BHK.$$

And since $EA : AM > \text{angle } E\Theta \Lambda : \text{angle } M\Theta \Lambda$,

as we shall prove in due course,

$\Theta M$ as radius cutting $\Theta E$ and $\Theta \Lambda$ produced, as in Eucl. Optic. 8 (v. vol. i. pp. 502-505); the proposition is equivalent to the formula $\tan a : \tan \beta > a : \beta$ if $\frac{1}{2}\pi > a > \beta$.  

389
GREEK MATHEMATICS

πρὸς ΛΜ ἡ ὑπὸ ΕΘΛ πρὸς τὴν ὑπὸ ΒΗΚ, ἡ ὑπὸ ΕΘΛ πρὸς τὴν ὑπὸ ΒΗΚ μείζων λόγον ἔχει ἢπερ πρὸς τὴν ὑπὸ ΜΘΛ. μείζων ἀρα ἡ ὑπὸ ΜΘΛ γωνία τῆς ὑπὸ ΒΗΚ. ἔστω δὲ καὶ ὀρθὴ ἡ πρὸς τῷ Λ ὀρθὴ τῇ πρὸς τῷ Κ ἴση. λοιπὴ ἀρα ἡ ὑπὸ ΗΒΚ μείζων ἐσται τῆς ὑπὸ ΘΜΛ. κελεσθω τῇ ὑπὸ ΗΒΚ ἴση ἡ ὑπὸ ΛΜΝ καὶ διήκθω ἡ ΛΘ ἐπὶ τὸ Ν. καὶ ἐπεὶ ἴση ἔστων ἡ ὑπὸ ΗΒΚ τῇ ὑπὸ ΝΜΛ, ἀλλὰ καὶ ἡ πρὸς τῷ Λ ἴση τῇ πρὸς τῷ Κ, ἔστω δὲ καὶ ἡ ΒΚ πλευρᾶ τῇ ΜΛ ἴση, ἴση ἀρα καὶ ἡ ΗΚ τῇ ΝΛ. μείζων ἀρα ἡ ΗΚ τῆς ΘΛ. μείζων ἀρα καὶ τὸ ὑπὸ τῆς ΑΒΓ περιμέτρου καὶ τῆς ΗΚ του ὑπὸ τῆς ΔΕΖ περιμέτρου καὶ τῆς ΘΛ. καὶ ἔστω τὸ μὲν ὑπὸ τῆς ΑΒΓ περιμέτρου καὶ τῆς ΗΚ διπλάσιον τοῦ ΑΒΓ πολυγώνου, ἐπεὶ καὶ τὸ ὑπὸ τῆς ΒΓ καὶ τῆς ΗΚ διπλάσιον ἔστων τοῦ ΗΒΓ τριγώνου. τὸ δὲ ὑπὸ τῆς ΔΕΖ περιμέτρου καὶ τῆς ΘΛ διπλάσιον τοῦ ΔΕΖ πολυγώνου. μείζων ἀρα τὸ ΑΒΓ πολύγωνον τοῦ ΔΕΖ.

Ibid. 358. 12-360. 3

Τούτου δεδειγμένου λέγω, ὅτι ἐὰν κύκλος εὐθυγράμμως ἵσωπλεύρω τε καὶ ἱσογωνίω ἵσοπεριμετρος ἦ, μείζων ἐσται ὁ κύκλος.

Κύκλος γὰρ ὁ ΑΒΓ ἵσωπλεύρω τε καὶ ἱσογωνίω τῷ ΔΕΖ εὐθυγράμμω ἵσοπεριμετρος ἐστὶν. λέγω, ὅτι μείζων ἐστὶν ὁ κύκλος.

Εἰλήφθω τού μὲν ΑΒΓ κύκλου κέντρου τὸ Η, τοῦ δὲ περὶ τὸ ΔΕΖ πολύγωνον περιγραφομένου τὸ Θ, καὶ περιγεγράφθω περὶ τὸν ΑΒΓ κύκλον.
and $EA : AM = \text{angle } E\Theta A : \text{angle } \Delta HK$,

$\therefore \text{angle } E\Theta A : \text{angle } \Delta HK > \text{angle } E\Theta A : M\Theta A$. 

$\therefore \text{angle } M\Theta A > \text{angle } \Delta HK$.

Now the right angle at $A$ is equal to the right angle at $K$. Therefore the remaining angle $\Delta HK$ is greater than the angle $\Theta MA$ [by Eucl. i. 32]. Let the angle $\Delta MN$ be placed equal to the angle $\Delta HK$, and let $\Delta \Theta$ be produced to $N$. Then since the angle $\Delta HK$ is equal to the angle $\Delta NMA$, and the angle at $A$ is equal to the angle at $K$, while $BK$ is equal to the side $MA$, therefore $HK$ is equal to $NA$ [Eucl. i. 26]. Therefore $HK > \Theta A$. Therefore the rectangle contained by the perimeter of $\Delta ABG$ and $HK$ is greater than the rectangle contained by the perimeter of $\Delta EZ$ and $\Theta A$. But the rectangle contained by the perimeter of $\Delta ABG$ and $HK$ is double of the polygon $\Delta ABG$, since the rectangle contained by $BG$ and $HK$ is double of the triangle $\Delta BG$ [Eucl. i. 41]; and the rectangle contained by the perimeter of $\Delta EZ$ and $\Theta A$ is double of the polygon $\Delta EZ$. Therefore the polygon $\Delta ABG$ is greater than $\Delta EZ$.

_Ibid._ 358. 12-360. 3

This having been proved, I say that _if a circle have an equal perimeter with an equilateral and equiangular rectilineal figure, the circle shall be the greater._

For let $\Delta ABG$ be a circle having an equal perimeter with the equilateral and equiangular rectilineal figure $\Delta EZ$. I say that the circle is the greater.

Let $H$ be the centre of the circle $\Delta ABG$, $\Theta$ the centre of the circle circumscribing the polygon $\Delta EZ$; and let there be circumscribed about the circle $\Delta ABG$ the
πολυγώνων ὁμοίων τῷ ΔΕΖ τῷ ΚΛΜ, καὶ ἐπεζεύχθω ἡ ἩΒ, καὶ κάθετος ἀπὸ τοῦ Θ ἐπὶ τὴν ΕΖ ήχθω ἡ ΘΝ, καὶ ἐπεζεύχθωσαν αἱ ἩΛ, ΘΕ.

ἐπεὶ οὖν ἡ τοῦ ΚΛΜ πολυγώνου περιμέτρου μείζων ἐστὶν τῆς τοῦ ΑΒΓ κύκλου περιμέτρου ὡς ἐν τῷ Περὶ σφαίρας καὶ κυλίνδρου Ἀρχιμήδης, ἦσθα δὲ ἡ τοῦ ΑΒΓ κύκλου περιμέτρου τῆς τοῦ ΔΕΖ πολυγώνου περιμέτρων, μείζων ἢρα καὶ ἡ τοῦ ΚΛΜ πολυγώνου περιμέτρους τῆς τοῦ ΔΕΖ πολυγώνου περιμέτρου. καὶ εἰσὶν ὁμοία τὰ πολυγώνα. μείζων ἢρα ἡ ΒΔ τῆς ΝΕ. καὶ ὁμοίου τοῦ ἩΛΒ τρίγωνον τῷ ΘΕΝ τριγώνω, ἐπεὶ καὶ τὰ 392
polygon $K\Lambda M$ similar to $\Delta EZ$, and let $HB$ be joined, and from $\Theta$ let $\Theta N$ be drawn perpendicular to $EZ$, and let $HA, \Theta E$ be joined. Then since the perimeter of the polygon $K\Lambda M$ is greater than the perimeter of the circle $AB\Gamma$, as Archimedes proves in his work *On the Sphere and Cylinder*,\(^a\) while the perimeter of the circle $AB\Gamma$ is equal to the perimeter of the polygon $\Delta EZ$, therefore the perimeter of the polygon $K\Lambda M$ is greater than the perimeter of the polygon $\Delta EZ$. And the polygons are similar; therefore $BA > NE$. And the triangle $HAB$ is similar to the triangle $\Theta EN$,

\(^a\) Prop. 1, *v. supra*, pp. 48-49.
GREEK MATHEMATICS

...polýgwna. meîζων ἀρα καὶ Ἡ Β θής ΘΕΝ. καὶ ἐστὶν ἴση ἡ τοῦ ΑΒΓ κύκλου περιμέτρου τῇ
toῦ ΔΕΖ πολυγώνου περιμέτρῳ. τὸ ἀρα ὑπὸ τῆς
περιμέτρου τοῦ ΑΒΓ κύκλου καὶ τῆς ἩΒ μεῖζὸν
ἐστὶν τοῦ ὑπὸ τῆς περιμέτρου τοῦ ΔΕΖ πολυ-
gώνου καὶ τῆς ΘΕΝ. ἄλλα τὸ μὲν ὑπὸ τῆς περι-
μέτρου τοῦ ΑΒΓ κύκλου καὶ τῆς ἩΒ διπλάσιον
toῦ ΑΒΓ κύκλου 'Ἀρχιμήδης ἔδειξεν, οὔ καὶ τὴν
dεῖξεν ἔξης ἐκθησόμεθα: τὸ δὲ ὑπὸ τῆς περιμέτρου
toῦ ΔΕΖ πολυγώνου καὶ τῆς ΘΕΝ διπλάσιον τοῦ
ΔΕΖ πολυγώνου. μεῖζων ἀρα ὁ ΑΒΓ κύκλος
toῦ ΔΕΖ πολυγώνου, ὁπερ ἔδει δεῖξαι.

Ibid. 364. 12-14

Δέγω δὴ καὶ ὅτι τῶν ἰσοπεριμέτρων εὐθυγράμ-
μων σχημάτων καὶ τὰς πλευρὰς ἰσοπληθεῖς
ἐχόντων τὸ μέγιστον ἰσόπλευρὸν τέ ἐστιν καὶ
ἰσογώνιον.

Ibid. 374. 12-14

Δέγω δὴ ὅτι καὶ ἡ σφαῖρα μεῖζων ἐστὶν πάντων
tῶν ἴσην ἐπιφάνειαν ἐχόντων στερεῶν σχημάτων,
προσχρησάμενος τοῖς ὑπὸ Ἀρχιμήδους δεδειγ-
mένοις ἐν τῷ Περὶ σφαίρας καὶ κυλίνδρου.

(d) Division of Zodiac Circle into 360 Parts:
Hypsicles

Hypscl. Anaph., ed. Manitius 5. 25-31

Τοῦ τῶν Ζωιδίων κύκλου εἰς τῇ περιφερείας ἴσας

* The proofs of these two last propositions are worked out
  by similar methods.
394
since the whole polygons are similar; therefore HB > θN. And the perimeter of the circle ABΓ is equal to the perimeter of the polygon ΔEZ. Therefore the rectangle contained by the perimeter of the circle ABΓ and HB is greater than the rectangle contained by the perimeter of the polygon ΔEZ and θN. But the rectangle contained by the perimeter of the circle ABΓ and HB is double of the circle ABΓ as was proved by Archimedes, whose proof we shall set out next; and the rectangle contained by the perimeter of the polygon ΔEZ and θN is double of the polygon ΔEZ [by Eucl. i. 41]. Therefore the circle ABΓ is greater than the polygon ΔEZ, which was to be proved.

*Ibid. 364. 12-14*

Now I say that, of all rectilineal figures having an equal number of sides and equal perimeter, the greatest is that which is equilateral and equiangular.

*Ibid. 374. 12-14*

Now I say that, of all solid figures having an equal surface, the sphere is the greatest; and I shall use the theorems proved by Archimedes in his work *On the Sphere and Cylinder*.

(d) Division of Zodiac Circle into 360 Parts:

Hypsicles


The circumference of the zodiac circle having been

* Des Hypsikles Schrift Anaphorikos nach Überlieferung und Inhalt kritisch behandelt, in Programm des Gymnasiuims zum Heiligen Kreuz in Dresden (Dresden, 1888), 10 Abt. 395
GREEK MATHEMATICS

diērημένου, ἕκαστη τῶν περιφερεῶν μοῖρα τοπικὴ καλεῖσθω. ὅμως δὴ καὶ τοῦ χρόνου, ἐν ὧν ὁ ζωδιακὸς ἀφ’ οὗ ἔτυχε σημείον ἐπὶ τὸ αὐτὸ σημεῖον παραγίγνεται, εἰς τὸ χρόνου ἰσοὺς διηρη-

(ε) Handbooks

(i.) Cleomedes

Cleom. De motu circ. ii. 6, ed. Ziegler 218. 8–224. 8

Τοιούτων δὲ τῶν περὶ τήν ἐκλειψιν τῆς σελήνης εἶναι ἐπιδειγμένων δοκεῖ ἐναντιοῦσαί τῷ λόγῳ τῷ κατασκευάζοντι ἐκλείπειν τήν σελήνην εἰς τήν

a Hypsicles, who flourished in the second half of the second century B.C., is the author of the continuation of Euclid’s Elements known as Book xiv. Diophantus attributed to him a definition of a polygonal number which is equivalent to the formula \( \frac{1}{2} n\{2 + (n - 1)(a - 2)\} \) for the \( n \)th \( a \)-gonal number.

The passage here cited is the earliest known reference in Greek to the division of the ecliptic into 360 degrees. This number appears to have been adopted by the Greeks from the Chaldaeans, among whom the zodiac was divided into twelve signs and each sign into thirty parts according to one system, sixty according to another (v. Tannery, Mémoires scientifiques, ii. pp. 256-268). The Chaldaeans do not, however, seem to have applied this system to other circles; Hipparchus is believed to have been the first to divide the 396
LATER DEVELOPMENTS IN GEOMETRY

divided into 360 equal arcs, let each of the arcs be called a *degree in space*, and similarly, if the time in which the zodiac circle returns to any position it has left be divided into 360 equal times, let each of the times be called a *degree in time*.

(e) Handbooks

(i.) Cleomedes

Cleomedes, *On the Circular Motion of the Heavenly Bodies* ii. 6, ed. Ziegler 218. 8–224. 8

Although these facts have been proved with regard to the eclipse of the moon, the argument that the moon suffers eclipse by falling into the shadow of the earth seems to be refuted by the stories told about paradoxical eclipses. For some say that an eclipse of the moon may take place even when both luminaries are seen above the horizon. This should make it clear that the moon does not suffer eclipse by circle in general into 360 degrees. The problem which Hypsicles sets himself in his book is: *Given the ratio between the length of the longest day and the length of the shortest day at any given place, to find how many time-degrees it takes any given sign to rise*. A number of arithmetical lemmas are proved.

b Cleomedes is known only as the author of the two books *Κυκλικὴ θεωρία μετεώρων*. This work is almost wholly based on Posidonius. He must therefore have lived after Posidonius and presumably before Ptolemy, as he appears to know nothing of Ptolemy's works. In default of better evidence, he is generally assigned to the middle of the first century B.C.

The passage explaining the measurement of the earth by Eratosthenes has already been cited (*supra*, pp. 266-273). This is the only other passage calling for notice.
GREEK MATHEMATICS

τῆς γῆς περιπτούσα, ἀλλ’ ἐτερον τρόπον. . . . οἱ παλαιότεροι τῶν μαθηματικῶν οὐτως ἐπεχείρουν λύειν τὴν ἀπορίαν ταύτην. ἔφασαν γὰρ, ὅτι . . . οἱ δ’ ἐπὶ γῆς ἐστῶτες οὐδὲν ἂν κωλύουντο ὅρων ἀμφοτέρους αὐτοὺς ἐπὶ τοῖς κυρτῶμαι τῆς γῆς ἐστῶτες. . . . τοιαύτην μὲν οὖν οἱ παλαιότεροι τῶν μαθηματικῶν τὴν τῆς προσαγομένης ἀπορίας λύσιν ἐποὐήσαντο. μὴ ποτὲ δ’ οὐχ ὑγίως εἰσὶν ἐννεγμένου. ἐφ’ ὑψοὺς μὲν γὰρ ἢ ὡς ἡμῶν γενομένη δύναιτ’ ἂν τοιτο παθεῖν, κωνοειδοῦς τοῦ ὀρίζοντος γινομένου πολὺ ἀπὸ τῆς γῆς ἐκ τὸν ἄερα ἡμῶν ἔξαρθθέντων, ἐπὶ δὲ τῆς γῆς ἐστῶτων οὐδαμῶς. εἰ γὰρ καὶ κύρτωμα ἐστίν, ἐφ’ οὖθεν ἀφαιρεῖται ἡμῶν ἢ ὡς ὑπὸ τοῦ μεγέθους τῆς γῆς. . . . ἀλλὰ πρῶτον μὲν ἀπαντητέον λέγοντας, ὅτι πέπλασται ὁ λόγος οὗτος ὑπὸ τῶν ἀπορίαν βουλομένων ἐμποίησαι τοῖς περὶ ταῦτα καταγινομένοις τῶν ἀστρολόγων καὶ φιλοσόφων. . . . πολλῶν δὲ καὶ παντοδαπῶν περὶ τὸν ἄερα παθῶν συνίστασθαι πεφυκότων ὑπὲρ εἰπ’ ἄδυνατον, ἢ δὴ καταδεδυκότος τοῦ ἥλιου καὶ ὑπὸ τὸν ὀρίζοντα οὖν τοῦς φαντασίαν ἡμῶν προσπεσεῖν ὡς μηδέπω καταδεδυκότος αὐτοῦ, ἢ νέφους παχυτέρου πρὸς τῇ δύσει οὖν καὶ λαμπρομομένου ὑπὸ τῶν ἥλιων ἀκτίνων καὶ ἥλιου ἡμῶν φαντασίαν ἀποπέμποντος ἢ ἀνθηλίου γενομένου. καὶ γὰρ

* i.e., the horizon would form the base of a cone whose vertex would be at the eye of the observer. He could thus look down on both the sun and moon as along the generators of a cone, even though they were diametrically opposite each other.

398
falling into the shadow of the earth, but in some other way. . . . The more ancient of the mathematicians tried to explain this difficulty after this fashion. They said that persons standing on the earth would not be prevented from seeing them both because they would be standing on the convexities of the earth. . . . Such is the solution of the alleged difficulty given by the more ancient of the mathematicians. But its soundness may be doubted. For, if our eye were situated on a height, the phenomenon in question might take place, the horizon becoming conical if we were raised sufficiently far above the earth, but it could in no wise happen if we stood on the earth. For though there might be some convexity where we stood, our sight itself becomes evanescent owing to the size of the earth. . . . The fundamental objection must first be made, that this story has been invented by certain persons wishing to make difficulty for the astronomers and philosophers who busy themselves with such matters. . . . Nevertheless, as the conditions which naturally arise in the air are many and various, it would not be impossible that, when the sun has just set and is below the horizon, we should receive the impression of its not having yet set, if there were a cloud of considerable density at the place of setting and if it were illumined by the solar rays and transmitted to us an image of the sun, or if there were a mock sun. For such images are often

Lit. "anthelion," defined in the Oxford English Dictionary as "a luminous ring or nimbus seen (chiefly in alpine or polar regions) surrounding the shadow of the observer's head projected on a cloud or fog bank opposite the sun." The explanation here tentatively put forward by Cleomedes is, of course, the true one.
τοιαύτα πολλὰ φαντάζεται ἐν τῷ ἀέρι, καὶ μάλιστα περὶ τὸν Πόντον.

(ii.) Theon of Smyrna

Ptol. Math. Syn. x. 1, ed. Heiberg i. pars ii. 296. 14-16

Ἐν μὲν γὰρ ταῖς παρὰ Θέωνος τοῦ μαθηματικοῦ δοθείσαι ἦμῖν εὑρομεν ἀναγεγραμμένην τήρησιν τῷ ἵστε Αδριανοῦ.

Theon Smyr., ed. Hiller 1. 1-2. 2

'Οτι μὲν οὖν οἶδ᾽ τε συνεῖναι τῶν μαθηματικῶς λεγομένων παρὰ Πλάτωνι μὴ καὶ αὐτὸν ἡσκημένον ἐν τῇ θεωρίᾳ ταύτῃ, πᾶς ἂν που ὄμολογησειν ὡς δὲ οὐδὲ τὰ ἄλλα ἀνωφελῆς οὐδὲ ἀνόνητος ἦ περὶ ταύτα ἐμπειρία, διὰ πολλῶν αὐτοῦ ἐμφανίζειν έοικε. τὸ μὲν οὖν συμπάσης γεωμετρίας καὶ συμπάσης μουσικῆς καὶ ἀστρονομίας ἐμπειρον γενόμενον τοῖς Πλάτωνοι συγγράμμασι  ἐντυγχάνει μακαριστὸν μὲν εἰ τῷ γένοιτο, οὐ μὴν εὐτορον οὐδὲ βάδιον ἄλλα πάντων πολλοῦ τοῦ ἐκ παίδων πόνου δεόμενον. ὥστε δὲ τοὺς διημαρτήκοτος τοῦ ἐν τοῖς μαθημασίς ἄσκηθηναι, ὄρεγομένους δὲ τῆς γνώσεως τῶν συγγραμμάτων αὐτοῦ μὴ παντάπασιν ὃν ποθοῦσι διαμαρτέτων, κεφαλαίωθε καὶ σύντομον ποιησόμεθα τῶν ἀναγκαίων καὶ ὃν δὲι μάλιστα τοῖς ἐντευξομένοις Πλάτωνι μαθηματικῶν θεωρημάτων παράδοσιν, ἀριθμητικῶν τε καὶ μουσικῶν καὶ γεωμετρικῶν τῶν τε κατὰ στερεομετρίαι καὶ ἀστρονομίαι, ὃν χωρὶς οὐχ
LATER DEVELOPMENTS IN GEOMETRY

seen in the air, and especially in the neighbourhood of Pontus.

(ii.) Theon of Smyrna

Ptolemy, *Syntaxis* x. 1, ed. Heiberg i. pars ii. 296. 14-16

For in the account given to us by Theon the mathematician we find recorded an observation made in the sixteenth year of Hadrian.ª

Theon of Smyrna, ed. Hiller 1. 1-2. 2

Everyone would agree that he could not understand the mathematical arguments used by Plato unless he were practised in this science; and that the study of these matters is neither unintelligent nor unprofitable in other respects Plato himself would seem to make plain in many ways. One who had become skilled in all geometry and all music and astronomy would be reckoned most happy on making acquaintance with the writings of Plato, but this cannot be come by easily or readily, for it calls for a very great deal of application from youth upwards. In order that those who have failed to become practised in these studies, but aim at a knowledge of his writings, should not wholly fail in their desires, I shall make a summary and concise sketch of the mathematical theorems which are specially necessary for readers of Plato, covering not only arithmetic and music and geometry, but also their application to stereometry and astronomy, for

GREEK MATHEMATICS

οἴόν τε εἶναι φησὶ τυχεῖν τοῦ ἀρίστου βίου, διὰ πολλῶν πάνυ δηλώσας ὡς οὐ χρὴ τῶν μαθημάτων ἀμελεῖν.

* By way of example, Theon proceeds to relate Plato's reply to the craftsmen about the doubling of the cube (v.
LATER DEVELOPMENTS IN GEOMETRY

without these studies, as he says, it is not possible to attain the best life, and in many ways he makes clear that mathematics should not be ignored.a

vol. i. p. 257), and also the *Epinomis*. Theon’s work, which has often been cited in these volumes, is a curious hotchpotch, containing little of real value to the study of Plato and no original work.
XXI. TRIGONOMETRY
XXI. TRIGONOMETRY

1. HIPPARCUS AND MENELAUS


Δέδεικται μὲν οὖν καὶ Ἰππάρχῳ πραγματεία τῶν ἐν κύκλῳ εὑθειῶν ἐν ἑβ βιβλίοις, ἐτι τε καὶ Μενελάῳ ἐν Ἐρ.

Heron, Metr. i. 22, ed. H. Schöne (Heron iii.) 58. 13-20

"Εστω ἐννάγωνον ἰσόπλευρον καὶ ἰσογώνιον τὸ ΑΒΓΔΕΖΘΗΚ, οὐ ἐκάστῃ τῶν πλευρῶν μονάδων ἵ. εὑρεῖν αὐτοῦ τὸ ἐμβαδὸν. περιγεγράφων περὶ αὐτὸ κύκλος, οὐ κέντρον ἐστω τὸ Λ, καὶ ἐπε-

* The beginnings of Greek trigonometry may be found in the science of sphaeric, the geometry of the sphere, for which v. vol. i. p. 5 n. b. It reached its culminating point in the Sphaerica of Theodosins.

Trigonometry in the strict sense was founded, so far as we know, by Hipparchus, the great astronomer, who was born at Nicaea in Bithynia and is recorded by Ptolemy to have made observations between 161 and 126 B.C., the most important of them at Rhodes. His greatest achievement was the discovery of the precession of the equinoxes, and he made a calculation of the mean lunar month which differs by less than a second from the present accepted figure. Unfortunately the only work of his which has survived is his early Commentary on the Phenomena of Eudoxus and Aratus. It
XXI. TRIGONOMETRY

1. HIPPARCUS AND MENELAUS

Theon of Alexandria, Commentary on Ptolemy’s Syntaxis
i. 10, ed. Rome, Studi e Testi, lxxii. (1936), 451. 4-5

An investigation of the chords in a circle is made by
Hipparchus in twelve books and again by Menelaus
in six.a

Heron, Metrics i. 22, ed. H. Schöne (Heron iii.) 58. 13-20

Let \( \triangle ABC \) be an equilateral and equi-
angular enneagon,\(^b\) whose sides are each equal to 10.
To find its area. Let there be described about it a
circle with centre \( A \), and let \( EA \) be joined and pro-
is clear, however, from the passage here cited, that he drew
up, as did Ptolemy, a table of chords, or, as we should say,
a table of sines; and Heron may have used this table (v. the
next passage cited and the accompanying note).

Menelaus, who also drew up a table of chords, is recorded
by Ptolemy to have made an observation in the first year of
Trajan’s reign (A.D. 98). He has already been encountered
(vol. i. pp. 348-349 and n. e) as the discoverer of a curve
called “paradoxical.” His trigonometrical work Sphaerica
has fortunately been preserved, but only in Arabic, which
will prevent citation here. A proof of the famous theorem
in spherical trigonometry bearing his name can, however, be
given in the Greek of Ptolemy (infra, pp. 458-463); and a
summary from the Arabic is provided by Heath, H.G.M. ii.
262-273.

\(^b\) i.e., a figure of nine sides.
GREEK MATHEMATICS

ζεύξθω ἥ ΕΔ καὶ ἐκβεβλήσθω ἐπὶ τὸ Μ, καὶ ἐπεζεύξθω ἥ ΜΖ. τὸ ἄρα ΕΖΜ τρίγωνον δοθέν ἐστιν τοῦ ἐνναγώνου. δεδεικταὶ δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθεῖῶν, ὅτι ἦ ΖΕ τῆς EM τρίτον μέρος ἐστὶν ὡς ἐγγίστα.

2. PTOLEMY

(a) General

Suidas, s.v. Πτολεμαῖος

Πτολεμαῖος, ὁ Κλαύδιος χρηματίσας, Ἀλέξ-ανδρεὺς, φιλόσοφος, γεγονός ἐπὶ τῶν χρόνων Μάρκου τοῦ βασιλέως. οντος ἐγραφεῖ Μηχανικὰ βιβλία γ, Περὶ φάσεων καὶ ἐπισημασιῶν ἀστέρων ἀπλανῶν βιβλία β, Ἄπλωσιν ἐπιφανείας φαίρας, Κανόνα πρόχειρον, τὸν Μέγαν ἀστρονόμον ἦτοι Σύνταξις καὶ ἄλλα.

* A similar passage (i. 24, ed. H. Schöne 62, 11-20) asserts that the ratio of the side of a regular hendecagon to the diameter of the circumscribing circle is approximately \( \frac{7}{25} \); and of this assertion also it is said δεδεικταὶ δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθεῖῶν. These are presumably the works of Hipparchus and Menelaus, though this opinion is controverted by A. Rome, "Premiers essais de trigonométrie rectiligne chez les Grecs" in L'Antiquité classique, t. 2 (1933), pp. 177-192. The assertions are equivalent to saying that sin 20° is approximately 0.333... and sin 16° 21' 49" is approximately 0.28.

* Nothing else is certainly known of the life of Ptolemy except, as can be gleaned from his own works, that he made observations between A.D. 125 and 141 (or perhaps 151). Arabian traditions add details on which too much reliance should not be placed. Suidas's statement that he was born 408
TRIGONOMETRY

duced to M, and let MZ be joined. Then the triangle EZM is given in the enneagon. But it has been proved in the works on chords in a circle that \( \frac{ZE}{EM} \) is approximately \( \frac{1}{3} \).^a

2. PTOLEMY

(a) General

Suidas, s.v. Ptolemaeus

Ptolemy, called Claudius, an Alexandrian, a philosopher, born in the time of the Emperor Marcus. He wrote Mechanics, three books, On the Phases and Seasons of the Fixed Stars, two books, Explanation of the Surface of a Sphere, A Ready Reckoner, the Great Astronomy or Syntaxis; and others.\(^b\)

in the time of the Emperor Marcus [Aurelius] is not accurate as Marcus reigned from A.D. 161 to 180.

Ptolemy’s Mechanics has not survived in any form; but the books On Balancings and On the Elements mentioned by Simplicius may have been contained in it. The lesser astronomical works of Ptolemy published in the second volume of Heiberg’s edition of Ptolemy include, in Greek, \( \text{Φάσεις ἀπλανῶν δότερων καὶ συναγωγῆς ἐπισημασιῶν καὶ Προχείρων κανόνων διάταξις καὶ ψηφοφορία,} \) which can be identified with two titles in Suidas’s notice. In the same edition is the Planisphaerium, a Latin translation from the Arabic, which can be identified with the \( \text{Ἀπλωσὶς ἐπιφανείας σφαῖρας} \) of Suidas; it is an explanation of the stereographic system of projection by which points on the heavenly sphere are represented on the equatorial plane by projection from a pole—circles are projected into circles, as Ptolemy notes, except great circles through the poles, which are projected into straight lines.

Allied to this, but not mentioned by Suidas, is Ptolemy’s Analemma, which explains how points on the heavenly sphere can be represented as points on a plane by means of orthogonal projection upon three planes mutually at right angles—
GREEK MATHEMATICS

Simpl. in De caelo iv. 4 (Aristot. 311 b 1), ed. Heiberg 710. 14-19

Πτολεμαῖος δὲ ὁ μαθηματικὸς ἐν τῷ Περὶ ῥοπῶν τὴν ἐναντίαν ἑχων τῷ Ἀριστοτέλει δόξαν πειρᾶται κατασκευάζειν καὶ αὐτός, ὅτι ἐν τῇ ἐαυτῶν χώρᾳ οὔτε τὸ ὕδωρ οὔτε ὁ ἀὴρ ἔχει βάρος. καὶ ὅτι μὲν τὸ ὕδωρ οὐκ ἔχει, δείκνυσιν ἐκ τοῦ τούς καταδύοντας μὴ αἰσθάνεσθαι βάρους τοῦ ἐπικειμένου ὕδατος, καίτοι τυνᾶς εἰς πολὺ καταδύοντας βάθος.

Ibid. i. 2, 269 a 9, ed. Heiberg 20. 11

Πτολεμαῖος ἐν τῷ Περὶ τῶν στοιχείων βιβλίων καὶ ἐν τοῖς Ὄπτικοῖς . . .

Ibid. i. 1, 268 a 6, ed. Heiberg 9. 21-27

Ὁ δὲ θαυμαστὸς Πτολεμαῖος ἐν τῷ Περὶ διαστάσεως μονοβιβλίῳ ἀπεδείξεν, ὅτι οὐκ εἶσι

the meridian, the horizontal and the "prime vertical." Only fragments of the Greek and a Latin version from the Arabic have survived; they are given in Heiberg's second volume.

Among the "other works" mentioned by Suidas are presumably the Inscription in Canobus (a record of some of Ptolemy's discoveries), which exists in Greek; the Ἰποθέσεις τῶν πλανωμένων, of which the first book is extant in Greek and the second in Arabic; and the Optics and the book On Dimension mentioned by Simplicius.

But Ptolemy's fame rests most securely on his Great Astronomy or Syntaxis as it is called by Suidas. Ptolemy himself called this majestic astronomical work in thirteen books the Μαθηματικὴ σύνταξις or Mathematical Collection. In due course the lesser astronomical works came to be called the Μικρὸς ἀστρονομοῦμενος (τόπος), the Little Astronomy, and the Syntaxis came to be called the Μεγάλη σύνταξις, or Great Collection. Later still the Arabs, combining their article Al
TRIGONOMETRY

Simplicius, *Commentary on Aristotle's De caelo iv. 4* (311 b 1), ed. Heiberg 710. 14-19

Ptolemy the mathematician in his work *On Balancings* maintains an opinion contrary to that of Aristotle and tries to show that in its own place neither water nor air has weight. And he proves that water has not weight from the fact that divers do not feel the weight of the water above them, even though some of them dive into considerable depths.

*Ibid. i. 2, 269 a 9, ed. Heiberg 20. 11*

Ptolemy in his book *On the Elements* and in his *Optics* . . .

*Ibid. i. 1, 268 a 6, ed. Heiberg 9. 21-27*

The gifted Ptolemy in his book *On Dimension* showed that there are not more than three dimen-

with the Greek superlative μεγιστος, called it Al-majisti; corrupted into *Almagest*, this has since been the favourite name for the work.

The *Syntaxis* was the subject of commentaries by Pappus and Theon of Alexandria. The trigonometry in it appears to have been abstracted from earlier treatises, but condensed and arranged more systematically.

Ptolemy’s attempt to prove the parallel-postulate has already been noticed (supra, pp. 372-383).

a Ptolemy’s *Optics* exists in an Arabic version, which was translated into Latin in the twelfth century by Admiral Eugenius Siculus (v. G. Govi, *L’ottica di Claudio Tolomeo di Eugenio Ammiraglio di Sicilia*); but of the five books the first and the end of the last are missing. Until the Arabic text was discovered, Ptolemy’s *Optics* was commonly supposed to be identical with the Latin work known as *De Speculis*; but this is now thought to be a translation of Heron’s *Catoptrica* by William of Moerbeke (v. infra, p. 502 n. a).
GREEK MATHEMATICS

πλείονες τῶν τριών διαστάσεις, ἐκ τοῦ δεῖν μὲν τὰς διαστάσεις ὁρισμένας εἶναι, τὰς δὲ ὁρισμένας διαστάσεις κατ' εὐθείας λαμβάνεσθαι καθέτους, τρεῖς δὲ μόνας πρὸς ὁρθὰς ἀλλήλαις εὐθείας δυνατὸν εἶναι λαβεῖν, δύο μὲν καθ’ ἂς τὸ ἐπίπεδον ὀρίζεται, τρίτην δὲ τὴν τὸ βάθος μετροῦσαι ὥστε, εἰ τὶς εἰη μετὰ τὴν τριχῆ διάστασιν ἄλλη, ἁμέτρος ἂν εἰη παντελῶς καὶ ἀόριστος.

(b) TABLE OF SINES

(i.) Introduction

Ptol. Math. Syn. i. 10, ed. Heiberg i. pars i. 31. 7–32. 9

1'. Περὶ τῆς πτηλικότητας τῶν ἐν τῷ κύκλῳ εὐθείων

Πρὸς μὲν οὖν τὴν εἰς ἐτοίμου χρῆσιν κανονικῆν τυνα μετὰ ταῦτα ἐκθεῖν ποιησόμεθα τῆς πτηλικότητας αὐτῶν τὴν μὲν περίμετρον εἰς τῇ τμήματα διελόντες, παρατηθέντες δὲ τὰς ὑπὸ τὰς καθ’ ἡμιμοίρους παραμεζήσεις τῶν περιφερειῶν ὑποτεινομένας εὐθείας, τούτους πόσων εἰσὶν τμημάτων ὡς τῆς διαμέτρου διὰ τὸ ἐξ αὐτῶν τῶν ἐπιλογισμῶν φανησόμενον ἐν τοῖς ἀριθμοῖς εὑχρηστον εἰς τὸ τμήμα τηρημένης. πρῶτον δὲ δείξομεν, πῶς ἂν ὡς ἐν μάλιστα δι’ ὀλίγων καὶ τῶν αὐτῶν θεωρημάτων εὐμεθέδευστον καὶ ταχείαν τὴν ἐπιβολὴν τὴν πρὸς τὰς πτηλικότητας αὐτῶν ποιήσει, ὅπως μὴ μόνον ἐκτεθείμενα τὰ μεγέθη τῶν εὐθείων ἔχωμεν ἀνεπιστάτως, ἀλλὰ καὶ διὰ τῆς ἐκ τῶν γραμμῶν μεθοδικῆς αὐτῶν συστάσεως τὸν ἔλεγχον εἰς εὐχεροῦς μεταχειριζόμεθα. καθόλου

412
sions; for dimensions must be determinate, and determinate dimensions are along perpendicular straight lines, and it is not possible to find more than three straight lines at right angles one to another, two of them determining a plane and the third measuring depth; therefore, if any other were added after the third dimension, it would be completely unmeasurable and undetermined.

(b) Table of Sines

(i.) Introduction

Ptolemy, Syntaxis i. 10, ed. Heiberg i. pars i. 31. 7–32. 9

10. On the lengths of the chords in a circle

With a view to obtaining a table ready for immediate use, we shall next set out the lengths of these [chords in a circle], dividing the perimeter into 360 segments and by the side of the arcs placing the chords subtending them for every increase of half a degree, that is, stating how many parts they are of the diameter, which it is convenient for the numerical calculations to divide into 120 segments. But first we shall show how to establish a systematic and rapid method of calculating the lengths of the chords by means of the uniform use of the smallest possible number of propositions, so that we may not only have the sizes of the chords set out correctly, but may obtain a convenient proof of the method of calculating them based on geometrical considera-
GREEK MATHEMATICS

"Εστω δὴ πρῶτον ήμικύκλιον τὸ ΑΒΓ ἐπὶ διαμέτρου τῆς ΛΔΓ περὶ κέντρου τὸ Δ, καὶ ἀπὸ τοῦ Δ τῇ ΑΓ πρὸς ὀρθὰς γωνίας ήχω η ἸΒ, καὶ τετμήσθω δίχα η ΔΓ κατὰ τὸ Ε, καὶ ἐπεζεύχθω η ΕΒ, καὶ κείσθω αὐτῇ ἵση η EZ, καὶ ἐπεζεύχθω η ΖΒ. λέγω, ὅτι η μὲν ΖΔ δεκαγώνου ἐστὶν πλευρά, ἡ δὲ ΒΖ πενταγώνου.

(iii.) sin 18° and sin 36°

Ibid. 32. 10–35. 16

By διὰ τῆς ἐκ τῶν γραμμῶν μεθοδικῆς συντάσεως Ptolemy meant more than a graphical method; the phrase indicates a rigorous proof by means of geometrical considerations, as will be seen when the argument proceeds; cf. the use of διὰ τῶν γραμμῶν infra, p. 434. It may be inferred, therefore, that when Hipparchus proved "by means of lines" (διὰ τῶν γραμμῶν, On the Phaenomena of Eudoxus and Aratus, ed. Manitius 148-150) certain facts about the risings of stars, he used rigorous, and not merely graphical calculations; in other words, he was familiar with the main formulae of spherical trigonometry.

i.e., ΖΔ is equal to the side of a regular decagon, and ΒΖ to the side of a regular pentagon, inscribed in the circle ΑΒΓ.
In general we shall use the sexagesimal system for the numerical calculations owing to the inconvenience of having fractional parts, especially in multiplications and divisions, and we shall aim at a continually closer approximation, in such a manner that the difference from the correct figure shall be inappreciable and imperceptible.

(ii.) $\sin 18^\circ$ and $\sin 36^\circ$

*Ibid.* 32. 10–35. 16

First, let $\triangle ABC$ be a semicircle on the diameter $A\Delta \Gamma$ and with centre $\Delta$, and from $\Delta$ let $\Delta B$ be drawn perpendicular to $A\Gamma$, and let $\Delta \Gamma$ be bisected at $E$, and let $EB$ be joined, and let $EZ$ be placed equal to it, and let $ZB$ be joined. I say that $ZA$ is the side of a decagon, and $BZ$ of a pentagon.
GREEK MATHEMATICS

"Επει γὰρ εὐθεία γραμμὴ ἢ ΔΓ τέτμηται δίξα κατὰ τὸ Ε, καὶ πρόσκειται τις αὐτῇ εὐθεία ἢ ΔΖ, τὸ ύπὸ τῶν ΓΖ καὶ ΖΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΔ τετραγώνου ἵσον ἐστὶν τῷ ἀπὸ τῆς ΕΖ τετραγώνῳ, τούτεστιν τῷ ἀπὸ τῆς BE, ἐπεὶ ὅσον ἐστὶν ἢ ΕΒ τῇ ΖΕ. ἀλλὰ τῷ ἀπὸ τῆς ΕΒ τετραγώνῳ ἵσα ἐστὶ τὰ ἀπὸ τῶν ΕΔ καὶ ΔΒ τετράγωνα. τὸ ἀρα ύπὸ τῶν ΓΖ καὶ ΖΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΔΕ τετραγώνου ἵσον ἐστὶν τοῖς ἀπὸ τῶν ΕΔ, ΔΒ τετραγώνους. καὶ κοινοὶ ἀφαιρεθῶσι τοῦ ἀπὸ τῆς ΕΔ τετραγώνου λοιπὸν τὸ ύπὸ τῶν ΓΖ καὶ ΖΔ ἵσον ἐστὶν τῷ ἀπὸ τῆς ΔΒ, τούτεστιν τῷ ἀπὸ τῆς ΔΓ· ἢ ΖΓ ἀρα ἀκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Δ. ἐπεὶ οὖν ἢ τοῦ ἐξαγώνου καὶ ἢ τοῦ δεκαγώνου πλευρά τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων ἐπὶ τῆς αὐτῆς εὐθείας ἀκρον καὶ μέσον λόγον τέμυνται, ἢ δὲ ΓΔ ἐκ τοῦ κέντρου οὖσα τῆς τοῦ ἐξαγώνου περιέχει πλευράν, ἢ ΔΖ ἀρα ἐστὶν ὅσον τῇ τοῦ δεκαγώνου πλευρᾶ. ὅμοιως δέ, ἐπεὶ ἢ τοῦ πενταγώνου πλευρά ὑπάρχει τὴν τε τοῦ ἐξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων, τοῦ δὲ ΒΔΖ ὀρθογώνιον τὸ ἀπὸ τῆς ΒΖ τετράγωνον ἵσον ἐστὶν τῷ τῇ τῆς ΒΔ, ἢτις ἐστὶν εξαγώνου πλευρά, καὶ τῷ ἀπὸ τῆς ΔΖ, ἢτις ἐστὶν δεκαγώνου πλευρά, ἢ ΒΖ ἀρα ὅσον ἐστὶν τῇ τοῦ πενταγώνου πλευρᾷ.

Ἐπεὶ οὖν, ὅσε ἐφη, ὑποτιθέμεθα τῇ τοῦ κύκλου διάμετρον τμημάτων ρκ, γίνεται διὰ τὰ προκείμενα ἢ μὲν ΔΕ ἡμίσεια οὖσα τῆς ἐκ τοῦ κέντρου

* Following the usual practice, I shall denote segments (τμήματα) of the diameter by \( r \), sixtieth parts of a τμήμα by \( 416 \)
TRIGONOMETRY

For since the straight line $\Delta \Gamma$ is bisected at $E$, and the straight line $\Delta Z$ is added to it,

$$\Gamma Z \cdot Z\Delta + E\Delta^2 = EZ^2$$  [Eucl. ii. 6]

$$= BE^2,$$

since $EB = ZE$.

But  

$$E\Delta^2 + \Delta B^2 = EB^2;$$  [Eucl. i. 47]

therefore  

$$\Gamma Z \cdot Z\Delta + E\Delta^2 = E\Delta^2 + \Delta B^2.$$

When the common term $E\Delta^2$ is taken away, the remainder  

$$\Gamma Z \cdot Z\Delta = \Delta B^2$$

i.e.,  

$$= \Delta \Gamma^2;$$

therefore $Z\Gamma$ is divided in extreme and mean ratio at $\Delta$ [Eucl. vi., Def. 3]. Therefore, since the side of the hexagon and the side of the decagon inscribed in the same circle when placed in one straight line are cut in extreme and mean ratio [Eucl. xiii. 9], and $\Gamma \Delta$, being a radius, is equal to the side of the hexagon [Eucl. iv. 15, coroll.], therefore $\Delta Z$ is equal to the side of the decagon. Similarly, since the square on the side of the pentagon is equal to the rectangle contained by the side of the hexagon and the side of the decagon inscribed in the same circle [Eucl. xiii. 10], and in the right-angled triangle $B\Delta Z$ the square on $BZ$ is equal [Eucl. i. 47] to the sum of the squares on $B\Delta$, which is a side of the hexagon, and $\Delta Z$, which is a side of the decagon, therefore $BZ$ is equal to the side of the pentagon.

Then since, as I said, we made the diameter $a$ consist of $120^\circ$, by what has been stated $\Delta E$, being half the numeral with a single accent, and second-sixtieths by the numeral with two accents. As the circular associations of the system tend to be forgotten, and it is used as a general system of enumeration, the same notation will be used for the squares of parts.
GREEK MATHEMATICS

τμημάτων λ και το ἀπ' αυτῆς τις, ἡ δὲ ΒΔ ἐκ τοῦ κέντρου οὖσα τμημάτων ξ καὶ το ἀπὸ αυτῆς γ, το δὲ ἀπὸ τῆς ΕΒ, toutéostin το ἀπὸ τῆς EZ, τῶν ἐπὶ το αὐτὸ, δφ. μήκει ἀρα ἐσται ἡ EZ τμημάτων ξ ἔν τε ἐγγιστα, καὶ λοιπη ἡ ΔΖ τῶν αὐτῶν λξ ἔν τε. ἡ ἀρα τοῦ δεκαγώνου πλευρά, ὑποτείνουσα δὲ περιφέρειαν τοιούτων λς, οἰων ἐστὶν ὁ κύκλος τξ, τοιούτων ἐσται λξ ἔν τε, οἰων ἡ διάμετρος πρ. πάλιν ἐπεὶ ἡ μὲν ΔΖ τμημάτων ἐστὶ λξ ἔν τε, τὸ δὲ ἀπὸ αὐτῆς ἀτὸς ἔν τε, ἐστὶ δὲ καὶ τὸ ἀπὸ τῆς ΔΒ τῶν αὐτῶν γ, α συνεθέντα ποιεῖ τὸ ἀπὸ τῆς ΒΖ τετράγωνον, δχαυς ἔν τε, μήκει ἀρα ἐσται ἡ ΒΖ τμημάτων ὁ λβ γ ἐγγιστα. καὶ ἡ τοῦ πενταγώνου ἀρα πλευρά, ὑποτείνουσα δὲ μοῖρας οβ, οἰων ἐστίν ὁ κύκλος τξ, τοιούτων ἐστίν ὁ λβ γ, οἰων ἡ διάμετρος πρ.

Φανερὸν δὲ αὐτόθεν, ὅτι καὶ ἡ τοῦ ἐξαγώνου πλευρά, ὑποτείνουσα δὲ μοῖρας ξ, καὶ ἠση οὖσα τῆ ἐκ τοῦ κέντρου, τμημάτων ἐστίν ξ. ὁμοίως δὲ, ἐπεὶ ἡ μὲν τοῦ τετραγώνου πλευρά, ὑποτείνουσα δὲ μοῖρας ζ, δυνάμει διπλασία ἐστίν τῆς ἐκ τοῦ κέντρου, ἡ δὲ τοῦ τριγώνου πλευρά, ὑποτείνουσα δὲ μοῖρας πρ, δυνάμει τῆς αὐτῆς ἐστίν τριπλασίων, τὸ δὲ ἀπὸ τῆς ἐκ τοῦ κέντρου τμημάτων ἐστίν γ, συναχθήσεται τὸ μὲν ἀπὸ τῆς τοῦ τετραγώνου πλευράς ζσ, τὸ δὲ ἀπὸ τῆς τοῦ τριγώνου Μ ὁ. ὅστε καὶ μήκει ἡ μὲν τὰς ζ μοῖρας ὑποτείνουσα εὐθεία τοιούτων ἐσται πὅ ἐν ἡ ἐγγιστα, οἰων ἡ
of the radius, consists of $30^\circ$ and its square of $900^2$, and $B\Delta$, being the radius, consists of $60^\circ$ and its square of $3600^2$, while $EB^2$, that is $EZ^2$, consists of $4500^2$; therefore $EZ$ is approximately $67^\circ 4' 55''$, and the remainder $\Delta Z$ is $37^\circ 4' 55''$. Therefore the side of the decagon, subtending an arc of $36^\circ$ (the whole circle consisting of $360^\circ$), is $37^\circ 4' 55''$ (the diameter being $120^\circ$). Again, since $\Delta Z$ is $37^\circ 4' 55''$, its square is $1375^2 4' 15''$, and the square on $\Delta B$ is $3600^2$, which added together make the square on $BZ$ $4975^2 4' 15''$, so that $BZ$ is approximately $70^\circ 32' 3''$. And therefore the side of the pentagon, subtending $72^\circ$ (the circle consisting of $360^\circ$), is $70^\circ 32' 3''$ (the diameter being $120^\circ$).

Hence it is clear that the side of the hexagon, subtending $60^\circ$ and being equal to the radius, is $60^\circ$. Similarly, since the square on the side of the square, subtending $90^\circ$, is double of the square on the radius, and the square on the side of the triangle, subtending $120^\circ$, is three times the square on the radius, while the square on the radius is $3600^2$, the square on the side of the square is $7200^2$ and the square on the side of the triangle is $10800^2$. Therefore the chord subtending $90^\circ$ is approximately $84^\circ 51' 10''$ (the diameter

---

* Theon's proof that $\sqrt{4500}$ is approximately $67^\circ 4' 55''$ has already been given (vol. i. pp. 56-61).
* This is, of course, the square itself; the Greek phrase is not so difficult. We could translate, "the second power of the side of the square," but the notion of powers was outside the ken of the Greek mathematician.
(iii.) \( \sin^2 \theta + \cos^2 \theta = 1 \)

Ibid. 35. 17–36. 12

Let \( AB \) be a chord of a circle subtending an angle \( \alpha \) at the centre \( O \), and let \( AKA' \) be drawn perpendicular to \( OB \) so as to meet \( OB \) in \( K \) and the circle again in \( A' \). Then

\[
\sin \alpha (= \sin AB) = \frac{AK}{AO} = \frac{1}{2} \frac{AA'}{AO}.
\]

And \( AA' \) is the chord subtended by double of the arc \( AB \), while Ptolemy expresses the lengths of chords as so many 120th parts of the diameter; therefore \( \sin \alpha \) is half the chord subtended by an angle \( 2\alpha \) at the centre, which is conveniently abbreviated by Heath to \( \frac{1}{2}(\text{crd. } 2\alpha) \), or, as we may alternatively express the relationship, \( \sin AB \) is "half the chord subtended by
TRIGONOMETRY

consisting of $120^\circ$), and the chord subtending $120^\circ$ is $103^p 55' 23''$.\(^a\)

(iii.) $\sin^2 \theta + \cos^2 \theta = 1$

_Ibid._ 35. 17-36. 12

The lengths of these chords have thus been obtained immediately and by themselves,\(^b\) and it will be thence clear that, among the given straight lines, the lengths are immediately given of the chords subtending the remaining arcs in the semicircle, by reason of the fact that the sum of the squares on these chords is equal to the square on the diameter; for example, since the chord subtending $36^\circ$ was shown to be $37^p 4' 55''$ and its square $1375^p 4' 15''$, while the square on the diameter is $14400^p$, therefore the square on the chord subtending the remaining $144^\circ$ in the semicircle is double of the arc AB," which is the Ptolemaic form; as Ptolemy means by this expression precisely what we mean by sin AB, I shall interpolate the trigonometrical notation in the translation wherever it occurs. It follows that $\cos a \equiv \sin(90 - a) \equiv \frac{1}{2} \text{crd.} (180^\circ - 2a)$, or, as Ptolemy says, "half the chord subtended by the remaining angle in the semicircle." Tan $a$ and the other trigonometrical ratios were not used by the Greeks.

In the passage to which this note is appended Ptolemy proves that

- side of decagon ($= \text{crd.} 36^\circ = 2 \sin 18^\circ$) = $37^p 4' 55''$,
- side of pentagon ($= \text{crd.} 72^\circ = 2 \sin 36^\circ$) = $70^p 32' 3''$,
- side of hexagon ($= \text{crd.} 60^\circ = 2 \sin 30^\circ$) = $60^p$,
- side of square ($= \text{crd.} 90^\circ = 2 \sin 45^\circ$) = $84^p 51' 10''$,
- side of equilateral triangle ($= \text{crd.} 120^\circ = 2 \sin 60^\circ$) = $103^p 55' 23''$.

\(^b\) _i.e._, not deduced from other known chords.
GREEK MATHEMATICS

Μ. γ κδ νε με, αυτή δὲ μήκει τῶν αυτῶν ριδ ξ Λξ ἐγγιστα, καὶ ἐπὶ τῶν ἄλλων ὁμοίως.

"Ον δὲ τρόπον ἀπὸ τούτων καὶ αἱ λοιπαὶ τῶν κατὰ μέρος δοθῆσονται, δείξομεν ἐφεξῆς προεκθέ-μενοι λημμάτιον εὐχρηστον πάνω πρὸς τὴν παροῦσαν πραγματειαν.

(iv.) "Ptolemy's Theorem"

Ibid. 36. 13–37. 18

"Εστῶ γὰρ κύκλος ἐγγεγραμμένον ἕχων τετρά-πλευρον τυχὸν τὸ ἈΒΓΔ, καὶ ἐπεζεύχθησαν αἱ ΑΓ καὶ ΒΔ. δεικτέον, ὅτι τὸ ὑπὸ τῶν ΑΓ καὶ ΒΔ περιεχόμενον ὀρθογώνιον ἵσον ἐστὶ συναμφό-τέρος τῷ τε ὑπὸ τῶν ΑΒ, ΔΓ καὶ τῷ ὑπὸ τῶν ΑΔ, ΒΓ.

Κείσθω γὰρ τῇ ὑπὸ τῶν ΔΒΓ γωνία ἤση ἡ ὑπὸ ΑΒΕ. εὰν οὖν κοινὴν προσθῶμεν τῇ ὑπὸ ΕΒΔ,

* i.e., crd. $144^\circ (=2 \sin 72^\circ) = 114^\circ 7' 37''$. If the given chord subtends an angle $2\theta$ at the centre, the chord subtended by the remaining arc in the semicircle subtends an angle $(180 - 2\theta)$, and the theorem asserts that

$$(\text{crd. } 2\theta)^2 + (\text{crd. } 180 - 2\theta)^2 = (\text{diameter})^2,$$

or

$$\sin^2 \theta + \cos^2 \theta = 1.$$
TRIGONOMETRY

13024° 55' 45" and the chord itself is approximately 114° 7' 37", and similarly for the other chords.¹

We shall explain in due course the manner in which the remaining chords obtained by subdivision can be calculated from these, setting out by way of preface this little lemma which is exceedingly useful for the business in hand.

(iv.) "Ptolemy's Theorem"

Ibid. 36. 13–37. 18

Let $ABG\Delta$ be any quadrilateral inscribed in a circle, and let $A\Gamma$ and $B\Delta$ be joined. It is required to prove that the rectangle contained by $A\Gamma$ and $B\Delta$ is equal to the sum of the rectangles contained by $AB$, $A\Gamma$ and $A\Delta$, $B\Gamma$.

For let the angle $ABE$ be placed equal to the angle $\Delta B\Gamma$. Then if we add the angle $EBA\Delta$ to both, the
GREEK MATHEMATICS

"έσται καὶ ἡ ὑπὸ ΑΒΔ γωνία ἵση τῇ ὑπὸ ΕΒΓ. ἔστων δὲ καὶ ἡ ὑπὸ ΒΔΑ τῇ ὑπὸ ΒΓΕ ἵση· τὸ γὰρ αὐτὸ τμήμα ὑποτείνουσι· ἴσογώνιον ἀρὰ ἔστιν τὸ ΑΒΔ τρίγωνον τῷ ΒΓΕ τριγώνῳ. ὥστε καὶ ἀνάλογον ἔστιν, ὡς ἡ ΒΓ πρὸς τὴν ΓΕ, οὕτως ἡ ΒΔ πρὸς τὴν ΔΑ. τὸ ἄρα ὑπὸ ΒΓ, ΑΔ ἵσον ἔστιν τῷ ὑπὸ ΒΔ, ΓΕ. πάλιν ἐπεὶ ἵση ἔστιν ἡ ὑπὸ ΑΒΕ γωνία τῇ ὑπὸ ΔΒΓ γωνία, ἔστων δὲ καὶ ἡ ὑπὸ ΒΑΕ ἵση τῇ ὑπὸ ΒΔГ, ἴσογώνιον ἀρὰ ἔστιν τὸ ΑΒΕ τρίγωνον τῷ ΒΔГ τριγώνῳ· ἀνάλογον ἀρὰ ἔστιν, ὡς ἡ ΒΑ πρὸς ΑΕ, ἡ ΒΔ πρὸς ΔΓ· τὸ ἄρα ὑπὸ ΒΑ, ΔΓ ἵσον ἔστιν τῷ ὑπὸ ΒΔ, ΑΕ. ἐδείχθη δὲ καὶ τὸ ὑπὸ ΒΓ, ΑΔ ἵσον τῷ ὑπὸ ΒΔ, ΓΕ· καὶ ὁλον ἄρα τὸ ὑπὸ ΑΓ, ΒΔ ἵσον ἔστιν συναμφοτέροις τῷ τε ὑπὸ ΑΒ, ΔΓ καὶ τῷ ὑπὲρ ΑΔ, ΒΓ· ὅπερ ἐδει δειξαί.

(v.) sin (θ – φ) = sin θ cos φ – cos θ sin φ

Ibid. 37. 19–39. 3

Τούτου προεκτεθέντος ἔστω ἡμικύκλιον τὸ ΑΒΓΔ ἐπὶ διαμέτρου τῆς ΑΔ, καὶ ἀπὸ τοῦ Α δύο διήκθωσαν αἱ ΑΒ, ΑΓ, καὶ ἔστω ἐκατέρα αὐτῶν δοθείσα τῷ μεγέθει, σῶν ἡ διάμετρος δοθείσα ρκ, καὶ ἐπεζεύξθω ἡ ΒΓ. λέγω, ὅτι καὶ αὐτὴ δέδοται.

Ἐπεζεύξθωσαν γὰρ αἱ ΒΔ, ΓΔ· δεδομέναι ἀρὰ εἰσὶν δηλονότι καὶ αὐταῖ διὰ τὸ λείπειν ἐκείνων εἰς τὸ ἡμικύκλιον. ἐπεὶ οὖν ἐν κύκλῳ τετράπλευρόν ἔστω τὸ ΑΒΓΔ, τὸ ἄρα ὑπὸ ΑΒ, ΓΔ μετὰ τοῦ 424
angle \( \angle ABA = \) the angle \( \angle EBF \). But the angle \( \angle BDA = \) the angle \( \angle BGE \) \([\text{Eucl. iii. 21}]\), for they subtend the same segment; therefore the triangle \( \triangle ABD \) is equiangular with the triangle \( \triangle BGE \).

\[
\therefore \quad \frac{\angle \Gamma}{\angle E} = \frac{\angle B}{\angle A}; \quad \text{[Eucl. vi. 4]}
\]

\[
\therefore \quad \frac{\angle B}{\angle A} = \frac{\angle \Gamma}{\angle E}. \quad \text{[Eucl. vi. 6]}
\]

Again, since the angle \( \angle ABE \) is equal to the angle \( \angle BGI \), while the angle \( \angle BAE \) is equal to the angle \( \angle BDA \) \([\text{Eucl. iii. 21}]\), therefore the triangle \( \triangle ABE \) is equiangular with the triangle \( \triangle BGE \);

\[
\therefore \quad \frac{\angle B}{\angle A} = \frac{\angle \Gamma}{\angle E}; \quad \text{[Eucl. vi. 4]}
\]

\[
\therefore \quad \frac{\angle B}{\angle A} = \frac{\angle \Gamma}{\angle E}. \quad \text{[Eucl. vi. 6]}
\]

But it was shown that

\[
\frac{\angle B}{\angle A} = \frac{\angle \Gamma}{\angle E};
\]

and \( \therefore \)

\[
\frac{\angle B}{\angle A} = \frac{\angle \Gamma}{\angle E}. \quad \text{[Eucl. ii. 1]}
\]

which was to be proved.

\[
(v.) \quad \sin (\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi
\]

\textit{Ibid. 37. 19–39. 3}

This having first been proved, let \( \triangle ABD \) be a semicircle having \( \Delta \) for its diameter, and from \( A \) let the two [chords] \( AB, AG \) be drawn, and let each of them be given in length, in terms of the \( 120^\circ \) in the diameter, and let \( \angle BGI \) be joined. I say that this also is given.

For let \( \angle BDA, \Gamma \) be joined; then clearly these also are given because they are the chords subtending the remainder of the semicircle. Then since \( \triangle ABD \) is a quadrilateral in a circle,
GREEK MATHEMATICS

υπὸ τῶν ΑΔ, ΒΓ ἵσον ἐστὶν τῷ υπὸ ΑΓ, ΒΔ. καὶ ἐστὶν τὸ τε υπὸ τῶν ΑΓ, ΒΔ δοθέν καὶ τὸ

υπὸ ΑΒ, ΓΔ· καὶ λοιπὸν ἀρά τὸ υπὸ ΑΔ, ΒΓ
dothéν ἐστιν. καὶ ἐστιν ἡ ΑΔ διάμετρος· δοθεῖσα
ἀρά ἐστιν καὶ ἡ ΒΓ εὐθεία.

Καὶ φανερὸν ἦμιν γέγονεν, ὅτι, ἐὰν δοθῶσιν δύο
περιφέρειαι καὶ αἱ ὑπ’ αὐτὰς εὐθείαι, δοθεῖσα
ἐσται καὶ ἡ τὴν ὑπεροχὴν τῶν δύο περιφερείων
ὑποτείνουσα εὐθεία. δῆλον δὲ, ὅτι διὰ τούτου τοῦ
θεωρήματος ἄλλας τε ὁὐκ ὀλίγας εὐθείας ἐγγρά-
ψομεν ἀπὸ τῶν ἐν ταῖς καθ’ αὐτὰς δεδομένων
ὑπεροχῶν καὶ ἐκαὶ τὴν ὑπὸ τὰς δώδεκα μοῖρας,
ἐπειδὴ ἤπερ ἔχομεν τὴν τε ὑπὸ τὰς ἕ καὶ τὴν ὑπὸ
tὰς οἷς.

426
TRIGONOMETRY

\[ AB \cdot \Gamma \Delta + \Lambda \Delta \cdot B\Gamma = A\Gamma \cdot B\Delta. \]

["Ptolemy's theorem"]

And \( A\Gamma \cdot B\Delta \) is given, and also \( AB \cdot \Gamma \Delta \); therefore the remaining term \( \Lambda \Delta \cdot B\Gamma \) is also given. And \( \Lambda \Delta \) is the diameter; therefore the straight line \( B\Gamma \) is given.\(^{\circ} \)

And it has become clear to us that, if two arcs are given and the chords subtending them, the chord subtending the difference of the arcs will also be given. It is obvious that, by this theorem we can inscribe\(^{b} \) many other chords subtending the difference between given chords, and in particular we may obtain the chord subtending 12°, since we have that subtending 60° and that subtending 72°.

\(^{\circ} \) If \( A\Gamma \) subtends an angle 2\( \theta \) and \( AB \) an angle 2\( \phi \) at the centre, the theorem asserts that

\[
\text{crd.} \ (2\theta - 2\phi) \cdot (\text{crd.} \ 180^\circ) = (\text{crd.} \ 2\theta) \cdot (\text{crd.} \ 180^\circ - 2\phi) - (\text{crd.} \ 2\phi) \cdot (\text{crd.} \ 180^\circ - 2\theta)
\]

\[ \text{i.e.,} \quad \sin (\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi. \]

\(^{b} \) Or "calculate," as we might almost translate \( \varepsilon \gamma \gamma \rho \alpha \phi \omicron \mu \epsilon \nu; \) cf. supra, p. 414 n. \( \alpha \) on \( \varepsilon \kappa \tau \omega \nu \gamma \rho \alpha \mu \mu \omega \nu. \)
Πάλιν προκείμενο δοθείσης τινός εὐθείας ἐν κύκλῳ τήν ὑπὸ τὸ ἡμισὺ τῆς ὑποτεινομένης περιφερείας εὐθείαν εὑρεῖν. καὶ ἔστω ἡμικύκλιον τὸ ἈΒΓ ἐπὶ διαμέτρου τῆς ΑΓ καὶ δοθείσα εὐθεία ἡ ΓΒ, καὶ ἡ ΓΒ περιφέρεια δίχα τετμῆσθω κατὰ τὸ Δ, καὶ ἐπεζεῦχθωσαν αἱ ἈΒ, ΑΔ, ΒΔ, ΔΓ, καὶ ἀπὸ τοῦ Δ ἐπὶ τὴν ΑΓ κάθετος ἡχθω ἡ ΔΖ. λέγω, ὅτι ἡ ΖΓ ἡμίσεια ἐστὶ τῆς τῶν ἈΒ καὶ ΑΓ ὑπεροχῆς.

Κεῖσθω γὰρ τῇ ἈΒ ἴση ἡ ΑΕ, καὶ ἐπεζεῦχθω ἡ ΔΕ. ἐπεὶ ἴση ἐστὶν ἡ ἈΒ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΔ, δύο δὴ αἱ ἈΒ, ΑΔ δύο ταῖς ΑΕ, ΑΔ ἵσαι εἶσιν ἑκατέρα ἑκατέρα. καὶ γωνία ἡ ὑπὸ ΒΑΔ γωνία τῇ ὑπὸ ΕΛΔ ἴση ἐστὶν· καὶ βάσις ἀρα ἡ ΒΔ βάσει τῇ ΔΕ ἴση ἐστὶν. ἀλλὰ ἡ ΒΔ τῇ ΔΓ ἴση ἐστὶν καὶ ἡ ΔΓ ἀρα τῇ ΔΕ ἴση ἐστὶν. ἐπεὶ οὖν ἴσοκελοῦς ὀντὸς τριγώνου τοῦ ΔΕΓ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἑκταὶ ἡ ΔΖ, ἴση ἐστὶν ἡ ΕΖ τῇ ΖΓ. ἀλλ' ἡ ΕΓ ὁλὴ ἡ ὑπεροχὴ ἐστὶν τῶν ἈΒ καὶ ΑΓ εὐθείῶν· ἡ ἀρα ΖΓ ἡμίσεια ἐστὶν τῆς τῶν αὐτῶν ὑπεροχῆς. ὥστε, ἐπεὶ τῆς ὑπο τὴν ΒΓ περιφέρειαν εὐθείας ὑποκειμένης.
Again, given any chord in a circle, let it be required to find the chord subtending half the arc subtended by the given chord. Let $AB\Gamma$ be a semicircle upon the diameter $A\Gamma$ and let the chord $\Gamma B$ be given, and let the arc $\Gamma B$ be bisected at $\Delta$, and let $AB$, $A\Delta$, $B\Delta$, $\Delta \Gamma$ be joined, and from $\Delta$ let $\Delta Z$ be drawn perpendicular to $A\Gamma$. I say that $Z\Gamma$ is half of the difference between $AB$ and $A\Gamma$.

For let $AE$ be placed equal to $AB$, and let $\Delta E$ be joined. Since $AB=AE$ and $A\Delta$ is common, [in the triangles $AB\Delta$, $AE\Delta$] the two [sides] $AB$, $A\Delta$ are equal to $AE$, $A\Delta$ each to each; and the angle $BA\Delta$ is equal to the angle $EA\Delta$ [Eucl. iii. 27]; and therefore the base $B\Delta$ is equal to the base $\Delta E$ [Eucl. i. 4]. But $B\Delta=\Delta \Gamma$; and therefore $\Delta \Gamma=\Delta E$. Then since the triangle $\Delta ET$ is isosceles and $\Delta Z$ has been drawn from the vertex perpendicular to the base, $EZ=Z\Gamma$ [Eucl. i. 26]. But the whole $ET$ is the difference between the chords $AB$ and $A\Gamma$; therefore $Z\Gamma$ is half of the difference. Thus, since the chord subtending the arc $B\Gamma$ is given, the chord $AB$ subtending the remainder.
GREEK MATHEMATICS

αὐτόθεν δέδοται καὶ ἡ λειπόσα εἰς τὸ ἡμικύκλιον ἡ ἈΒ, δοθήσεται καὶ ἡ ΖΓ ἡμίσεια οὐσα τῆς τῶν ΑΓ καὶ ΑΒ ὑπεροχῆς. ἀλλ' ἐπεὶ ἐν ὀρθογωνίως τῷ ΑΓΔ καθέτου ἀκθείσης τῆς ΖΔ ἱσογωνίου γί-νεται τὸ ΛΔΓ ὀρθογώνιον τῷ ΔΓΖ, καὶ ἔστω, ὡς ἡ ΑΓ πρὸς ΓΔ, ἡ ΓΔ πρὸς ΓΖ, τὸ ἀρα ὑπὸ τῶν ΑΓ, ΓΖ περιεχόμενον ὀρθογώνιον ἱσον ἔστω τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ. ὀδηθεύτε ὑπὸ τῶν ΑΓ, ΓΖ. ὀδηθεύτε ἀρα ἔστω καὶ τὸ ἀπὸ τῆς ΓΔ τετρά-γωνου. ὥστε καὶ μὴ κεῖ ἡ ΓΔ εὐθεία δοθήσεται τὴν ἡμίσειαν ὑποτείνουσα τῆς ΒΓ περιφερείας.

Καὶ διὰ τούτου ἐπὶ πάλιν τοῦ θεωρήματος ἀλλ' τε ληφθήσονται πλεῖστα κατὰ τὰς ἡμίσειας τῶν προεκτεθειμένων, καὶ δὴ καὶ ἀπὸ τῆς τὰς ἴδια μοίρας ὑποτείνουσας εὐθείας ἡ τε ὑπὸ τὰς ξ καὶ ἡ ὑπὸ τὰς γ καὶ ἡ ὑπὸ τὴν μίαν ἡμίσοι καὶ ἡ ὑπὸ τὸ ἡμίσον τέταρτον τῆς μιᾶς μοίρας. εὐθράκομεν δὲ ἐκ τῶν ἐπιλογισμῶν τὴν μὲν ὑπὸ τὴν μίαν ἡμίσον μοίραν τοιούτων αὐτῶν ὡς ἐξ οἷά εὐγνωστα, οἰων ἔστιν ἡ διάμετρος τῆς τῶν αὐτῶν Ω μὲν ἵ.

(vii.) \[ \cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \]

_Ibid._ 41. 4–43. 5

Πάλιν ἔστω κύκλος ὁ ΑΒΓΔ περὶ διάμετρον μὲν τὴν ΑΔ, κέντρον δὲ τὸ Ζ, καὶ ἀπὸ τοῦ Α ἀπειλήφθωσαν δύο περιφερείας δοθεῖσαι κατὰ τοῦ ἔξης αἱ ΑΒ, ΒΓ, καὶ ἐπεζεύξωσαν αἱ ΑΒ, ΒΓ ὑπ' αὐτῶν εὐθείαι καὶ αὐταὶ δεδομέναι. λέγω ὅτι, ἕαν ἐπεζεύξωμεν τὴν ΑΓ, δοθήσεται καὶ αὐτή.

* If ΒΓ subtends an angle 2θ at the centre the proposition asserts that

430
of the semicircle is immediately given, and $Z\Gamma$ will also be given, being half of the difference between $\Delta \Gamma$ and $AB$. But since the perpendicular $\Delta Z$ has been drawn in the right-angled triangle $\Delta \Gamma \Delta$, the right-angled triangle $\Delta \Delta \Gamma$ is equiangular with $\Delta \Gamma Z$ [Eucl. vi. 8], and

$$\Delta \Gamma : \Gamma \Delta = \Gamma \Delta : \Gamma Z,$$

and therefore

$$\Delta \Gamma \cdot \Gamma Z = \Gamma \Delta^2.$$

But $\Delta \Gamma \cdot \Gamma Z$ is given; therefore $\Gamma \Delta^2$ is also given. Therefore the chord $\Gamma \Delta$, subtending half of the arc $\Delta \Gamma$, is also given.\(^a\)

And again by this theorem many other chords can be obtained as the halves of known chords, and in particular from the chord subtending $12^\circ$ can be obtained the chord subtending $6^\circ$ and that subtending $3^\circ$ and that subtending $1\frac{1}{2}^\circ$ and that subtending $\frac{1}{2}^\circ + \frac{1}{4}^\circ (= \frac{3}{4}^\circ)$. We shall find, when we come to make the calculation, that the chord subtending $1\frac{1}{2}^\circ$ is approximately $1^p \, 34' \, 15''$ (the diameter being $120^p$) and that subtending $\frac{3}{4}^\circ$ is $0^p \, 47' \, 8''$.\(^b\)

(vii.) \[\cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi\]

*Ibid.* 41. 4-43. 5

Again, let $AB\Delta \Gamma$ be a circle about the diameter $\Delta \Delta$ and with centre $Z$, and from $A$ let there be cut off in succession two given arcs $AB$, $\Delta \Gamma$, and let there be joined $AB$, $\Delta \Gamma$, which, being the chords subtending them, are also given. I say that, if we join $\Delta \Gamma$, it also will be given.

$$(\text{crd. } \theta)^2 = \frac{1}{2}(\text{crd. } 180^\circ) \cdot ((\text{crd. } 180^\circ) - \text{crd. } 180^\circ - 2\theta)$$

i.e., \[\sin^2 \frac{1}{2} \theta = \frac{1}{4}(1 - \cos \theta).\]

\(^a\) The symbol in the Greek for $O$ should be noted; \(v\). vol. i. p. 47 n. a.
Διήχθω γὰρ διὰ τοῦ Β διάμετρος τοῦ κύκλου η ΒΖΕ, καὶ ἐπεξεύχθωσαν αἱ ΒΔ, ΔΓ, ΓΕ, ΔΕ·
δῆλον δὴ αὐτόθεν, ὅτι διὰ μὲν τὴν ΒΓ δοθῆσεται καὶ η ΓΕ, διὰ δὲ τὴν ΑΒ δοθῆσεται η τε ΒΔ καὶ
η ΔΕ. καὶ διὰ τὰ αὐτὰ τοῖς ἐμπροσθεν, ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστιν τὸ ΒΓΔΕ, καὶ διηγοῦμέναι
eἰσιν αἱ ΒΔ, ΓΕ, τὸ ὑπὸ τῶν διηγομένων περι-
εχόμενον ὀρθογώνιον ἵσον ἐστὶν συναμφοτέρους
toῖς ὑπὸ τῶν ἀπεναντίων· ὥστε, ἐπεὶ δεδομένου
τοῦ ὑπὸ τῶν ΒΔ, ΓΕ δέδοται καὶ τὸ ὑπὸ τῶν ΒΓ,
ΔΕ, δέδοται ἂρα καὶ τὸ ὑπὸ ΒΕ, ΓΔ. δέδοται
dὲ καὶ η ΒΕ διάμετρος, καὶ λοιπὴ η ΓΔ ἐσται
dεδομένη, καὶ διὰ τοῦτο καὶ η λείπουσα εἰς τὸ
ἡμικύκλιον η ΓΑ· ὥστε, ἐὰν δοθῶσιν δύο περι-
φέρεια καὶ αἱ ὑπ’ αὐτὰς εὑθεῖαι, δοθῆσεται καὶ
η συναμφοτέρας τὰς περιφερείας κατὰ σύνθεσιν
ὑποτείνουσα εὑθεῖα διὰ τούτου τοῦ θεωρήματος.

* If AB subtends an angle $2\theta$ and BG an angle $2\phi$ at the
centre, the theorem asserts that

$$(\text{crd. } 180^\circ) \cdot (\text{crd. } 180^\circ - 2\theta - 2\phi) = (\text{crd. } 180^\circ - 2\theta) \cdot (\text{crd. } 180^\circ - 2\phi),$$

i.e.,

$$\cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi.$$
For through B let BZE, the diameter of the circle, be drawn, and let $\Delta A$, $\Delta \Gamma, \Gamma E, \Delta E$ be joined; it is then immediately obvious that, by reason of $\Gamma E$ being given, $\Gamma E$ is also given, and by reason of $AB$ being given, both $B\Delta$ and $\Delta E$ are given. And by the same reasoning as before, since $B\Gamma\Delta E$ is a quadrilateral in a circle, and $B\Delta, \Gamma E$ are the diagonals, the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides. And so, since $B\Delta. \Gamma E$ is given, while $B\Gamma. \Delta E$ is also given, therefore $BE. \Gamma \Delta$ is given. But the diameter $BE$ is given, and [therefore] the remaining term $\Gamma \Delta$ will be given, and therefore the chord $\Gamma A$ subtending the remainder of the semicircle \(a\); accordingly, if two arcs be given, and the chords subtending them, by this theorem the chord subtending the sum of the arcs will also be given.
Φανερόν δὲ, ὅτι συντιθέντες άεί μετά τῶν προ-
εκτεθειμένων πασῶν τὴν ὑπὸ α .parseDouble(6.4, 0) μοίραν καὶ
τὰς συναπτομένας ἐπιλογιζόμενοι πάσας ἀπλῶς
ἐγγράφομεν, ὅσα δις γνώμεναι τρίτον μέρος
ἐξουσιώτερον, καὶ μόνατε ἐπὶ περιλειφθήσονται αἱ μεταξὺ
tῶν ἀνὰ α .parseDouble(6.4, 0) μοίραν διαστημάτων δύο καθ' ἕκαστον ἐσώμεναι, ἐπειδὴ ἕνερ
καθ' ἡμιμοιρίων ποιούμεθα τὴν ἐγγραφήν. ὅστε, ἐὰν τὴν ὑπὸ τὸ
ἡμιμοιρίον εὐθείαν εὑρωμεν, αὐτὴ κατὰ τε τὴν
σύνθεσιν καὶ τὴν ὑπεροχὴν τὴν πρὸς τὰς τὰ
dιαστημάτα περιεχοῦσα καὶ δεδομένα εὐθείας
καὶ τὰς λοιπὰς τὰς μεταξὺ πάσας ἡμῖν συνανα-
πληρώσει. ἐπεὶ δὲ δοθείσης τινὸς εὐθείας ὡς τῆς
ὐπὸ τὴν α .parseDouble(6.4, 0) μοίραν ἡ τὸ τρίτον τῆς αὐτῆς περι-
φερείας ὑποτείνουσα διὰ τῶν γραμμῶν οὐ δίδοται
πως· εἰ δὲ γε δυνατὸν ἂν, εἰχομεν ἂν αὐτὸθεν καὶ
τὴν ὑπὸ τὸ ἡμιμοιρίον· πρῶτον μεθοδεύσομεν
tὴν ὑπὸ τὴν α .parseDouble(6.4, 0) μοίραν ἀπὸ τε τῆς ὑπὸ τὴν α .parseDouble(6.4, 0) μοίραν καὶ τῆς ὑπὸ α .parseDouble(6.4, 0)
d' ὑποτεθέμενοι λημμάτιον, ὅ, κἂν μὴ πρὸς τὸ καθόλου δύνησαι τὰς τηλι-
kότητας ὄριζειν, ἐπὶ γε τῶν οὕτως ἐλαχίστων τὸ
πρὸς τὰς ὃρισμένας ἀπαράλλακτον δύνατ' ἂν
συντηρεῖν.

(viii.) Method of Interpolation

Ibid. 43. 6–46. 20

Δέγω γαρ, ὅτι ἐὰν ἐν κύκλῳ διαχωρῶσω ἀνίσοι
dύο εὐθείαι, ἡ μείζων πρὸς τὴν ἐλάσσονα ἐλάσσονα
λόγων ἔχει ἦπερ ἡ ἐπὶ τῆς μείζωνος εὐθείας περι-
φέρεια πρὸς τὴν ἐπὶ τῆς ἐλάσσονος.

'Εστώ γαρ κύκλος ὁ ΑΒΓΔ, καὶ διήχωσαν ἐν
αὐτῷ δύο εὐθείαι ἀνίσοι ἐλάσσων μὲν ἡ ΑΒ,
It is clear that, by continually putting next to all known chords a chord subtending \(1\frac{1}{2}^\circ\) and calculating the chords joining them, we may compute in a simple manner all chords subtending multiples of \(1\frac{1}{2}^\circ\), and there will still be left only those within the \(1\frac{1}{2}^\circ\) intervals—two in each case, since we are making the diagram in half degrees. Therefore, if we find the chord subtending \(\frac{1}{2}^\circ\), this will enable us to complete, by the method of addition and subtraction with respect to the chords bounding the intervals, both the given chords and all the remaining, intervening chords. But when any chord subtending, say, \(1\frac{1}{4}^\circ\), is given, the chord subtending the third part of the same arc is not given by the [above] calculations—if it were, we should obtain immediately the chord subtending \(\frac{1}{2}^\circ\); therefore we shall first give a method for finding the chord subtending \(1^\circ\) from the chord subtending \(1\frac{1}{2}^\circ\) and that subtending \(\frac{3}{4}^\circ\), assuming a little lemma which, even though it cannot be used for calculating lengths in general, in the case of such small chords will enable us to make an approximation indistinguishable from the correct figure.

(viii.) Method of Interpolation

Ibid. 43. 6–46. 20

For I say that, if two unequal chords be drawn in a circle, the greater will bear to the less a less ratio than that which the arc on the greater chord bears to the arc on the lesser.

For let \(\Delta\Gamma\Delta\) be a circle, and in it let there be drawn two unequal chords, of which \(\Delta\) is the lesser
GREEK MATHEMATICS

μεῖζων δὲ ἡ ΒΓ. λέγω, ὅτι ἡ ΓΒ εὐθεία πρὸς τὴν ΒΑ εὐθείαν ἐλάσσονα λόγον ἔχει ἦπερ ἡ ΒΓ περιφέρεια πρὸς τὴν ΒΑ περιφέρειαν.

Τετμήσθω γὰρ ἡ ὑπὸ ΑΒΓ γωνία δίχα ὑπὸ τῆς ΒΔ, καὶ ἐπεξεύχθωσαν ἡ τε ΑΕΓ καὶ ἡ ΑΔ καὶ ἡ ΓΔ. καὶ ἐπεὶ ἡ ὑπὸ ΑΒΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΒΕΔ εὐθείας, ἵση μὲν ἐστὶν ἡ ΓΔ εὐθεία τῇ ΑΔ, μεῖζων δὲ ἡ ΓΕ τῆς ΕΔ. ἡχθω δὴ ἀπὸ τοῦ Δ κάθετος ἐπὶ τὴν ΑΕΓ ἡ ΔΖ. ἐπεὶ τοῖνυν μεῖζων ἐστὶν ἡ μὲν ΑΔ τῆς ΕΔ, ἡ δὲ ΕΔ τῆς ΔΖ, ὁ ἀρὰ κέντρῳ μὲν τῷ Δ, διαστήματι δὲ τῷ ΔΕ γραφόμενος κύκλος τὴν μὲν ΑΔ τεμεῖ, ὑπερπερσεῖται δὲ τὴν ΔΖ. γεγράφθω δὴ ὁ ΗΕΘ, καὶ ἐκβεβλήσθω ἡ ΔΖΘ. καὶ ἐπεὶ ὁ μὲν ΔΕΘ τομεὺς μεῖζων ἐστὶν τοῦ ΔΕΖ τριγώνου, τὸ δὲ ΔΕΑ τρίγωνον μεῖζον τοῦ ΔΕΗ τομέως, τὸ ἀρὰ ΔΕΖ

* Lit. "let ΔΖΘ be produced."
and $\Gamma \Theta$ the greater. I say that 

$$\Gamma \Theta : \Theta A < \text{arc } \Gamma \Theta : \text{arc } \Theta A.$$  

For let the angle $\Theta A \Theta$ be bisected by $\Theta A$, and let 

$$\Theta A \Theta\text{ and } \Theta A \Delta\text{ and } \Theta \Gamma \Delta\text{ be joined. Then since the angle } \Theta A \Theta \text{ is bisected by the chord } \Theta E \Delta, \text{ the chord } \Theta \Delta = \Delta \Delta [\text{Eucl. iii. 26, 29}], \text{ while } \Theta E > \Theta A [\text{Eucl. vi. 3}].$$  

Now let $\Delta Z$ be drawn from $\Delta$ perpendicular to $\Theta A \Theta$. Then since $\Delta Z > \Theta A$, and $\Theta A > \Delta Z$, the circle described with centre $\Delta$ and radius $\Theta E$ will cut $\Delta Z$, and will fall beyond $\Delta Z$. Let [the arc] $\Theta E \Theta$ be described, and let $\Delta Z$ be produced to $\Theta$. Then since 

sector $\Delta E \Theta > \text{triangle } \Delta E Z,$  
and 

$\text{triangle } \Delta E A > \text{sector } \Delta E H,$

VOL. II P 437
GREEK MATHEMATICS

τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον ἐλάσσονα λόγον ἔχει ἥπερ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ. ἀλλ' ὃς μὲν τὸ ΔΕΖ τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον, οὔτως ἡ ΕΖ εὐθεῖα πρὸς τὴν ΕΑ, ὡς δὲ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ τομέα, οὔτως ἡ ὑπὸ ΖΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. ἡ ἀρα ΖΕ εὐθεῖα πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ὑπὸ ΖΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. καὶ συνθέντι ἀρα ἡ ΖΑ εὐθεία πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ὑπὸ ΖΔΑ γωνία πρὸς τὴν ὑπὸ ΑΔΕ. καὶ τὰν ἡγομένων τὰ διπλάσια, ἡ ΓΑ εὐθεία πρὸς τὴν ΑΕ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ὑπὸ ΓΔΑ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. καὶ διελόντι ἡ ΓΕ εὐθεία πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ὑπὸ ΓΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. ἀλλ' ὃς μὲν ἡ ΓΕ εὐθεία πρὸς τὴν ΕΑ, οὔτως ἡ ΓΒ εὐθεία πρὸς τὴν ΒΑ, ὡς δὲ ἡ ὑπὸ ΓΔΒ γωνία πρὸς τὴν ὑπὸ ΒΔΑ, οὔτως ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ. ἡ ΓΒ ἀρα εὐθεία πρὸς τὴν ΒΑ ἐλάσσονα λόγον ἔχει ἥπερ ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ περι-
φέρειαν.

Τούτου δὴ οὖν ὑποκειμένου ἐστὶν κύκλος ὁ ΑΒΓ, καὶ διήχωσαν ἐν αὐτῷ δύο εὐθείαι ἡ τε ΑΒ καὶ ἡ ΑΓ, ὑποκείσθω δὲ πρῶτον ἡ μὲν ΑΒ ὑποτείνουσα μιᾶς μοῖρας ζ' δ', ἡ δὲ ΑΓ μοῖραν ἀ. ἐπεὶ ἡ ΑΓ εὐθεία πρὸς τὴν ΒΑ εὐθεῖαι ἐλάσσονα λόγον ἔχει ἥπερ ἡ ΑΓ περιφέρεια πρὸς τὴν ΑΒ, ἡ δὲ ΑΓ περιφέρεια ἐπίτριτος ἐστὶν τῆς ΑΒ, ἡ ΓΑ ἀρα εὐθεία τῆς ΒΑ ἐλάσσων ἐστὶν ἡ ἐπίτριτος. ἀλλὰ ἡ ΑΒ εὐθεία ἐδειχθη σωτότων ο μὲ ἡ, οἰων ἐστὶν ἡ διάμετρος ἰκ. ἡ ἀρα ΓΑ
TRIGONOMETRY

\[ \therefore \text{triangle } \Delta EZ : \text{triangle } \Delta EA < \text{sector } \Delta E \Theta : \text{sector } \Delta EH. \]

But \( \text{triangle } \Delta EZ : \text{triangle } \Delta EA = \text{EZ} : \text{EA} \)

[Eucl. vi. 1]

and \( \text{sector } \Delta E \Theta : \text{sector } \Delta EH = \text{angle } Z \Delta E : \text{angle } E \Delta A. \)

\[ \therefore \text{ZE} : \text{EA} < \text{angle } Z \Delta E : \text{angle } E \Delta A. \]

\[ \therefore \text{componendo, } \text{ZA} : \text{EA} < \text{angle } Z \Delta A : \text{angle } A \Delta E; \]

and, by doubling the antecedents,

\[ \Gamma \text{A} : \text{AE} < \text{angle } \Gamma \Delta A : \text{angle } E \Delta A; \]

and \( \text{dirimendo, } \Gamma \text{E} : \text{EA} < \text{angle } \Gamma \Delta E : \text{angle } E \Delta A. \)

But \( \Gamma \text{E} : \text{EA} = \Gamma \text{B} : \text{BA}, \)

[Eucl. vi. 3]

and \( \text{angle } \Gamma \Delta B : \text{angle } B \Delta A = \text{arc } \Gamma \text{B} : \text{arc } \text{BA}; \)

[Eucl. vi. 33]

\[ \therefore \Gamma \text{B} : \text{BA} < \text{arc } \Gamma \text{B} : \text{arc } \text{BA}. \]

On this basis, then, let \( \Delta BG \) be a circle, and in it let there be drawn the two chords \( \text{AB} \) and \( \text{AG}, \) and let it first be supposed that \( \text{AB} \) subtends an angle of \( \frac{1}{2} \degree \)

and \( \text{AG} \) an angle of \( 1 \degree. \) Then since

\[ \text{AG} : \text{BA} < \text{arc } \text{AG} : \text{arc } \text{AB}, \]

while \( \text{arc } \text{AG} = \frac{4}{3} \text{ arc } \text{AB}, \)

\[ \therefore \text{GA} : \text{BA} < \frac{4}{3}. \]

But the chord \( \text{AB} \) was shown to be \( 0^\circ 47' 8'' \) (the diameter being \( 120^\circ \)); therefore the chord \( \text{GA} \)

\[ \text{If the chords } \Gamma \text{B, BA subtend angles } 2 \theta, 2\phi \text{ at the centre, this is equivalent to the formula,} \]

\[ \frac{\sin \theta}{\sin \phi} < \frac{\theta}{\phi}, \]

where \( \theta < \phi < \frac{1}{2} \pi. \)

439
GREEK MATHEMATICS

εὐθεία ἐλάσσων ἐστὶν τῶν αὐτῶν ἃ β ὑ· ταὐτα γὰρ ἐπιτριτά ἐστὶν ἐγχιστα τῶν ο̄ μζ η.

Πάλιν ἐπὶ τῆς αὐτῆς καταγραφῆς ἡ μὲν ΑΒ εὐθεία ὑποκείσθω ὑποτείνουσα μοῖραν ἃ, ἡ δὲ ΑΓ μοῖραν ἃ Λ. κατὰ τὰ αὐτὰ δή, ἐπεὶ ἡ ΑΓ περιφέρεια τῆς ΑΒ ἐστὶν ἡμιολία, ἡ ΓΑ ἄρα εὐθεία τῆς ΒΑ ἐλάσσων ἐστὶν ἡ ἡμιόλιος. ἀλλὰ τὴν ΑΓ ἀπεδείξαμεν τοιούτων οὐσαν ἃ λδ ἰς, οἷων ἐστὶν ἡ διάμετρος ῥκ· ἡ ἄρα ΑΒ εὐθεία μεῖζων ἐστὶν τῶν αὐτῶν ἃ β ὑ· τούτων γὰρ ἡμιολία ἐστὶν τὰ προκείμενα ἃ λδ ἰς. ὡστε, ἐπεὶ τῶν αὐτῶν ἑδείχθη καὶ μεῖζων καὶ ἐλάσσων ἡ τὴν μίαν μοῖραν ὑποτείνουσα εὐθεία, καὶ ταὐτὴν δῆλον· ὅτι ἐξομεν τοιούτων ἃ β ὑ ἐγχιστα, οἷων ἐστὶν ἡ διάμετρος ῥκ, καὶ διὰ τὰ προδεδειγμένα καὶ τὴν ύπὸ τὸ ἡμιορίουν, ἦτε εὑρίσκεται τῶν αὐτῶν

440
<1^\circ 2' 50"; for this is approximately four-thirds of 0^\circ 47' 8".

Again, with the same diagram, let the chord AB be supposed to subtend an angle of 1^\circ, and A\Gamma an angle of 1\frac{1}{2}^\circ. By the same reasoning,
since \[ \text{arc } A\Gamma = \frac{3}{2} \cdot \text{arc } AB, \]
\[ \therefore \frac{\Gamma A}{BA} < \frac{3}{2}. \]
But we have proved A\Gamma to be 1^\circ 34' 15" (the diameter being 120") ; therefore the chord AB > 1^\circ 2' 50" ; for 1^\circ 34' 15" is one-and-a-half times this number. Therefore, since the chord subtending an angle of 1^\circ has been shown to be both greater and less than [approximately] the same [length], manifestly we shall find it to have approximately this identical value 1^\circ 2' 50" (the diameter being 120") , and by what has been proved before we shall obtain the chord subtending \frac{1}{2}^\circ, which is found to be approximately
Ο ά γι γιουστα. και συναναπληρωθήσεται τά λοιπά, ώσ εφαμεν, διαστήματα έκ μεν τής πρός τήν μίαν ήμισυ μοιράν λόγου ἐνεκεν ώσ ἐπὶ τοῦ πρῶτου διαστήματος συνθέσεως τοῦ ήμιμοιρίου δεικνυμένης τῆς ὑπὸ τᾶς β μοίρας, ἐκ δὲ τῆς ὑπεροχῆς τῆς πρὸς τᾶς γ μοίρας καὶ τῆς ὑπὸ τᾶς β Λ διδομένης. ὥσαυτως δὲ καὶ ἐπί τῶν λοιπῶν.

(ix.) The Table

Ibid. 46. 21-63. 46

Ἡ μὲν οὖν πραγματείᾳ τῶν ἐν τῷ κύκλῳ εὐθείων οὖτως ἄν οἴμαι βάστα μεταχειρισθείη. ἦν δὲ, ὡς ἐφη, ἐφ' ἐκάστης τῶν χρειῶν ἐξ ἐτοιμοῦ τᾶς πηλικότητας ἐξωμεν τῶν εὐθείων ἐκκειμένας, κανόνια ύποτάξομεν ἀνὰ στίχους μὲ διὰ τὸ σύμμετρον, ὅτα μὲν πρώτα μέρη περιέξει τᾶς πηλικότητας τῶν περιφερειῶν καθ' ήμιμοιρίον παρηγηγημένας, τὰ δὲ δεύτερα τὰς τῶν παρακειμένων ταῖς περιφερειας εὐθείων πηλικότητας ὡς τῆς διαμέτρου τῶν ἁ τιμημάτων ύποκειμένης, τὰ δὲ τρίτα τὸ λ' μέρος τῆς καθ' ἐκαστον ήμιμοιρίων τῶν εὐθείων παραυξήσεως, ἢν ἔχοντες καὶ τὴν τοῦ ἐνὸς ἐξηκοστοῦ μέσην ἐπιβολῆν ἀδιαφοροῦσαν πρός αἰσθησιν τῆς ἀκριβοῦς καὶ τῶν μεταξὺ τοῦ ἡμίσους μερῶν ἐξ ἐτοιμοῦ τᾶς ἐπιβαλλούσας πηλικότητας ἐπιλογίζεσθαι δυνάμεθα. εὐκατανόητον δ', ὅτι διὰ τῶν αὐτῶν καὶ προκειμένων θεωρημάτων, καὶ ἐν δισταγμῷ γενόμεθα γραφικῆς ἀμαρτίας περὶ τινα τῶν ἐν τῷ κανονίῳ παρακειμένων εὐθείων, ῥαδίαν ποιησόμεθα τὴν τε ἐξέτασιν καὶ τὴν 442
TRIGONOMETRY

0° 31' 25''. The remaining intervals may be completed, as we said, by means of the chord subtending 12°—in the case of the first interval, for example, by adding 2° we obtain the chord subtending 2°, and from the difference between this and 3° we obtain the chord subtending 21°, and so on for the remainder.

(ix.) The Table

Ibid. 46. 21-63. 46

The theory of the chords in the circle may thus, I think, be very easily grasped. In order that, as I said, we may have the lengths of all the chords in common use immediately available, we shall draw up tables arranged in forty-five symmetrical rows.a

The first section will contain the magnitudes of the arcs increasing by half degrees, the second will contain the lengths of the chords subtending the arcs measured in parts of which the diameter contains 120, and the third will give the thirtieth part of the increase in the chords for each half degree, in order that for every sixtieth part of a degree we may have a mean approximation differing imperceptibly from the true figure and so be able readily to calculate the lengths corresponding to the fractions between the half degrees. It should be well noted that, by these same theorems now before us, if we should suspect an error in the computation of any of the chords in the table, b we can easily make a test and

a As there are 360 half degrees in the table, the statement appears to mean that the table occupied eight pages each of 45 rows; so Manitius, Des Claudius Ptolemäus Handbuch der Astronomie, 1er Bd., p. 35 n. a.

b Such an error might be accumulated by using the approximations for 1° and 2°; but, in fact, the sines in the table are generally correct to five places of decimals.

443
GREEK MATHEMATICS

επανόρθωσιν ήτοι ἀπὸ τῆς ὑπὸ τὴν διπλασίαν τῆς ἐπιξητουμένης ἢ τῆς πρὸς ἄλλας τινὰς τῶν δεδομένων ὑπεροχῆς ἢ τῆς τὴν λείπουσαν εἰς τὸ ἡμικύκλιον περιφέρειαν ὑποτεινούσης εὐθείας. καὶ ἐστίν ἢ τοῦ κανονίου καταγραφῆ τοιαῦτη.

ια’. Κανόνιον τῶν ἐν κύκλῳ εὐθείῶν

<table>
<thead>
<tr>
<th>περιφερειῶν</th>
<th>ευθείῶν</th>
<th>ἐξηκοστῶν</th>
</tr>
</thead>
<tbody>
<tr>
<td>Λ′</td>
<td>ο</td>
<td>λα</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>β</td>
</tr>
<tr>
<td>a Λ′</td>
<td>a</td>
<td>λδ</td>
</tr>
<tr>
<td>β</td>
<td>β</td>
<td>ε</td>
</tr>
<tr>
<td>β Λ′</td>
<td>β</td>
<td>λζ</td>
</tr>
<tr>
<td>γ</td>
<td>γ</td>
<td>η</td>
</tr>
<tr>
<td>γ Λ′</td>
<td>γ</td>
<td>λθ</td>
</tr>
<tr>
<td>δ</td>
<td>δ</td>
<td>ια</td>
</tr>
<tr>
<td>δ Λ′</td>
<td>δ</td>
<td>μβ</td>
</tr>
</tbody>
</table>

| ξ           | ξ       | ο           | ο           | ο           | ο           | νδ           | κα           |

| ροζ-       | ριθ     | νε         | λη         | ο           | ο           | β            | γη           |
| ροζ Λ′     | ριθ     | νζ         | λθ         | ο           | ο           | a            | μζ           |
| ροζ ζ       | ριθ     | νη         | ιη         | ο           | ο           | a            | νδ           |
| ροθ         | ριθ     | νθ         | νε         | ο           | ο           | νζ           | κα           |
| ροθ Λ′      | ριθ     | μθ         | νζ         | ο           | ο           | ριθ          | θ            |
| ροθ θ       | ριθ     | ο          | νδ         | ο           | ο           | ο            | θ            |
| ρπ         | ρκ      | νθ         | νζ         | ο           | ο           | ο            | θ            |
TRIGONOMETRY

apply a correction, either from the chord subtending double of the arc which is under investigation, or from the difference with respect to any others of the given magnitudes, or from the chord subtending the remainder of the semicircular arc. And this is the diagram of the table:

11. Table of the Chords in a Circle

<table>
<thead>
<tr>
<th>Arcs</th>
<th>Chords</th>
<th>Sixtieths</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0°</td>
</tr>
<tr>
<td>1°</td>
<td>0°</td>
<td>31°</td>
</tr>
<tr>
<td>1'</td>
<td>1°</td>
<td>2</td>
</tr>
<tr>
<td>1½</td>
<td>1°</td>
<td>34</td>
</tr>
<tr>
<td>2</td>
<td>2°</td>
<td>5</td>
</tr>
<tr>
<td>2½</td>
<td>2°</td>
<td>37</td>
</tr>
<tr>
<td>3</td>
<td>3°</td>
<td>8</td>
</tr>
<tr>
<td>3½</td>
<td>3°</td>
<td>39</td>
</tr>
<tr>
<td>4</td>
<td>4°</td>
<td>11</td>
</tr>
<tr>
<td>4½</td>
<td>4°</td>
<td>42</td>
</tr>
</tbody>
</table>

|      |        | 60 | 60 | 0 | 0 | 0 | 54 | 21 |

|      |        |    |    |    |    |    |    |    |
| 176  | 119    | 55 | 38 | 0 | 0 | 2 | 3  |
| 176½ | 119    | 56 | 39 | 0 | 0 | 1 | 47 |
| 177  | 119    | 57 | 32 | 0 | 0 | 1 | 30 |
| 177½ | 119    | 58 | 18 | 0 | 0 | 1 | 17 |
| 178  | 119    | 58 | 55 | 0 | 0 | 0 | 57 |
| 178½ | 119    | 59 | 24 | 0 | 0 | 0 | 41 |
| 179  | 119    | 59 | 44 | 0 | 0 | 0 | 25 |
| 179½ | 119    | 59 | 56 | 0 | 0 | 0 | 9  |
| 180  | 120    | 0  | 0  | 0 | 0 | 0 | 0  |
GREEK MATHEMATICS

(c) MENELAUS'S THEOREM

(i.) Lemmas

Ibid. 68. 14–74. 8

ιγ’. Προλαμβανόμενα εἰς τὰς σφαιρικὰς δείξεις

Ἀκολούθου δ’ ὄντος ἀποδείξαι καὶ τὰς κατὰ μέρος γινομένας πηλικότητας τῶν ἀπολαμβανόμενων περιφερειῶν μεταξὺ τοῦ τε ἱσημερινοῦ καὶ τοῦ διὰ μέσων τῶν Ζωδίων κύκλου τῶν γραφομένων μεγίστων κύκλων διὰ τῶν τοῦ ἱσημερινοῦ πόλων προεκθησόμεθα λημμάτια βραχέα καὶ εὐχρηστα, δι’ ὅν τὰς πλεῖστας σχεδὸν δείξεις τῶν σφαιρικῶς θεωρουμένων, ὡς ἐνι μάλιστα, ἀπλούστερον καὶ μεθοδικῶτερον πουησόμεθα.

Εἰς δύο δὴ εὑθείας τὰς ΑΒ καὶ ΑΓ διαχθεῖσαι δύο εὑθείαι ἡ τε ΒΕ καὶ ἡ ΓΔ τεμνέτωσαν ἀλλήλας 446
13. Preliminary matter for the spherical proofs

The next subject for investigation being to show the lengths of the arcs, intercepted between the celestial equator and the zodiac circle, of great circles drawn through the poles of the equator, we shall set out some brief and serviceable little lemmas, by means of which we shall be able to prove more simply and more systematically most of the questions investigated spherically.

Let two straight lines $BE$ and $\Gamma \Delta$ be drawn so as to meet the straight lines $AB$ and $\Delta \Gamma$ and to cut one
GREEK MATHEMATICS

κατὰ τὸ Ζ σημεῖον. λέγω, ὅτι ὁ τῆς ΓΑ πρὸς ΑΕ λόγος συνήπται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΖΒ πρὸς ΒΕ.

"Ηχθω γὰρ διὰ τοῦ Ε τῆς ΓΔ παράλληλος ἡ ΕΗ. ἐπεὶ παράλληλοι εἰσιν αἱ ΓΔ καὶ ΕΗ, ὁ τῆς ΓΑ πρὸς ΕΑ λόγος ὁ αὐτὸς ἐστὶν τῷ τῆς ΓΔ πρὸς ΕΗ. ἔξωθεν δὲ ἡ ΖΔ· ὁ ἀρα τῆς ΓΔ πρὸς ΕΗ λόγος συγκείμενος ἐσται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΔΖ πρὸς ΗΕ· ὥστε καὶ ὁ τῆς ΓΑ πρὸς ΑΕ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΔΖ πρὸς ΗΕ. ἐστὶν δὲ καὶ ὁ τῆς ΔΖ πρὸς ΗΕ λόγος ὁ αὐτὸς τῷ τῆς ΖΒ πρὸς ΒΕ διὰ τὸ παραλλήλους πάλιν εἰναι τὰς ΕΗ καὶ ΖΔ· ὁ ἀρα τῆς ΓΑ πρὸς ΑΕ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΔ πρὸς ΔΖ καὶ τοῦ τῆς ΖΒ πρὸς ΒΕ· ὅπερ προεκεῖτο δεῖξαι.

Κατὰ τὰ αὐτὰ δὲ δειχθῆσεται, ὅτι καὶ κατὰ διαίρεσιν ὁ τῆς ΓΕ πρὸς ΕΑ λόγος συνήπται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ, διὰ τοῦ Α τῆς ΕΒ παραλλήλου ἀχθείσης καὶ

* Lit. “the ratio of ΓΑ to ΑΕ is compounded of the ratio of ΓΔ to ΔΖ and ΖΒ to ΒΕ.”
another at the point $Z$. I say that
\[ \Gamma A : AE = (\Gamma \Delta : \Delta Z)(ZB : BE). \]

For through $E$ let $EH$ be drawn parallel to $\Gamma \Delta$. Since $\Gamma \Delta$ and $EH$ are parallel,
\[ \Gamma A : EA = \Gamma \Delta : EH. \] [Eucl. vi. 4]
But $Z\Delta$ is an external [straight line];
\[ \therefore \quad \Gamma \Delta : EH = (\Gamma \Delta : \Delta Z)(\Delta Z : HE); \]
\[ \therefore \quad \Gamma A : AE = (\Gamma \Delta : \Delta Z)(\Delta Z : HE). \]
But $\Delta Z : HE = ZB : BE$; [Eucl. vi. 4]
by reason of the fact that $EH$ and $Z\Delta$ are parallels;
\[ \therefore \quad \Gamma A : AE = (\Gamma \Delta : \Delta Z)(ZB : BE); \quad (1) \]
which was set to be proved.

With the same premises, it will be shown by transformation of ratios that
\[ \Gamma E : EA = (\Gamma Z : \Delta Z)(\Delta B : BA), \]

\[ a \text{ parallel to } EB \text{ being drawn through } A \text{ and } \Gamma \Delta \]
GREEK MATHEMATICS

προσεκβληθείσας ἐπ' αὐτὴν τῆς ΓΔΗ. ἐπεὶ γὰρ πάλιν παράλληλός ἐστὶν ἡ ΑΗ τῇ ΕΖ, ἔστω, ὥς ἡ ΓΕ πρὸς ΕΑ, ἡ ΓΖ πρὸς ΖΗ. ἀλλὰ τῆς ΖΔ ἐξωθεν λαμβανομένης ὁ τῆς ΓΖ πρὸς ΖΗ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΖΔ καὶ τοῦ τῆς ΔΖ πρὸς ΖΗ· ἔστω δὲ ὁ τῆς ΔΖ πρὸς ΖΗ λόγος ὁ αὐτὸς τῷ τῆς ΔΒ πρὸς ΒΑ διὰ τὸ εἰς παραλλήλους τὰς ΑΗ καὶ ΖΒ διήχθαι τὰς ΒΑ καὶ ΖΗ· ὁ ἀρα τῆς ΓΖ πρὸς ΖΗ λόγος συνήπται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ. ἀλλὰ τῷ τῆς ΓΖ πρὸς ΖΗ λόγῳ ὁ αὐτὸς ἔστων ὁ τῆς ΓΕ πρὸς ΕΑ· καὶ ὁ τῆς ΓΕ ἀρα πρὸς ΕΑ λόγος σύγκειται ἐκ τε τοῦ τῆς ΓΖ πρὸς ΔΖ καὶ τοῦ τῆς ΔΒ πρὸς ΒΑ· ὁπερ ἐδει δεῖξαι.

Πάλιν ἔστω κύκλος ὁ ΑΒΓ, οὗ κέντρον τὸ Δ, καὶ εἰληφθὼς ἐπὶ τὰς περιφερείας αὐτοῦ τυχόντα 450
being produced to it. For, again, since AH is parallel to EZ,
\[ \Gamma E : EA = \Gamma Z : ZH. \]  
[Eucl. vi. 2]

But, an external straight line ZΔ having been taken,
\[ \Gamma Z : ZH = (\Gamma Z : Z\Delta)(\Delta Z : ZH); \]
and
\[ \Delta Z : ZH = \Delta B : BA, \]
by reason of BA and ZH being drawn to meet the parallels AH and ZB;
\[ \therefore \Gamma Z : ZH = (\Gamma Z : \Delta Z)(\Delta B : BA). \]
But \[ \Gamma Z : ZH = \Gamma E : EA; \]  
[supra]
and \[ \therefore \Gamma E : EA = (\Gamma Z : \Delta Z)(\Delta B : BA); \]  
(2)
which was to be proved.

Again, let AΒΓ be a circle with centre Δ, and let

there be taken on its circumference any three points

451
GREEK MATHEMATICS

tría σημεία τὰ Α, Β, Γ, ὄστε ἐκατέραν τῶν ΑΒ, ΒΓ περιφερεῖῶν ἐλάσσονα εἶναι ἡμικυκλίου καὶ ἐπὶ τῶν ἔξῆς δὲ λαμβανομένων περιφερειῶν τὸ ομοιὸν ὑπακονέσθω καὶ ἐπεζεύχωσαν αἱ ΑΓ καὶ ΔΕΒ. λέγω, ὦτι ἐστὶν, ὡς ἡ ὑπὸ τὴν διπλῆν τῆς ΑΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ, οὕτως ἡ ΑΕ εὐθεία πρὸς τὴν ΕΓ εὐθείαν.

"Ἡχθῶσαν γὰρ κάθετοι ἀπὸ τῶν Α καὶ Γ σημεῖων ἐπὶ τὴν ΔΒ ἢ τε ΑΖ καὶ ἢ ΓΗ. ἐπεὶ παράλληλος ἐστὶν ἡ ΑΖ τῇ ΓΗ, καὶ διήκται εἰς αὐτὰς εὐθεία ἡ ΑΕΓ, ἐστὶν, ὡς ἡ ΑΖ πρὸς τὴν ΓΗ, οὕτως ἡ ΑΕ πρὸς ΕΓ. ἀλλ' ὁ αὐτὸς ἐστὶν λόγος ὁ τῆς ΑΖ πρὸς ΓΗ καὶ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ· ἡμίσεια γὰρ ἐκατέρα ἐκατέρας· καὶ ὁ τῆς ΑΕ ἄρα πρὸς ΕΓ λόγος ὁ αὐτὸς ἐστὶν τῷ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ· ὅπερ ἐδεί δεῖξαι.

Παρακολουθεὶ δ' αὐτόθεν, ὦτι, καὶ δοθῶσιν ἡ τε ΑΓ ὅλη περιφέρεια καὶ ὁ λόγος ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΑΒ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΒΓ, δοθησεται καὶ ἐκατέρα τῶν ΑΒ καὶ ΒΓ περιφερειῶν. ἐκτεθείσης γὰρ τῆς αὐτῆς καταγραφῆς ἐπεζεύχωσῃ ἡ ΑΔ, καὶ ἦχθω ἀπὸ τοῦ Δ κάθετος ἐπὶ τὴν ΑΕΓ ἢ ΔΖ. ὦτι μὲν οὖν τῆς ΑΓ περί

452
TRIGONOMETRY

A, B, Γ, in such a manner that each of the arcs AB, BG is less than a semicircle; and upon the arcs taken in succession let there be a similar relationship; and let ΑΓ be joined and ΔΕΒ. I say that

the chord subtended by double of the arc AB:
the chord subtended by double of the arc BG

\([i.e., \sin AB : \sin BG'] = \AE : \EG\).

For let perpendiculars AZ and ΓΗ be drawn from the points A and Γ to ΔΒ. Since AZ is parallel to ΓΗ, and the straight line ΑΕΓ has been drawn to meet them,

AZ : ΓΗ = \AE : \EG. \quad [\text{Eucl. vi. 4}]

But \ AZ : ΓΗ = the chord subtended by double of the arc AB:
the chord subtended by double of the arc BG,

for each term is half of the corresponding term;

and therefore

\AE : \EG = the chord subtended by double of the arc AB:
the chord subtended by double of the arc BG. \quad . \quad . \quad . \quad (3)

\[= \sin AB : \sin BG\],

which was to be proved.

It follows immediately that, if the whole arc ΑΓ be given, and the ratio of the chord subtended by double of the arc AB to the chord subtended by double of the arc BG \([i.e. \sin AB : \sin BG]\), each of the arcs AB and BG will also be given. For let the same diagram be set out, and let ΑΔ be joined, and from Δ let ΔΖ be drawn perpendicular to ΑΕΓ. If the arc

* v. supra, p. 420 n. a.
utherford thes dodeis thes te upo AΔΖ γωνία tēn ἡμι-
seian autēs ὑποτεύνουσα δεδομένη ἐσται καὶ
όλον τὸ AΔΖ τρίγωνον, δῆλον· ἐπεὶ δὲ τῆς ΑΓ

εὐθεῖας ὅλης δεδομένης ὑπόκειται καὶ ὁ τῆς ΑΕ
πρὸς ΕΓ λόγος ὁ αὐτὸς ὃν τῷ τῆς ὑπὸ τὴν διπλήν
tῆς AB πρὸς τὴν ὑπὸ τὴν διπλήν τῆς ΒΓ, ἢ τε
ΑΕ ἐσται δοθεῖσα καὶ λοιπῇ ἡ ΖΕ. καὶ διὰ τούτο
καὶ τῆς ΔΖ δεδομένης δοθήσεται καὶ ἢ τε ὑπὸ
ΕΔΖ γωνία τοῦ ΕΔΖ ὀρθωγώνιον καὶ ὅλη ἢ ὑπὸ
ΑΔΒ· ὅστε καὶ ἢ τε AB περιφερεία δοθήσεται
καὶ λοιπῇ ἡ ΒΓ· ὅπερ ἐδει δεῖξαι.

Πάλιν ἐστὶν κύκλος ὁ ABΓ περὶ κέντρον τὸ Δ,
καὶ ἐπὶ τῆς περιφερείας αὐτοῦ εἰλήφθω τρία
σημεῖα τὰ A, B, Γ, ὥστε ἐκατέρων τῶν AB, ΑΓ
περιφερείων ἐλάσσονα εἶναι ἡμικυκλίου· καὶ ἐπὶ
454
TRIGONOMETRY

$\triangle$ is given, it is then clear that the angle $\triangle AZ$, subtending half the same arc, will also be given and therefore the whole triangle $\triangle AZ$; and since the whole chord $\triangle A$ is given, and by hypothesis

$AE : E\Gamma = $ the chord subtended by double of the arc $\triangle AB$,

the chord subtended by double of the arc $\triangle B\Gamma$,

[i.e. $\sin \triangle AB : \sin \triangle B\Gamma$],

therefore $AE$ will be given [Eucl. Dat. 7], and the remainder $Z\Gamma$. And for this reason, $\Gamma Z$ also being given, the angle $E\Gamma Z$ will be given in the right-angled triangle $E\Gamma Z$, and [therefore] the whole angle $\triangle A\Gamma B$; therefore the arc $\triangle AB$ will be given and also the remainder $\triangle B\Gamma$; which was to be proved.

Again, let $\triangle AB\Gamma$ be a circle about centre $\triangle$, and let

three points $A$, $B$, $\Gamma$ be taken on its circumference so that each of the arcs $\triangle AB$, $\triangle A\Gamma$ is less than a semicircle ;
GREEK MATHEMATICS

tων ἐξής δὲ λαμβανομένων περιφερειῶν τὸ ὁμοιόν ὑπακοὐέσθω. καὶ ἐπιζευχθεῖσα ἢ τε ΔΑ καὶ ἡ ΓΒ ἐκβεβλήθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Ε σημεῖον. λέγω, ὅτι ἐστὶν, ὡς ἡ ὑπὸ τὴν διπλῆν τῆς ΓΑ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς AB, οὔτως ἡ ΓΕ εὐθεία πρὸς τὴν BE.

"Ὅμοιως γὰρ τῷ προτέρῳ λημματιῶ, ἐὰν ἀπὸ τῶν B καὶ Γ ἀγάγωμεν καθέτους ἐπὶ τὴν ΔΑ τὴν τε ΒΖ καὶ τὴν ΓΗ, ἔσται διὰ τὸ παραλλήλους αὐτὰς εἶναι, ὡς ἡ ΓΗ πρὸς τὴν ΒΖ, οὔτως ἡ ΓΕ πρὸς τὴν EB· ὅστε καὶ, ὡς ἡ ὑπὸ τὴν διπλῆν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς AB, οὔτως ἡ ΓΕ πρὸς τὴν EB· ὅπερ ἔδει δεῖξαι.

Καὶ ἐνταῦθα δὲ αὐτὸθεν παρακολουθεῖ, διότι, κἂν ἡ ΓΒ περιφέρεια μόνη δοθῇ, καὶ ὁ λόγος ὁ τῆς ὑπὸ τὴν διπλῆν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς AB δοθῇ, καὶ ἡ AB περιφέρεια δοθῇ-σεται. πάλιν γὰρ ἐπὶ τῆς ὁμοίας καταγραφῆς ἐπιζευχθείσας τῆς ΔΒ καὶ καθέτου ἀκτείσης ἐπὶ τὴν BG τῆς ΔΖ ἡ μὲν ὑπὸ ΒΔΖ γωνία τῆς ἡμι-σειαν ὑποτείνουσα τῆς BG περιφερείας ἐσται 456
and upon the arcs taken in succession let there be a similar relationship; and let $\Delta A$ be joined and let $\Gamma B$ be produced so as to meet it at the point $E$. I say that

the chord subtended by double of the arc $\Gamma A$:
the chord subtended by double of the arc $AB$

\[ [i.e., \sin \Gamma A : \sin AB] = \Gamma E : BE. \]

For, as in the previous lemma, if from $B$ and $\Gamma$ we draw $BZ$ and $\Gamma H$ perpendicular to $\Delta A$, then, by reason of the fact that they are parallel,

\[ \Gamma H : BZ = \Gamma E : EB. \quad [\text{Eucl. vi. 4}] \]

\[ \therefore \text{the chord subtended by double of the arc } \Gamma A : \]
\[ \text{the chord subtended by double of the arc } AB \]

\[ [i.e., \sin \Gamma A : \sin AB] = \Gamma E : EB; \quad . \quad . \quad . \quad (4) \]

which was to be proved.

And thence it immediately follows why, if the arc $\Gamma B$ alone be given, and the ratio of the chord subtended by double of the arc $\Gamma A$ to the chord subtended by double of the arc $AB$ $[i.e., \sin \Gamma A : \sin AB]$, the arc $AB$ will also be given. For again, in a similar diagram let $\Delta B$ be joined and let $\Delta Z$ be drawn perpendicular to $\beta \Gamma$; then the angle $\beta \Delta Z$ subtended by half the arc $\beta \Gamma$ will be given; and therefore the
GREEK MATHEMATICS

δεδομένη· καὶ ὀλον ἄρα τὸ ΒΔΖ ὀρθογώνιον. ἔπει δὲ καὶ ὁ τῆς ΓΕ πρὸς τὴν ΕΒ λόγος δέδοται καὶ ἐτι ἡ ΓΒ εὐθεία, δοθῆσαται καὶ ἡ τε ΕΒ καὶ ἐτι ὅλη ἡ ΕΒΖ· ὦστε καὶ, ἔπει ἡ ΔΖ δέδοται, δοθῆσαται καὶ ἡ τε ύπὸ ΕΔΖ γωνία τοῦ αὐτοῦ ὀρθογώνιου καὶ λοιπῇ ἡ ύπὸ ΕΔΒ. ὦστε καὶ ἡ ΑΒ περιφέρεια ἔσται δεδομένη.

(ii.) The Theorem

Ibid. 74. 9-76. 9

Τούτων προληφθέντων γεγράφθωσαν ἐπὶ σφαιρικῆς ἐπιφανείας μεγίστων κύκλων περιφέρειαν, ὦστε εἰς δύο τὰς ΑΒ καὶ ΑΓ δύο γραφεῖσας τὰς ΒΕ καὶ ΓΔ τέμνειν ἀλλήλας κατὰ τὸ Ζ σημεῖον· ἐστω δὲ ἐκάστῃ αὐτῶν ἐλάσσων ἡμικυκλίου· τὸ δὲ αὐτὸ καὶ ἐπὶ πασῶν τῶν καταγραφῶν ὑποκούεσθω.

Λέγω δὴ, ὅτι ὁ τῆς ύπὸ τὴν διπλῆν τῆς ΓΕ περιφέρειας πρὸς τὴν ύπὸ τὴν διπλῆν τῆς ΕΑ λόγος συνήπται ἕκ τε τοῦ τῆς ύπὸ τὴν διπλῆν τῆς ΓΖ πρὸς τὴν ύπὸ τὴν διπλῆν τῆς ΖΔ καὶ τοῦ τῆς ύπὸ τὴν διπλῆν τῆς ΔΒ πρὸς τὴν ύπὸ τὴν διπλῆν τῆς ΒΑ.

Εἰλήφθω γὰρ τὸ κέντρον τῆς σφαίρας καὶ ἔστω τὸ Η, καὶ ἄχθωσαν ἀπὸ τοῦ Η ἐπὶ τὰς Β, Ζ, Ε τομάς τῶν κύκλων ἡ τε ΗΒ καὶ ἡ ΗΖ καὶ ἡ ΗΕ, καὶ ἐπιζευγθέεσα ἡ ΑΔ ἐκβεβλήσθω καὶ συμπεπτέτω τῇ ΗΒ ἐκβληθεῖσα καὶ αὐτῇ κατὰ τὸ Θ σημεῖον, ὁμοίως δὲ ἐπιζευγθέεσαι αἱ ΔΓ καὶ ΑΓ τεμνέτωσαν τὰς ΗΖ καὶ ΗΕ κατὰ τὸ Κ καὶ Λ 458
TRIGONOMETRY

whole of the right-angled triangle $B\Delta Z$. But since the ratio $\Gamma E : EB$ is given and also the chord $\Gamma B$, therefore $EB$ will also be given and, further, the whole [straight line] $EBZ$; therefore, since $\Delta Z$ is given, the angle $E\Delta Z$ in the same right-angled triangle will be given, and the remainder $E\Delta B$. Therefore the arc $AB$ will be given.

(ii.) The Theorem

_Ibid._ 74. 9–76. 9

These things having first been grasped, let there be described on the surface of a sphere arcs of great circles such that the two arcs $BE$ and $\Gamma \Delta$ will meet the two arcs $AB$ and $\Delta \Gamma$ and will cut one another at the point $Z$; let each of them be less than a semicircle; and let this hold for all the diagrams.

Now I say that the ratio of the chord subtended by double of the arc $\Gamma E$ to the chord subtended by double of the arc $EA$ is compounded of (a) the ratio of the chord subtended by double of the arc $\Gamma Z$ to the chord subtended by double of the arc $Z\Delta$, and (b) the ratio of the chord subtended by double of the arc $\Delta B$ to the chord subtended by double of the arc $BA$,

$$\left[i.e., \frac{\sin \Gamma E}{\sin EA} = \frac{\sin \Gamma Z}{\sin Z\Delta} \cdot \frac{\sin \Delta B}{\sin BA}\right].$$

For let the centre of the sphere be taken, and let it be $H$, and from $H$ let $HB$ and $HZ$ and $HE$ be drawn to $B$, $Z$, $E$, the points of intersection of the circles, and let $A\Delta$ be joined and produced, and let it meet $HB$ produced at the point $\Theta$, and similarly let $\Delta \Gamma$ and $\Delta \Gamma$ be joined and cut $HZ$ and $HE$ at $K$ and the point.
GREEK MATHEMATICS

σημείων· ἐπὶ μιᾶς δὴ γίνεται εὐθεῖας τὰ Θ, Κ, Λ σημεία διὰ τὸ ἐν δυσὶν ἀμα εἶναι ἐπίπεδοι τῷ τε τοῦ ΛΓΔ τριγώνου καὶ τῷ τοῦ ΒΖΕ κύκλου, ἤτις

ἐπιζευγθείσα ποιεῖ εἰς δύο εὐθεῖας τὰς ΘΑ καὶ ΓΛ διηγμένας τὰς ΘΛ καὶ ΓΔ τεμνούσας ἀλλήλας κατὰ τὸ Κ σημείων· ὁ ἀρὰ τῆς ΓΛ πρὸς ΛΑ λόγος συνήπται ἐκ τοῦ τῆς ΓΚ πρὸς ΚΔ καὶ τοῦ τῆς ΔΘ πρὸς ΘΑ. ἀλλ' ὡς μὲν ἡ ΓΛ πρὸς ΛΑ, οὐτῶς ἡ ὑπὸ τὴν διπλῆν τῆς ΓΕ πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΕΑ περιφερείας, ὡς δὲ ἡ ΓΚ πρὸς ΚΔ, οὐτῶς ἡ ὑπὸ τὴν διπλῆν τῆς ΓΖ περιφερείας πρὸς τὴν ὑπὸ τὴν διπλῆν τῆς ΖΔ, ὡς δὲ ἡ ΘΔ

460
TRIGONOMETRY

Then the points $\Theta$, $K$, $\Lambda$ will lie on one straight line because they lie simultaneously in two planes, that of the triangle $\Delta\Gamma\Lambda$ and that of the circle $BZE$, and therefore we have straight lines $\Theta\Lambda$ and $\Gamma\Delta$ meeting the two straight lines $\Theta\Lambda$ and $\Gamma\Lambda$ and cutting one another at the point $K$; therefore

$$\Gamma\Lambda : \Lambda\Lambda = (\Gamma K : K\Delta)(\Delta\Theta : \Theta\Lambda).$$  

[by (2)]

But $\Gamma\Lambda : \Lambda\Lambda$ = the chord subtended by double of the arc $\Gamma\Theta$:

the chord subtended by double of the arc $\Theta\Lambda$

[i.e., $\sin \Gamma\Theta : \sin \Theta\Lambda$],

while $\Gamma K : K\Delta$ = the chord subtended by double of the arc $\Gamma Z$:

the chord subtended by double of the arc $Z\Delta$  

[i.e., $\sin \Gamma Z : \sin Z\Delta$],

461
GREEK MATHEMATICS

πρὸς ΘΑ, οὕτως ἢ ὑπὸ τὴν διπλὴν τῆς ΔΒ περι-
ϕερεῖας πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΒΑ. καὶ ὁ
λόγος ἀρα ὁ τῆς ὑπὸ τὴν διπλὴν τῆς ΓΕ πρὸς τὴν
ὑπὸ τὴν διπλὴν τῆς ΕΑ συνήπται ἐκ τε τοῦ τῆς
ὑπὸ τὴν διπλὴν τῆς ΓΖ πρὸς τὴν ὑπὸ τὴν διπλὴν
τῆς ΖΔ καὶ τοῦ τῆς ὑπὸ την διπλὴν τῆς ΔΒ πρὸς
τὴν ὑπὸ τὴν διπλὴν τῆς ΒΑ.

Κατὰ τὰ αὐτὰ δὴ καὶ ὡσπερ ἐπὶ τῆς ἑπιπέδου
καταγραφῆς τῶν εὐθειῶν δεῖκνυται, ὅτι καὶ ὁ τῆς
ὑπὸ τὴν διπλὴν τῆς ΓΑ πρὸς τὴν ὑπὸ τὴν διπλὴν
τῆς ΕΑ λόγος συνήπται ἐκ τε τοῦ τῆς ὑπὸ τὴν
dιπλὴν τῆς ΓΔ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΔΖ
καὶ τοῦ τῆς ὑπὸ τὴν διπλὴν τῆς ΖΒ πρὸς τὴν ὑπὸ
tὴν διπλὴν τῆς ΒΕ· ἀπερ προέκειτο δεῖξαι.

* From the Arabic version, it is known that “Menelaus’s Theorem” was the first proposition in Book iii. of his Sphaerica, and several interesting deductions follow.
TRIGONOMETRY

and \( \Theta \Delta : \Theta A = \) the chord subtended by double of the arc \( \Delta B \):

the chord subtended by double of the arc \( BA \) \[[by (4)]\]

[i.e., \( \sin \Delta B : \sin BA \)],

and therefore the ratio of the chord subtended by double of the arc \( IE \) to the chord subtended by double of the arc \( EA \) is compounded of (a) the ratio of the chord subtended by double of the arc \( \Gamma Z \) to the chord subtended by double of the arc \( Z \Delta \), and (b) the ratio of the chord subtended by double of the arc \( \Delta B \) to the chord subtended by double of the arc \( BA \),

\[
\left[ \text{i.e.,} \right. 
\frac{\sin \Gamma E}{\sin EA} = \frac{\sin \Gamma Z}{\sin Z \Delta} \cdot \frac{\sin \Delta B}{\sin BA}.
\]

Now with the same premises, and as in the case of the straight lines in the plane diagram [by (1)], it is shown that the ratio of the chord subtended by double of the arc \( \Gamma A \) to the chord subtended by double of the arc \( EA \) is compounded of (a) the ratio of the chord subtended by double of the arc \( \Gamma \Delta \) to the chord subtended by double of the arc \( \Delta Z \), and (b) the ratio of the chord subtended by double of the arc \( ZB \) to the chord subtended by double of the chord \( BE \),

\[
\left[ \text{i.e.,} \right. 
\frac{\sin \Gamma A}{\sin EA} = \frac{\sin \Gamma \Delta}{\sin \Delta Z} \cdot \frac{\sin ZB}{\sin BE}.
\]

which was set to be proved.*
XXII. MENSURATION:
HERON OF ALEXANDRIA
XXII. MENSURATION: HERON OF ALEXANDRIA

(a) Definitions

Heron, Deff., ed. Heiberg (Heron iv.) 14. 1-24

Kai tâ mev pro tēs geωμετρικῆs stoιχειώσεωs
techvologouμeνa ὑπογράφων soi kai ὑποτυπούμενοs,
δω ἔχει μάλιστα συντόμωs, Διονύσιος λαμ-
πρότατε, τήν τε ἀρχὴν καὶ τήν ὅλην σύνταξιν
ποιήσομαι κατὰ τήν τοῦ Εὐκλείδου τοῦ Στοιχειω-
toῦ tῆs ἐν γεωμετρία theωρίας διδασκαλίαν ὀἶμαι
γὰρ οὗτωs ὦ ὑμὸν τὰs ἐκεῖνoν πραγματείαs

* The problem of Heron’s date is one of the most disputed questions in the history of Greek mathematics. The only details certainly known are that he lived after Apollonius, whom he quotes, and before Pappus, who cites him, say between 150 B.C. and A.D. 250. Many scraps of evidence have been thrown into the dispute, including the passage here first cited; for it is argued that the title λαμπρότατος corresponds to the Latin clarissimus, which was not in common use in the third century A.D. Both Heiberg (Heron, vol. v. p. ix) and Heath (II.G.M. ii. 306) place him, however, in the third century A.D., only a little earlier than Pappus.

The chief works of Heron are now definitively published in five volumes of the Teubner series. Perhaps the best known are his Pneumática and the Automata, in which he shows how to use the force of compressed air, water or steam; they are of great interest in the history of physics, and have led some to describe Heron as “the father of the turbine,” but
In setting out for you as briefly as possible, O most excellent Dionysius, a sketch of the technical terms premised in the elements of geometry, I shall take as my starting point, and shall base my whole arrangement upon, the teaching of Euclid, the writer of the elements of theoretical geometry; for in this way I think I shall give you a good general understanding, as they have no mathematical interest they cannot be noticed here. Heron also wrote a Belopoeica on the construction of engines of war, and a Mechanics, which has survived in Arabic and in a few fragments of the Greek.

In geometry, Heron’s elaborate collection of Definitions has survived, but his Commentary on Euclid’s Elements is known only from extracts preserved by Proclus and an-Nairizi, the Arabic commentator. In mensuration there are extant the Metrica, Geometrica, Stereometrica, Geodaesia, Mensurae and Liber Geôponicus. The Metrica, discovered in a Constantinople ms. in 1896 by R. Schöne and edited by his son H. Schöne, seems to have preserved its original form more closely than the others, and will be relied on here in preference to them. Heron’s Dioptra, describing an instrument of the nature of a theodolite and its application to surveying, is also extant and will be cited here.

For a full list of Heron’s many works, v. Heath, H.G.M. ii. 308-310.
The first definition is that of Euclid i. Def. 1, the third in effect that of Plato, who defined a point as ἀρχὴ γραμμῆς (Aristot. Metaph. 992 a 20); the second is reminiscent of Nicomachus, Arith. Introd. ii. 7. 1, v. vol. i. pp. 86-89.
MENSURATION: HERON OF ALEXANDRIA

not only of Euclid's works, but of many others pertaining to geometry. I shall begin, then, with the point.

1. A point is that which has no parts, or an extremity without extension, or the extremity of a line, and, being both without parts and without magnitude, it can be grasped by the understanding only. It is said to have the same character as the moment in time or the unit having position. It is the same as the unit in its fundamental nature, for they are both indivisible and incorporeal and without parts, but in relation to surface and position they differ; for the unit is the beginning of number, while the point is the beginning of geometrical being—but a beginning by way of setting out only, not as a part of a line, in the way that the unit is a part of number—and is prior to geometrical being in conception; for when a point moves, or rather is conceived in motion, a line is conceived, and in this way a point is the beginning of a line and a surface is the beginning of a solid body.

Ibid. 60. 22-62. 9

97. A spire is generated when a circle revolves and returns to its original position in such a manner that its centre traces a circle, the original circle remaining at right angles to the plane of this circle; this same curve is also called a ring. A spire is open when there is a gap, continuous when it touches at one point, and self-crossing when the revolving circle cuts itself.

GREEK MATHEMATICS

(a) Measurement of Areas and Volumes

(i.) Area of a Triangle Given the Sides

Heron, Metr. i. 8, ed. H. Schöne (Heron iii.) 18. 12–24. 21

"Εστι δὲ καθολικὴ μέθοδος ὦστε τριών πλευρῶν δοθεισῶν οὐδηποτὸν τριγώνου τὸ ἐμβαδόν εὑρεῖν χωρίς καθέτον· οἱν ἐστωσαν αἱ τοῦ τριγώνου πλευραί μονάδων ζ, η, θ. σύνθες τὰ ζ καὶ τὰ η καὶ τὰ θ· γίγνεται κδ. τούτων λαβὲ τὸ ἡμισὺ· γίγνεται ιβ. ἀφελε τὰς ζ μονάδας· λοιπαὶ ε. πάλιν ἀφελε ἀπὸ τῶν ιβ τὰς η· λοιπαὶ δ. καὶ ἐτὶ τὰς θ· λοιπαὶ γ. ποίησον τὰ ιβ ἐπὶ τὰ ε· γίγνονται ζ. ταῦτα ἐπὶ τὸν δ· γίγνονται γμ· ταῦτα ἐπὶ τὸν γ· γίγνεται ψκ· τούτων λαβὲ πλευρὰν καὶ ἔσται τὸ ἐμβαδόν τοῦ τριγώνου. ἐπεὶ οὖν αἱ ψκ ῥητὴν τὴν πλευρὰν οὐκ ἔχουσι, ληψόμεθα μετὰ διαφόρου ἐλαχίστον τὴν πλευρὰν οὔτως· ἐπεὶ οἱ συνεγχῖζων τῷ ψκ τετράγωνὸς ἐστὶν ὁ ψκδ καὶ πλευρὰν ἔχει τὸν κζ, μέρους τὰς ψκ εἰς τὸν κζ· γίγνεται κς καὶ τρίτα δύο· πρόσθες τὰς κζ· γίγνεται γν τρίτα δύο. τούτων τὸ ἡμισὺ· γίγνεται κς ζγ'. ἔσται ἀρα τοῦ ψκ ἡ πλευρὰ ἐγγίστα τὰ κς ζγ'. τὰ γὰρ κς ζγ' εφ' ἐαυτὰ γίγνεται ψκ λς'. ὥστε τὸ διάφορον μονάδων ἐστὶ μόριον λς'. ἐὰν δὲ βουλῶμεθα

470
MENSURATION: HERON OF ALEXANDRIA

Certain special curves are generated by sections of these spires. But the square rings are prismatic sections of cylinders; various other kinds of prismatic sections are formed from spheres and mixed surfaces.*

(b) MEASUREMENT OF AREAS AND VOLUMES

(i.) Area of a Triangle Given the Sides

Heron, Metrica i. 8, ed. H. Schöne (Heron iii.) 18. 12–24. 21

There is a general method for finding, without drawing a perpendicular, the area of any triangle whose three sides are given. For example, let the sides of the triangle be 7, 8 and 9. Add together 7, 8 and 9; the result is 24. Take half of this, which gives 12. Take away 7; the remainder is 5. Again, from 12 take away 8; the remainder is 4. And again 9; the remainder is 3. Multiply 12 by 5; the result is 60. Multiply this by 4; the result is 240. Multiply this by 3; the result is 720. Take the square root of this and it will be the area of the triangle. Since 720 has not a rational square root, we shall make a close approximation to the root in this manner. Since the square nearest to 720 is 729, having a root 27, divide 27 into 720; the result is $26\frac{2}{3}$; add 27; the result is $53\frac{2}{3}$. Take half of this; the result is $26\frac{1}{2} + \frac{1}{3}(=26\frac{5}{6})$. Therefore the square root of 720 will be very nearly $26\frac{5}{6}$. For $26\frac{5}{6}$ multiplied by itself gives $720\frac{1}{3}$; so that the difference is $\frac{1}{36}$. If we wish to make the difference less than $\frac{1}{36}$,

* The passage should be read in conjunction with those from Proclus cited supra, pp. 360-365; note the slight difference in terminology—self-crossing for interlaced.
If a non-square number $A$ is equal to $a^2 \pm b$, Heron's method gives as a first approximation to $\sqrt{A}$,

$$a_1 = \frac{1}{2} \left( a + \frac{A}{a} \right),$$

and as a second approximation,

$$a_2 = \frac{1}{2} \left( a_1 + \frac{A}{a_1} \right).$$

An equivalent formula is used by Rhabdas (v. vol. 1. p. 30 n. b) and by a fourteenth century Calabrian monk Barlaam, who wrote in Greek and who indicated that the process could be continued indefinitely. Several modern writers have used the formula to account for Archimedes' approximations to $\sqrt{3}$ (v. vol. i. p. 322 n. a).

Heron had previously shown how to do this.
instead of 729 we shall take the number now found, 720\(\frac{1}{5}\), and by the same method we shall find an approximation differing by much less than \(\frac{1}{5}\).\(^a\)

The geometrical proof of this is as follows: *In a triangle whose sides are given to find the area.* Now it is possible to find the area of the triangle by drawing one perpendicular and calculating its magnitude,\(^b\) but let it be required to calculate the area without the perpendicular.

Let \(\Delta\) be the given triangle, and let each of

\[AB, BG, GA\] be given; to find the area. Let the
GREEK MATHEMATICS

dόν. ἔγγεγράφθω εἰς τὸ τριγώνον κύκλος ὁ ΔΕΖ, οὗ κέντρον ἔστω τὸ Η, καὶ ἔπεζεύχθωσαν αἱ ΑΗ, ΒΗ, ΓΗ, ΔΗ, ΕΗ, ΖΗ. τὸ μὲν ἁρα ὑπὸ ΒΓ, ΕΗ διπλάσιον ἔστι τοῦ ΒΗΓ τριγώνου, τὸ δὲ ὑπὸ ΓΑ, ΖΗ τοῦ ΑΓΗ τριγώνου, (τὸ δὲ ὑπὸ ΑΒ, ΔΗ τοῦ ΑΒΗ τριγώνου) ¹. τὸ ἁρα ὑπὸ τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου καὶ τῆς ΕΗ, τουτέστι τῆς ἐκ τοῦ κέντρου τοῦ ΔΕΖ κύκλου, διπλάσιον ἐστὶ τοῦ ΑΒΓ τριγώνου. ἐκβεβλήσθω ἡ ΓΒ, καὶ τῇ ΑΔ ἵση κείσθω ἡ ΒΘ· ἡ ἁρα ΓΒΘ ἡμίσεια ἐστὶ τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου διὰ τὸ ἵσην εἶναι τὴν μὲν ΑΔ τῇ ΑΖ, τὴν δὲ ΔΒ τῇ ΒΕ, τὴν δὲ ΖΓ τῇ ΓΕ. τὸ ἁρα ὑπὸ τῶν ΓΘ, ΕΗ ὕσσον ἐστὶ τῷ ΑΒΓ τριγώνῳ. ἀλλὰ τὸ ὑπὸ τῶν ΓΘ, ΕΗ πλευρά ἐστιν τοῦ ἀπὸ τῆς ΓΘ ἐπὶ τὸ ἀπὸ τῆς ΕΗ· ἐσται ἁρα τοῦ ΑΒΓ τριγώνου τὸ ἐμβαδὸν ἐφ’ ἐαυτὸ γενόμενον ὕσσον τῷ ἀπὸ τῆς ΘΓ ἐπὶ τὸ ἀπὸ τῆς ΕΗ. ἡχθω τῇ μὲν ΓΗ πρὸς ὀρθᾶς ἡ ΗΛ, τῇ δὲ ΓΒ ἡ ΒΛ, καὶ ἐπεζεύχθω ἡ ΓΛ. ἐπεὶ οὖν ὀρθή ἐστιν ἐκατέρα τῶν ὑπὸ ΓΗΔ, ΓΒΛ, ἐν κύκλω ἁρα ἐστὶ τῷ ΓΗΒΛ τετράπλευρον· αἱ ἁραι ὑπὸ ΓΗΒ, ΓΛΒ δυσὶν ὀρθαῖς εἰσιν ὕσιν. εἰσὶν δὲ καὶ αἱ ὑπὸ ΓΗΒ, ΑΗΔ δυσὶν ὀρθαῖς ἤσαι διὰ τὸ δίχα τετμήσαν τὰς πρὸς τῷ ἡγιασάς ταῦς ΑΗ, ΒΗ, ΓΗ καὶ ἱσας εἶναι τὰς ὑπὸ τῶν ΓΗΒ, ΑΗΔ ταῖς ὑπὸ τῶν ΑΗΓ, ΔΗΒ καὶ τὰς πάσας τέτρασιν ὀρθαίς ἤσαι εἰσιν· ἵση ἁρα ἐστὶν ἡ ὑπὸ ΑΗΔ τῇ ὑπὸ ΓΛΒ. ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΔΗ ὀρθῇ τῇ ὑπὸ ΓΒΛ ἵση· ὅμοιον ἁρα ἐστὶ τῷ ΑΗΔ τριγώνου τῷ ΓΒΛ τριγώνῳ. ὡς ἁρα ἡ ΒΓ πρὸς

¹ τὸ δὲ . . . τριγώνου: these words, along with several
474
circle $\Delta EZ$ be inscribed in the triangle with centre $H$ [Eucl. iv. 4], and let $AH$, $BH$, $GH$, $\Delta H$, $EH$, $ZH$ be joined. Then

$$BG \cdot EH = 2 \cdot \text{triangle } BHG, \quad [\text{Eucl. i. 41}$$

$$GA \cdot ZH = 2 \cdot \text{triangle } AHG, \quad [\text{ibid.}$$

$$AB \cdot \Delta H = 2 \cdot \text{triangle } ABH. \quad [\text{ibid.}$$

Therefore the rectangle contained by the perimeter of the triangle $ABG$ and $EH$, that is the radius of the circle $\Delta EZ$, is double of the triangle $ABG$. Let $B'G$ be produced and let $B\Theta$ be placed equal to $AA'$; then $G'\Theta$ is half of the perimeter of the triangle $ABG$ because $AA' = AZ, \Delta B = BE, ZG' = G'E$ [by Eucl. iii. 17].

Therefore

$$G\Theta \cdot EH = \text{triangle } ABG. \quad [\text{ibid.}$$

But

$$G\Theta \cdot EH = \sqrt{G\Theta^2 \cdot EH^2};$$

therefore

$$(\text{triangle } ABG)^2 = \Theta G^2 \cdot EH^2.$$ 

Let $HA$ be drawn perpendicular to $GH$ and $BA$ perpendicular to $GB$, and let $GA$ be joined. Then since each of the angles $GHA, GBA$ is right, a circle can be described about the quadrilateral $\Gamma HBA$ [by Eucl. iii. 31]; therefore the angles $GHB, \Gamma AB$ are together equal to two right angles [Eucl. iii. 22]. But the angles $GHB, A\Delta$ are together equal to two right angles because the angles at $H$ are bisected by $AH, BH, GH$ and the angles $GHB, A\Delta$ together with $AHG, \Delta HB$ are equal to four right angles; therefore the angle $A\Delta$ is equal to the angle $\Gamma AB$. But the right angle $A\Delta H$ is equal to the right angle $\Gamma BA$; therefore the triangle $A\Delta$ is similar to the triangle $\Gamma BA$.

Other obvious corrections not specified in this edition, were rightly added to the text by a fifteenth-century scribe.
GREEK MATHEMATICS

THE, Ἡ ΑΔ πρὸς ΔΗ, τοιτέστων Ἡ ΒΘ πρὸς ΕΗ, καὶ εὐαλλάξ, ὡς Ἡ ΓΒ πρὸς ΒΘ, Ἡ ΒΑ πρὸς ΕΗ, τοιτεστὸ Ἡ ΒΚ πρὸς ΚΕ διὰ τὸ παράλληλον εἶναι τὴν ΒΑ τῇ ΕΗ, καὶ συνδέτω, ὡς Ἡ ΓΘ πρὸς ΒΘ, οὔτως Ἡ ΒΕ πρὸς ΕΚ. ὡστε καὶ ὡς τὸ ἀπὸ τῆς ΓΘ πρὸς τὸ ὑπὸ τῶν ΓΘ, ΘΒ, οὔτως τὸ ὑπὸ ΒΕΓ πρὸς τὸ ὑπὸ ΓΕΚ, τοιτεστὶ πρὸς τὸ ἀπὸ ΕΗ. εὖ ὀρθογώνιῳ γὰρ ἀπὸ τῆς ὀρθῆς ἐπὶ τῆς βάσις κάθετος ἦκται Ἡ ΕΗ. ὡστε τὸ ἀπὸ τῆς ΓΘ ἐπὶ τὸ ἀπὸ τῆς ΕΗ, οὐ πλευρὰ ἣν τὸ ἐμβαδὸν τοῦ ΑΒΓ τριγώνου, ἵσον ἔσται τῷ ὑπὸ ΓΘΒ ἐπὶ τὸ ὑπὸ ΓΕΒ. καὶ ἔστι δοθεῖσα ἐκάστῃ τῶν ΓΘ, ΘΒ, ΒΕ, ΓΕ. ἡ μὲν γὰρ ΓΘ ἡμίσεια ἔστι τῆς περιμέτρου τοῦ ΑΒΓ τριγώνου, ἡ δὲ ΒΘ ἡ ὑπεροχή, ἢ ὑπερέχει ἡ ἡμίσεια τῆς περιμέτρου τῆς ΓΒ, ἡ δὲ BE ἡ ὑπεροχή, ἢ ὑπερέχει ἡ ἡμίσεια τῆς περιμέτρου τῆς ΑΓ, ἡ δὲ ΕΓ ἡ ὑπεροχή, ἢ ὑπερέχει ἡ ἡμίσεια τῆς περιμέτρου τῆς ΑΒ, ἐπειδὴ περ ἵσο ἔστιν ἡ μὲν ΕΓ τῇ ΓΖ, ἡ δὲ ΒΘ τῇ ΑΖ, ἐπεὶ καὶ τῇ ΑΔ ἐστὶν ἵση. δοθὲν ἄρα καὶ τὸ ἐμβαδὸν τοῦ ΑΒΓ τριγώνου.

(ii.) Volume of a Spire

Ibid. ii. 13, ed. H. Schöne (Heron iii.) 126. 10–130. 3.

"Ἑστὼ γάρ τις ἐν ἐπιπέδῳ εὐθεῖα ἡ AB καὶ δύο τυχόντα ἐπ᾽ αὐτῆς σημεῖα. εἰλήφθω δὲ ΒΓΔΕ <κύκλος> ὁρθὸς ὡς πρὸς τὸ ὑποκείμενον ἐπίπεδον, ἐν ὃ ἔστιν ἡ AB εὐθεῖα, καὶ μένοντος τοῦ A

1 κύκλος add. H. Schöne.
MENSURATION: HERON OF ALEXANDRIA

Therfore \[ BI : BA = \Delta \Delta : \Delta H \]
\[ = B\Theta : EH, \]
and permutando, \[ \Gamma B : B\Theta = BA : EH \]
\[ = BK : KE, \]
because \( BA \) is parallel to \( EH \),
and componendo \[ \Gamma \Theta : B\Theta = BE : EK ; \]
therefore \[ \Gamma \Theta^2 : \Gamma \Theta \cdot \Theta B = BE \cdot \Theta \Gamma : \Gamma E \cdot EK, \]
i.e. \[ = BE \cdot \Gamma \Theta : EH^2, \]
for in a right-angled triangle \( EH \) has been drawn from the right angle perpendicular to the base; therefore \( \Gamma \Theta^2 \cdot EH^2 \), whose square root is the area of the triangle \( AB\Gamma \), is equal to \( (\Gamma \Theta \cdot \Theta B)(\Gamma E \cdot EB) \).
And each of \( \Gamma \Theta, \Theta B, BE, \Gamma E \) is given; for \( \Gamma \Theta \) is half of the perimeter of the triangle \( AB\Gamma \), while \( B\Theta \) is the excess of half the perimeter over \( \Gamma B \), \( BE \) is the excess of half the perimeter over \( \Gamma \Gamma \), and \( \Theta \Gamma \) is the excess of half the perimeter over \( AB \), inasmuch as \( \Theta \Gamma = \Gamma Z \), \( B\Theta = \Delta \Delta = \Delta Z \). Therefore the area of the triangle \( AB\Gamma \) is given.

(ii.) Volume of a Spire

*Ibid.* ii. 13, ed. H. Schöne (Heron iii.) 126. 10–130. 3

Let \( AB \) be any straight line in a plane and \( A, B \) any two points taken on it. Let the circle \( B\Gamma \Delta \Theta \) be taken perpendicular to the plane of the horizontal, in which lies the straight line \( AB \), and, while the point

* If the sides of the triangle are \( a, b, c \), and \( s = \frac{1}{2}(a + b + c) \), Heron's formula may be stated in the familiar terms,

area of triangle = \( \sqrt{s(s-a)(s-b)(s-c)} \).

Heron also proves the formula in his *Dioptra* 30, but it is now known from Arabian sources to have been discovered by Archimedes.
σημείου περιφερέσθω κατά το ἐπίπεδον Ἡ ἈΒ, ἀρκετοὶ εἰς τὸ αὐτὸ ἀποκατασταθῆ συμπεριφερο-
μένου καὶ τοῦ ΒΓΔΕ κύκλου ὀρθὸν διαμένοντος
πρὸς τὸ ὑποκείμενον ἐπίπεδον. ἀπογεννήσει ἀρα
τινὰ ἐπιφάνειαν Ἡ ΒΓΔΕ περιφέρεια, ἣν ἡ
σπειρικὴν καλοῦσιν κἂν μὴ ἡ ἦ δὲ ὅλος ὁ κύκλος,
ἀλλὰ τμῆμα αὐτοῦ, πάλιν ἀπογεννήσει τὸ τοῦ
κύκλου τμῆμα σπειρικῆς ἐπιφανείας τμῆμα, καθ-
ἀπερ εἰς καὶ αἱ ταῖς κίοσιν ὑποκείμεναι σπεῖραι·
τριῶν γὰρ οὕσω ἐπιφανεῖσσ᾽ ἐν τῷ καλομένῳ
ἀναγραφεῖ, ὃν δὴ τινες καὶ ἐμβολέα καλοῦσιν, δύο
μὲν κοιλων τῶν ἄκρων, μᾶς δὲ μέσης καὶ κυρτῆς,
ἀμα περιφερομέναι αἱ τρεῖς ἀπογεννώσι τὸ εἶδος
τῆς τοῖς κίοσιν ὑποκείμενης σπείρας.

Δέον οὖν ἐστω τὴν ἀπογεννηθείσαν σπεῖραν ὑπὸ
tοῦ ΒΓΔΕ κύκλου μετρήσαι. δεδόσθω ἢ μὲν
ἈΒ μονάδων κ, ἢ δὲ ΒΓ διαμέτρος μονάδων ἦβ.

εἴληφθω τὸ κέντρον τοῦ κύκλου τοῦ Ζ, καὶ ἀπὸ τῶν
Α, Ζ τῷ ὑποκείμενῳ ἐπιπέδῳ πρὸς ὀρθᾶς ἥχθωσαν
αἱ ΔΖΕ, ἩΑΘ. καὶ διὰ τῶν Δ, Ε τῇ ἈΒ παράλ-
478
A remains stationary, let AB revolve in the plane until it concludes its motion at the place where it started, the circle $B\Gamma\Delta E$ remaining throughout perpendicular to the plane of the horizontal. Then the circumference $B\Gamma\Delta E$ will generate a certain surface, which is called spiric; and if the whole circle do not revolve, but only a segment of it, the segment of the circle will again generate a segment of a spiric surface, such as are the spirae on which columns rest; for as there are three surfaces in the so-called anagrapheus, which some call also emboleus, two concave (the extremes) and one (the middle) convex, when the three are moved round simultaneously they generate the form of the spira on which columns rest.\(^a\)

Let it then be required to measure the spire generated by the circle $B\Gamma\Delta E$. Let $AB$ be given as 20, and the diameter $B\Gamma$ as 12. Let $Z$ be the centre of the circle, and through \(^b\) A, Z let $HA\Theta$, $\Delta ZE$ be drawn perpendicular to the plane of the horizontal. And through $\Delta$, $E$ let $\Delta H$, $E\Theta$ be drawn parallel to

\(^a\) The ἀναγραφεύς or ἐμβολεύς is the pattern or templet for applying to an architectural feature, in this case an Attic-Ionic column-base. The Attic-Ionic base consists essentially of two convex mouldings, separated by a concave one. In practice, there are always narrow vertical ribbons between the convex mouldings and the concave one, but Heron ignores them. In the templet, there are naturally two concave surfaces separated by a convex, and the kind of figure Heron had in mind appears to be that here illustrated. I am indebted to Mr. D. S. Robertson, Regius Professor of Greek in the University of Cambridge, for help in elucidating this passage.

\(^b\) Lit. "from."
GREEK MATHEMATICS

ληλοι ἡχθωσαν αἱ ΔΗ, ΕΘ. δέδεικται δὲ Διονυ-
σοδώρῳ ἐν τῷ Περὶ τῆς σπείρας ἑπιγραφομένῳ,
ὅτι ὃν λόγον ἔχει ὁ ΒΓΔΕ κύκλος πρὸς τὸ ἡμῖν
tου ΔΕΘ parαλληλογράμμου, τούτοιν ἔχει καὶ
ἡ γεννηθείσα σπείρα ὑπὸ τοῦ ΒΓΔΕ κύκλου πρὸς
tοῦ κύλινδρον, οὐ ἄξων μὲν ἑστὶν ὁ ΗΘ, ἡ δὲ ἔκ
tου κέντρου τῆς βάσεως ἡ ΕΘ. ἔπει οὖν ἡ ΒΓ
μονάδων ἰβ ἑστὶν, ἡ ἀρα ΖΓ ἑσται μονάδων ἵ.
ἐστὶ δὲ καὶ ἡ ΑΓ μονάδων ἱ. ἑσται ἀρα ἡ ΑΖ
μονάδων ἰδ, τούτου οὐ ΕΘ, ἦτις ἑστὶν ἐκ του
cέντρου τῆς βάσεως τοῦ εἰρημένου κύλινδρον·
dοθεὶς ἀρα ἑστὶν ὁ κύκλος· ἀλλὰ καὶ ὁ ἄξων δοθεὶς·
ἑστὶ γὰρ μονάδων ἰβ, ἔπει καὶ ἡ ΔΕ. ὡστε
dοθεὶς καὶ ὁ εἰρημένος κύλινδρος· καὶ ἑστὶ τὸ ΔΘ
parαλληλόγραμμον (δοθέν)². ὡστε καὶ τὸ ἡμῖν
αὐτοῦ· ἀλλὰ καὶ ὁ ΒΓΔΕ κύκλος· δοθεῖσα γὰρ
ἡ ΓΒ διάμετρος. λόγος ἀρα τοῦ ΒΓΔΕ κύκλου
πρὸς τὸ ΔΘ parαλληλόγραμμον δοθεῖς· ὡστε καὶ
tῆς σπείρας πρὸς τοῦ κύλινδρον λόγος ἑστὶ δοθεῖς.
καὶ ἑστὶ δοθεὶς ὁ κύλινδρος· δοθὲν ἀρα καὶ τὸ
στερεὸν τῆς σπείρας.

Συντεθήσεται δὴ ἀκολούθως τῇ ἀναλύει ὰὐτῶς.
ἀφελε ἀπὸ τῶν κ τὰ ἰβ· λοιπὰ ἱ· καὶ πρόσθες τὰ
κ· γίγνεται κη· καὶ μέτρησον κύλινδρον, οὐ ἡ μὲν
diάμετρος τῆς βάσεως ἑστὶ μονάδων κη, τὸ δὲ
ὕψος ἰβ· καὶ γίγνεται τὸ στερεὸν αὐτοῦ ἧτξβ.
καὶ μέτρησον κύκλον, οὐ διάμετρός ἑστὶ μονάδων
ἰβ· γίγνεται τὸ ἐμβαδὸν αὐτοῦ, καθὼς ἐμάθομεν,
ριγ ζ· καὶ λαβὲ τῶν κη τὸ ἡμῖν· γίγνεται ἰδ.
ἐπὶ τὸ ἡμῖν τῶν ἰβ· γίγνεται πδ· καὶ πολλα-
² δοθέν add. H. Schöne.
MENSURATION: HERON OF ALEXANDRIA

AB. Now it is proved by Dionysodorus in the book which he wrote On the Spire that the circle \( \Gamma \Delta E \) bears to half of the parallelogram \( \Delta E \Theta \) the same ratio as the spire generated by the circle \( \Gamma \Delta E \) bears to the cylinder having \( \Delta E \Theta \) for its axis and \( E \Theta \) for the radius of its base. Now, since \( \Gamma \) is 12, \( Z \Gamma \) will be 6. But \( A \Gamma \) is 8; therefore \( A Z \) will be 14, and likewise \( E \Theta \), which is the radius of the base of the aforesaid cylinder. Therefore the circle is given; but the axis is also given; for it is 12, since this is the length of \( \Delta E \). Therefore the aforesaid cylinder is also given; and the parallelogram \( \Delta \Theta \) is given, so that its half is also given. But the circle \( \Gamma \Delta E \) is also given; for the diameter \( \Gamma B \) is given. Therefore the ratio of the circle \( \Gamma \Delta E \) to the parallelogram is given; and so the ratio of the spire to the cylinder is given. And the cylinder is given; therefore the volume of the spire is also given.

Following the analysis, the synthesis may thus be done. Take 12 from 20; the remainder is 8. And add 20; the result is 28. Let the measure be taken of the cylinder having for the diameter of its base 28 and for height 12; the resulting volume is 7392. Now let the area be found of a circle having a diameter 12; the resulting area, as we learnt, is 113\( \frac{1}{2} \). Take the half of 28; the result is 14. Multiply it by the half of 12; the result is 84. Now multiply

\[ \pi r^2 : a \]

\[ a \] for Dionysodorus v. supra, p. 162 n. a and p. 364 n. a.

If \( \Delta E = H \Theta = 2r \) and \( E \Theta = a \), then the volume of the spire bears to the volume of the cylinder the ratio \( 2 \pi a : 2r \cdot \pi a^2 \) or \( \pi r : a \), which, as Dionysodorus points out, is identical with the ratio of the circle to half the parallelogram, that is, \( \pi r^2 : ra \) or \( \pi r : a \).
GREEK MATHEMATICS

πλασιάσας τὰ ἰττζβ ἐπὶ τὰ πίγ γ' καὶ τὰ γενόμενα
παράβαλε παρὰ τὸν πδ. γίγνεται ,θ'γυς γ'. τοσ-
ούτου ἐσταὶ τὸ στερεὸν τῆς σπείρας.

(iii.) Division of a Circle

Ibid. iii. 18, ed. H. Schöne (Heron iii.) 172. 13–174. 2

Τὸν δοθέντα κύκλον διελεύν εἰς τρία ἵσα δυσών
ἐυθείας. τὸ μὲν οὖν πρόβλημα ὀτι οὐ βητόν ἔστιν,
δήλου, τῆς εὐχρηστίας δὲ ἐνεκεν διελούμεν αὐτὸν
ὡς ἐγγιστα ὀστώ. ἐστο ὃ δοθεῖς κύκλος, οὗ
κέντρω τὸ Α, καὶ ἐνημοσθὼ εἰς αὐτὸν τρίγωνον
ἰούτλευρον, οὗ πλευρά ή ΒΓ, καὶ παράλληλος αὐτῇ
ἡχθω ή ΔΑΕ καὶ ἐπεζεύξθωσαν αἱ ΒΔ, ΔΓ. λέγω,
ὅτι τὸ ΔΒΓ τμῆμα τρίτον ἐγγιστά ἐστι μέρος τοῦ
ολού κύκλου. ἐπεζεύξθωσαν γὰρ αἱ ΒΑ, ΑΓ. ὃ
ἀρα ΑΒΓΖΒ τομεύς τρίτον ἐστὶ μέρος τοῦ ὀλοῦ
κύκλου. καὶ ἐστιν ἵσον τὸ ΑΒΓ τρίγωνον τῷ
ΒΓΔ τριγώνῳ. τὸ ἄρα ΒΔΓΖ σχήμα τρίτον
μέρος ἐστὶ τοῦ ὀλοῦ κύκλου, ὃ δὴ μείζον ἐστὶν
αὐτοῦ τὸ ΔΒΓ τμῆμα ἀνεπαισθήτου ὄντος ὡς
πρὸς τὸν ὀλον κύκλου. ὰμοῖος δὲ καὶ ἐτέραν
482
7392 by $113\frac{1}{7}$ and divide the product by 84; the result is $9956\frac{4}{7}$. This will be the volume of the spire.

(iii.) Division of a Circle

Ibid. iii. 18, ed. H. Schöne (Heron iii.) 172. 13–174. 2

To divide a given circle into three equal parts by two straight lines. It is clear that this problem is not rational, and for practical convenience we shall make the division as closely as possible in this way. Let the given circle have A for its centre, and let there be inserted in it an equilateral triangle with side $BG$, and let $DAE$ be drawn parallel to it, and let $BA, \Delta$ be joined. I say that the segment $\Delta B\Gamma$ is approximately a third part of the whole circle. For let $BA, \Delta \Gamma$ be joined. Then the sector $AB\Gamma ZB$ is a third part of the whole circle. And the triangle $AB\Gamma$ is equal to the triangle $B\Gamma \Delta$ [Eucl. i. 37]; therefore the figure $B\Delta \Gamma Z$ is a third part of the whole circle, and the excess of the segment $\Delta B\Gamma$ over it is negligible in comparison with the whole circle. Similarly, if we
πλευρὰν ἵσοπλεύρου τριγώνου ἐγγράψαντες ἀφε- λοῦμεν ἔτερον τρίτον μέρος: ὡστε καὶ τὸ κατα- λειπόμενον τρίτον μέρος ἔσται [μέρος]¹ τοῦ ὀλου κύκλου.

(iv.) Measurement of an Irregular Area

Heron, Diopt. 23, ed. H. Schöne (Heron iii.) 260. 18–264. 15

Τὸ δοθὲν χωρίον μετρῆσαι διὰ διάστρας. ἔστω τὸ δοθὲν χωρίον περιεχόμενον ὑπὸ γραμμῆς ἀτάκτου τῆς ΑΒΓΔΕΖΗΘ. ἐπεὶ οὖν ἐμάδομεν διὰ τῆς κατασκευασθείσης διάστρας διάγειν πάση τῇ δοθείσῃ εὐθείᾳ (ἐτέραν)² πρὸς ὀρθάς, ἔλαβον τι σημεῖον ἐπὶ τῆς περιεχούσης τὸ χωρίον γραμμῆς 484
MENSURATION: HERON OF ALEXANDRIA

inscribe another side of the equilateral triangle, we may take away another third part; and therefore the remainder will also be a third part of the whole circle.  

(iv.) Measurement of an Irregular Area

Heron, *Dioptra* b 23, ed. H. Schöne (Heron iii.)

260. 18–264. 15

*To measure a given area by means of the dioptra.* Let the given area be bounded by the irregular line $AB\Delta EZH\Theta$. Since we learnt to draw, by setting the dioptra, a straight line perpendicular to any other straight line, I took any point B on the line en-

---

a Euclid, in his book *On Divisions of Figures* which has partly survived in Arabic, solved a similar problem—*to draw in a given circle two parallel chords cutting off a certain fraction of the circle*; Euclid actually takes the fraction as one-third. The general character of the third book of Heron’s *Metrics* is very similar to Euclid’s treatise.

It is in the course of this book (iii. 20) that Heron extracts the cube root of 100 by a method already noted (vol. i. pp. 60–63).

b The dioptra was an instrument fulfilling the same purposes as the modern theodolite. An elaborate description of the instrument prefaces Heron’s treatise on the subject, and it was obviously a fine piece of craftsmanship, much superior to the “parallactic” instrument with which Ptolemy had to work—another piece of evidence against an early date for Heron.

---

1 $\mu\varepsilon\rho\sigma$ om. H. Schöne.
2 eτέραν add. H. Schöne.
GREEK MATHEMATICS

tο Β, καὶ ἕγαγον εὐθεὶαν τυχοῦσαν διὰ τῆς διώπτρας τὴν ΒΗ, καὶ ταύτη πρὸς ὀρθὰς τὴν ΒΓ, (καὶ ταύτη) ἔτεραν πρὸς ὀρθὰς τὴν ΓΖ, καὶ ὁμοίως τῇ ΓΖ πρὸς ὀρθὰς τὴν ΖΘ. καὶ ἐλαβον ἐπὶ τῶν ἀχθεισῶν εὐθειῶν συνεχῆ σημεία, ἐπὶ μὲν τῆς ΒΗ τὰ Κ, Α, Μ, Ν, Ε, Θ. ἐπὶ δὲ τῆς ΒΓ τὰ Π, Ρ. ἐπὶ δὲ τῆς ΓΖ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, ω. ἐπὶ δὲ τῆς ΖΘ τὰ ζ, ζ. καὶ ἀπὸ τῶν ληφθέντων σημείων ταῖς εὐθείαις, ἐφ' ὅν ἑστὶ τὰ σημεία, πρὸς ὀρθὰς ἕγαγον τὰς ΚΛ, ΔΑ, ΜΑ, ΝΒ, Ε, Γ, Θ, Δ, Π, Ε, Ρ, Ζ, ΣΖ, Τ, Η, Υ, Θ, Ψ, Δ, ΧΜ, ΨΜ, ΩΕ, ΖΜ, ζΜ οὕτως ὡστε [τὰς ἐπὶ]² τὰ πέρατα τῶν ἀχθεισῶν πρὸς ὀρθὰς [ἐπιζευγνυμένας]³ ἀπολαμβάνειν γραμμὰς ἀπὸ τῆς περιεχοῦσας τὸ χωρίον γραμμῆς σύνεγγυς εὐθείας· καὶ τούτων γενθέντων ἔσται δυνατὸν τὸ χωρίον μετρεῖν. τὸ μὲν γὰρ ΒΓΖΜ παραλληλόγραμμον ὀρθογώνιων ἐστιν· ἔπειτα τὰς πλευρὰς ἀλύσει ἡ σχοινίων βεβασανισμένω, τοιτέστιν μῆτ' ἐκτείνεσθαι μῆτε συστέλλεσθαι δυναμένω, μετρήσωντες ἔξομεν τὸ ἐμβαδὸν τοῦ παραλληλογράμμου. τὰ δ' ἐκτὸς τούτοις τρίγωνα ὀρθογώνια καὶ τραπέζια ὁμοίως μετρήσωμεν, ἔχοντες τὰς πλευρὰς αὐτῶν· ἔσται γὰρ τρίγωνα μὲν ὀρθογώνια τὰ ΒΚΛ, ΒΠ, Ε, ΓΡ, Ζ, ΓΣΖ, ЗΩΕ, ЗМ, ΘΗΜ· τὰ δὲ λοιπὰ τραπέζια ὀρθογώνια. τὰ μὲν οὖν τρίγωνα μετρεῖται τῶν περὶ τὴν ὀρθὴν γωνίαν πολλαπλασιαζομένων ἐπὶ ἄλληλα· καὶ τοῦ γενομένου τὸ ἡμισυ. τὰ δὲ τραπεζία· συναμφότερων τῶν παραλλήλων τὸ ἡμισυ ἐπὶ τὴν ἐπὶ αὐτῶς κάθετον οὖσαν, οἷον 486
closing the area, and by means of the dioptra drew any straight line BH, and drew BG perpendicular to it, and drew another straight line GZ perpendicular to this last, and similarly drew ZΘ perpendicular to GZ. And on the straight lines so drawn I took a series of points—on BH taking K, Λ, Μ, N, Ε, O, on BG taking Π, P, on GZ taking Σ, T, Υ, Φ, X, Ψ, Ω, and on ZΘ taking ζ, ζ. And from the points so taken on the straight lines designated by the letters, I drew the perpendiculars Kζ, ΛΑ, ΜΑ, ΝΒ, ΕΓ, ΟΔ, ΠΕ, Ρζ, ΣΖ, ΤΗ, ΥΘ, ΦΔ, ΧΜ, ΨΜ, ΩΕ, ζΜ, ζΜ in such a manner that the extremities of the perpendiculars cut off from the line enclosing the area approximately straight lines. When this is done it will be possible to measure the area. For the parallelogram BΓΖΜ is right-angled; so that if we measure the sides by a chain or measuring-rod, which has been carefully tested so that it can neither expand nor contract, we shall obtain the area of the parallelogram. We may similarly measure the right-angled triangles and trapezia outside this by taking their sides; for BKζ, BΠΕ, ΓΡζ, ΓΣζ, ΖΩΕ, Ζζζ, ΘΗΜ are right-angled triangles, and the remaining figures are right-angled trapezia. The triangles are measured by multiplying together the sides about the right angle and taking half the product. As for the trapezia—take half of the sum of the two parallel sides and multiply it by the perpendicular upon

1 καὶ ταύτη add. H. Schöne.
2 τὰς ἐπὶ om. H. Schöne.
3 ἐπιζευγνυμένας om. H. Schöne.
GREEK MATHEMATICS

τῶν Κ>, ΑΔ τὸ ἡμισυ ἐπὶ τὴν ΚΛ· καὶ τῶν λοιπῶν δὲ ὁμοίως. ἔσται ἀρα μεμετρημένον ὅλον τὸ χωρίον διὰ τε τοῦ μέσου παραλληλογράμμου καὶ τῶν ἐκτὸς αὐτῶν τριγώνων καὶ τραπεζίων. εἰς τῇ τύχῃ ποτὲ μεταξύ αὐτῶν τῶν ἁρθεισῶν πρὸς όρθὰς τὰς τοῦ παραλληλογράμμου πλευρὰς καμπύλη γραμμὴ μὴ συνεγγίζουσα εὐθεία (οἷον μεταξύ τῶν Ε,Γ, Ο,Δ γραμμῆς ἧ ,Γ,Δ), ἀλλὰ περιφερεῖ, μετρήσομεν οὕτως· ἁγαγόντες τῇ

Ο,Δ πρὸς όρθὰς τὴν ΔΜ, καὶ ἐπ’ αὐτῆς λαβόντες σημεῖα συνεχῆ τὰ Μ, Μ, καὶ ἀπ’ αὐτῶν πρὸς όρθὰς ἁγαγόντες τῇ Μ,Δ τὰς ΜΜ, ΜΜ, ὡστε τὰς μεταξύ τῶν ἁρθεισῶν σύνεγγυς εὐθείας εἶναι, τάλιν μετρήσομεν τὸ τε ΜΕΟ,Δ παραλληλόγραμμον καὶ τὸ MM,Δ τρίγωνον, καὶ τὸ ΓΜΜΜ τραπεζίον, καὶ ἐπὶ τὸ ἑτερον τραπεζίον, καὶ ἔσομεν τὸ περιεχόμενον χωρίον ύπὸ τε τῆς ΓΜΜ,Δ γραμμῆς καὶ τῶν ΓΕ, ΥΟ, ΥΟ εὐθειών μεμετρημένον.

(c) Mechanics

Heron, Diopt. 87, ed. H. Schöne (Heron iii.) 306. 22–312. 22

Τῇ δοθείσῃ δυνάμει τὸ δοθὲν βάρος κινῆσαι

1 τῇ add. H. Schöne. 2 ΥΟ add. H. Schöne.

* Heron's Mechanics in three books has survived in Arabic, but has obviously undergone changes in form. It begins with the problem of arranging toothed wheels so as to move 488.
them, as, for example, half of $K \gamma$, $\Lambda \Lambda$ by $K \Lambda$; and similarly for the remainder. Then the whole area will have been measured by means of the parallelogram in the middle and the triangles and trapezia outside it. If perchance the curved line between the perpendiculars drawn to the sides of the parallelogram should not approximate to a straight line (as, for example, the curve $\Gamma \Delta$ between $\Xi, \Gamma, O, \Delta$), but to an arc, we may measure it thus: Draw $\Delta \overline{M}$ perpendicular to $O, \Delta$, and on it take a series of points $\theta, \eta, \varsigma$ and from them draw $M \overline{M}, \overline{M} M$ perpendicular to $\overline{M} \Delta$, so that the portions between the straight lines so drawn approximate to straight lines, and again we can measure the parallelogram $\overline{M} \Xi O, \Delta$ and the triangle $\overline{M} \Delta$, and the trapezium $\Gamma M \overline{M} M$, and also the other trapezium, and so we shall obtain the area bounded by the line $\Gamma \overline{M} \Delta$ and the straight lines $\Gamma \Xi, \Xi O, O \Delta$.

(c) **Mechanics**

Heron, *Dioptia* 37, ed. H. Schöne (Heron iii.) 306. 22–312. 22

*With a given force to move a given weight by the given weight by a given force.* This account is the same as that given in the passage here reproduced from the *Dioptia*, and it is obviously the same as the account found by Pappus (viii. 19, ed. Hultsch 1060. 1–1068. 23) in a work of Heron's (now lost) entitled *Baroualcos* ("weight-lifter")—though Pappus himself took the ratio of force to weight as $4:160$ and the ratio of successive diameters as $2:1$. It is suggested by Heath (*H.G.M.* ii. 346–347) that the chapter from the
GREEK MATHEMATICS

diā τυμπάνων ὀδοντωτῶν παραθέσεως. κατε-
σκευάσθω πῆγμα καθάπερ γλωσσόκομον· εἰς τοὺς
μακροὺς καὶ παράλληλους τοίχους διακείσθωσαν
ἀξονες παράλληλοι ἑαυτοῖς, ἐν διαστήμασι κείμενοι

ὡστε τὰ συμφυὴ αὐτοῖς ὀδοντωτὰ τύμπανα παρα-
κείσθαι καὶ συμπεπλέχθαι ἀλλήλοις, καθὰ μέλλομεν
δηλοῦν. ἔστω τὸ εἰρημένον γλωσσόκομον τὸ
ἈΒΓΔ, ἐν ὧν ἄξων ἔστω διακείμενος, ὡς εἰρηταί,
καὶ δυνάμενος εὐλύτως στρέφεσθαι, ὁ ἙΖ. τούτῳ
δὲ συμφυὲς ἔστω τύμπανον ὀδοντωμένον τὸ ἘΘ
ἐχον τὴν διάμετρον, ἐὰν τὸν, πενταπλασίονα τῆς
τοῦ ἙΖ ἄξονος διαμέτρου. καὶ ἵνα ἐπὶ παραδείγ-
ματος τὴν κατασκευὴν ποιησόμεθα, ἔστω τὸ μὲν
ἀγόμενον βάρος ταλάντων χιλίων, ἡ δὲ κινώσα
δύναμις ἔστω ταλάντων ἐ, τούτεστιν ὁ κινών
ἀνθρώπος ἡ παιδόριον, ὡστε δύνασθαι καὶ ἐαυτῶν
ἀνευ μηχανῆς ἐλκειν τάλαντα ἐ. οὐκοῦν ἔδω τὰ
ἐκ τοῦ φορτίου ἐκδεδεμένα ὁπλα διὰ τῶν ὁπῆς

490
MENSURATION: HERON OF ALEXANDRIA

Juxtaposition of toothed wheels.¹ Let a framework be prepared like a chest; and in the long, parallel walls let there lie axles parallel one to another, resting at such intervals that the toothed wheels fitting on to them will be adjacent and will engage one with the other, as we shall explain. Let ΑΒΓΔ be the aforesaid chest, and let EZ be an axle lying in it, as stated above, and able to revolve freely. Fitting on to this axle let there be a toothed wheel ΗΘ whose diameter, say, is five times the diameter of the axle EZ. In order that the construction may serve as an illustration, let the weight to be raised be 1000 talents, and let the moving force be 5 talents, that is, let the man or slave who moves it be able by himself, without mechanical aid, to lift 5 talents. Then if the rope holding the load passes through some aperture in

Βαρούλκος was substituted for the original opening of the Mechanics, which had become lost.

Other problems dealt with in the Mechanics are the paradox of motion known as Aristotle’s wheel, the parallelogram of velocities, motion on an inclined plane, centres of gravity, the five mechanical powers, and the construction of engines. Edited with a German translation by L. Nix and W. Schmidt, it is published as vol. ii. in the Teubner Heron.

¹ Perhaps “rollers.”

¹ τῆς add. Vincentius.
GREEK MATHEMATICS

οὕσης⁴ ἐν τῷ ΑΒ τοῖχῳ ἐπειληθῇ περὶ τὸν ΕΖ ἀξόνα (.....)² κατειλύμενα τὰ ἐκ τοῦ φορτίου ὀπλα κινῆσει το βάρος³. ἦν δὲ κινήθη τὸ ΗΘ τύμπανου, (debita)μει⁴ ύπάρχειν πλέον ταλάντων διακοσίων, διὰ τὸ τὴν διάμετρον τοῦ τυμπάνου τῆς διαμέτρου τοῦ ἀξόνος, ὡς ύπεθέμεθα, πενταπλῆν (eivan)⁵. τάτα γὰρ ἀπεδείχθη ἐν ταῖς τῶν ἐ δυνάμεων ἀποδείξεων. ἀλλ' (.....)⁶ ἔχομεν τὶ τὴν δύναμιν ταλάντων διακοσίων, ἀλλὰ πέντε. γεγονέτω οὖν έτερος ἀξών (παράλληλος)⁷ διακειμένος τῷ EZ, ὁ ΚΛ, ἔχων συμφυές τύμπανον ωδοντωμένον τὸ MN. ωδοντώδες δὲ καὶ τὸ ΗΘ τύμπανον, ὡστε ἐναρμόζειν ταῖς ωδοντώσεις τοῦ MN τυμπάνου. τῷ δὲ αὐτῷ ἀξών τῷ ΚΛ συμφυές τύμπανον τὸ ΞΟ, ἔχον ὄμοιός τὴν διάμετρον πενταπλασίων τῆς τοῦ MN τυμπάνου διαμέτρου. διὰ δὴ τοῦτο δεῖσε τὸν βουλόμενον κινεῖν διὰ τοῦ ΞΟ τυμπάνου τὸ βάρος ἔχειν δύναμιν ταλάντων μ, ἐπειδήπερ τῶν ὅ ταλάντων τὸ πέμπτον ἐστὶ ταλάντα μ. πάλιν οὖν παρακείσθω (τῷ ΞΟ τυμπάνῳ ωδοντωμένῳ)⁸ τύμπανον ωδοντωθέν ἐτερον (τῷ ΠΡ, καὶ ἔστω τῷ)⁹ τυμπάνῳ ωδοντωμένῳ τῷ ΠΡ συμφυές ἐτερον τύμπανον τὸ ΣΤ¹⁰ ἔχον ὄμοιός πενταπλῆν τὴν διάμετρον τῆς ΠΡ τυμπάνου διαμέτρου. ἡ δὲ ἀνάλογος ἐσται δύναμις¹¹ τοῦ ΣΤ τυμπάνου ἢ ἔχουσα τὸ βάρος ταλάντων ἦ.

¹ ὁπῆς ὀὔης add. Hultsch et H. Schöne.
² After ἀξόνα there is a lacuna of five letters.
³ τὰ ἐκ τοῦ φορτίου ὀπλα κινῆσει το βάρος H. Schöne, τὰ ἐκ τοῦ φορτίου ἐπλακών εν τοι το βάρος cod.
⁴ debitae—"septem litteris madore absumptis, supplevi dubitanter," H. Schöne.
⁵ eivan add. H. Schöne.

492
the wall AB and is coiled round the axle EZ, the rope holding the load will move the weight as it winds up. In order that the wheel HΘ may be moved, a force of more than 200 talents is necessary, owing to the diameter of the wheel being, as postulated, five times the diameter of the axle; for this was shown in the proofs of the five mechanical powers.\(^6\) We have [not, however . . .] a force of 200 talents, but only of 5. Therefore let there be another axle KA, lying parallel to EZ, and having the toothed wheel MN fitting on to it. Now let the teeth of the wheel HΘ be such as to engage with the teeth of the wheel MN. On the same axle KA let there be fitted the wheel ΞO, whose diameter is likewise five times the diameter of the wheel MN. Now, in consequence, anyone wishing to move the weight by means of the wheel ΞO will need a force of 40 talents, since the fifth part of 200 talents is 40 talents. Again, then, let another toothed wheel ΠΠ lie alongside the toothed wheel ΞO, and let there be fitted to the toothed wheel ΠΠ another toothed wheel ΣΤ whose diameter is likewise five times the diameter of the wheel ΠΠ; then the force needing to be applied to the wheel ΣΤ will be 8 talents; but the force actually available

\(^6\) The wheel and axle, the lever, the pulley, the wedge and the screw, which are dealt with in Book ii. of Heron’s *Mechanics.*

\(^{11}\) \textit{ἀνάλογος ἐσται δύναμις}—so H. Schöne completes the lacuna.
GRÈK MATHÈMATICÈS

ἀλλ' ἡ ὑπάρχουσα ἡμῖν δύναμις δέδοται ταλάντων ἐ. ὁμοίως ἐτέρον παρακείσθω τύμπανον ὀδοντωμένον τὸ ΓΦ τῷ ΣΤ ὀδοντωθέντι τούδε τοῦ ΓΦ τυμπάνου τῷ ἄξονι συμφυὲς ἔστω τύμπανον τὸ ΨΤ ὀδοντωμένον, ὥστ' ἡ διάμετρος πρὸς τὴν τοῦ ΓΦ τυμπάνου διάμετρου λόγον ἐχέτω, δὲ τὰ ὀκτὼ τάλαντα πρὸς τὰ τῆς δοθείσης δυνάμεως τάλαντα ἐ.

Καὶ τούτων παρασκευασθέντων, ἐὰν ἑπινόησομεν τὸ ΑΒΓΔ (γλωσσόκομον) ἑτέρων κείμενον, καὶ ἐκ μὲν τοῦ ΕΖ ἄξονος τὸ βάρος ἐξάψωμεν, ἐκ δὲ τοῦ ΨΤ τυμπάνου τὴν ἐλκουσαν δύναμιν, οὐδοπότερον αὐτῶν κατενεχθήσεται, εὐλύτως στρεφόμενων τῶν ἄξωνων, καὶ τῆς τῶν τυμπάνων παραθέσεως καλῶς ἁρμοζοῦσης, ἀλλ' ὡσπερ ξυγοῦ τινος ἱσορροπῆσε ἡ δύναμις τῶν βάρει. ἐὰν δὲ ἐνὶ αὐτῶν προσθῶμεν ὀλίγον ἔτερον βάρος, καταρρέψει καὶ ἑνεχθήσεται ἐφ' ὁ προσετέθη βάρος, ὡστε ἐὰν ἐν τῶν ἵ ταλάντων δυνάμει < . . . . . > εἰ τέχνη μναίαδον προσετέθη βάρος, κατακρατήσει καὶ ἐπιστάσαται τὸ βάρος. ἀντὶ δὲ τῆς προσθέσεως τούτων παρακείσθω κοχλίας ἔχων τὴν ἐλικὰ ἁρμοστὴν τοὺς ὅδοις τοῦ τυμπάνου, στρεφόμενος εὐλύτως περὶ τόρμους ἐνότας ἐν τρήμασι στρογγύλους, ὅπ' ὁ μὲν ἐτερος ὑπερεχέτω εἰς τὸ ἐκτὸς μέρος τοῦ γλωσσοκόμου κατὰ τὸν ΓΔ (τοῖχον τοῦ παρακείμενον) τῷ κοχλίῳ· ἡ ἀρὰ ὑπεροχὴ τετραγωνισθεῖσα λαβέτω χειρολάβην τὴν ζζ, δι' ἃς ἐπιλαμβανόμενός τις καὶ ἐπιστρέφων ἐπιστρέφει τοὺς κοχλίαν καὶ τὸ ΨΤ τύμπανον, ὡστε καὶ τὸ ΓΦ συμφυὲς αὐτῶ. διὰ δὲ τούτῳ καὶ τὸ παρακείμενον τὸ ΣΤ ἐπιστραφήσεται, καὶ τὸ συμφυὲς αὐτῷ τὸ ΠΡ, καὶ τὸ τούτῳ παρακείμενον τὸ ΞΩ, 494
to us is 5 talents. Let there be placed another toothed wheel ΥΦ engaging with the toothed wheel ΣΤ; and fitting on to the axle of the wheel ΥΦ let there be a toothed wheel ΧΨ, whose diameter bears to the diameter of the wheel ΥΦ the same ratio as 8 talents bears to the given force 5 talents.

When this construction is done, if we imagine the chest ΑΒΓΔ as lying above the ground, with the weight hanging from the axle ΕΖ and the force raising it applied to the wheel ΧΨ, neither of them will descend, provided the axles revolve freely and the juxtaposition of the wheels is accurate, but as in a beam the force will balance the weight. But if to one of them we add another small weight, the one to which the weight was added will tend to sink down and will descend, so that if, say, a mina is added to one of the 5 talents in the force it will overcome and draw the weight. But instead of this addition to the force, let there be a screw having a spiral which engages the teeth of the wheel, and let it revolve freely about pins in round holes, of which one projects beyond the chest through the wall ΓΔ adjacent to the screw; and then let the projecting piece be made square and be given a handle ζς. Anyone who takes this handle and turns, will turn the screw and the wheel ΧΨ, and therefore the wheel ΥΦ joined to it. Similarly the adjacent wheel ΣΤ will revolve, and ΠΠ joined to it, and then the adjacent wheel ΞΟ, and then MN fitting

1 γλωσσόκομον add. H. Schöne.
2 After δυνάμει is a lacuna of seven letters.
3 In Schöne’s text δὲ is printed after τοῦτῳ.
4 τοῖχον τὸν παρακείμενον add. H. Schöne.
GREEK MATHEMATICS

(d) Optics: Equality of Angles of Incidence and Reflection


'Απεδείξε γάρ ὁ μηχανικὸς Ἰων ἐν τοῖς αὐτοῦ Κατοπτρικοῖς, ὅτι αἱ πρὸς ἵσας γωνίας κλώμεναι εὑθεῖαι ἐλάχισται εἰσὶ πασῶν τῶν ἀπὸ τῆς αὐτῆς καὶ ὀμοιομερούς γραμμῆς πρὸς τὰ αὐτὰ κλωμένων [πρὸς ἀνίσους γωνίας].

Olympiod. In Meteor. iii. 2 (Aristot. 371 b 18), ed. Stüve 212. 5-213. 21

'Επείδη γάρ τούτο ὠμολογημένον ἐστὶ παρὰ πᾶσιν, ὅτι οὐδὲν μάτην ἑργάζεται ἡ φύσις οὐδὲ ματαιοπονεῖ, ἕαν μὴ δώσωμεν πρὸς ἵσας γωνίας γίνεσθαι τὴν ἀνάκλασιν, πρὸς ἀνίσους ματαιοπονεῖ

1 τῆν add. H. Schöne.
2 πασῶν G. Schmidt, τῶν μέσων codd.
3 πρὸς ἀνίσους γωνίας om. R. Schöne.
MENSURATION: HERON OF ALEXANDRIA

on to this last, and then the adjacent wheel ΗΘ, and so finally the axle EZ fitting on to it; and the rope, winding round the axle, will move the weight. That it will move the weight is obvious because there has been added to the one force that moving the handle which describes a circle greater than that of the screw; for it has been proved that greater circles prevail over lesser when they revolve about the same centre.

(d) Optics: Equality of Angles of Incidence and Reflection

Damianus, On the Hypotheses in Optics 14, ed. R. Schöne 20. 12-18

For the mechanician Heron showed in his Catoptrica that of all [mutually] inclined straight lines drawn from the same homogenous straight line [surface] to the same [points], those are the least which are so inclined as to make equal angles. In his proof he says that if Nature did not wish to lead our sight in vain, she would incline it so as to make equal angles.

Olympiodorus, Commentary on Aristotle’s Meteora iii. 2 (371 b 18), ed. Stüve 212. 5-213. 21

For this would be agreed by all, that Nature does nothing in vain nor labours in vain; but if we do not grant that the angles of incidence and reflection are equal, Nature would be labouring in vain by following

* Damianus, or Heliodorus, of Larissa (date unknown) is the author of a small work on optics, which seems to be an abridgement of a large work based on Euclid’s treatise. The full title given in some mss.—Δαμιανοῦ φιλοσόφου τοῦ Ἡλιοδότου Λαρισαίου Περὶ ὄπτικῶν ὑποθέσεων βιβλία β leaves uncertain which was his real name.
GREEK MATHEMATICS

ἡ φύσις, καὶ ἀντὶ τοῦ διὰ βραχείας περιόδου
θάσαι τὸ ὅρωμενον τὴν ὄψιν, διὰ μακρᾶς περιόδου
τοῦτο φανῆσεται καταλαμβάνουσα. εὐρεθήσονται
γὰρ αἱ τὰς ἀνίσους γυνίας περιέχουσαι εὐθείαι,
αἰτίνες ἀπὸ τῆς ὀψεως [περιέχουσαι].3 φέρονται
πρὸς τὸ κάτοπτρον κάκειθεν πρὸς τὸ ὅρωμενον,
μεῖζονες οὖσαι τῶν τὰς ἰσασ γυνίας περιεχουσῶν
εὐθείῶν. καὶ ὦτο τοῦτο ἀληθὲς, δῆλον ἐντεῦθεν.

Ὑποκείσθω γὰρ τὸ κάτοπτρον εὐθεία τῆς ἦ ΑΒ,
καὶ ἔστω τὸ μὲν ὅρων Γ, τὸ δὲ ὅρωμενον τὸ Δ,
tὸ δὲ Ε σημεῖον τοῦ κατόπτρου, ἐν ὧ προσπη
tποὺσα ἢ ὄψις ἀνακλᾶται πρὸς τὸ ὅρωμενον, ἔστω,

καὶ ἑπέζευξθω ἦ ΓΕ, ΕΔ. λέγω ὅτι ἦ ὑπὸ ΑΕΓ
γυνία ἴση ἐστὶ τῇ ὑπὸ ΔΕΒ.

498
MENSURATION: HERON OF ALEXANDRIA unequal angles, and instead of the eye apprehending the visible object by the shortest route it would do so by a longer. For straight lines so drawn from the eye to the mirror and thence to the visible object as to make unequal angles will be found to be greater than straight lines so drawn as to make equal angles. That this is true, is here made clear.

For let the straight line AB be supposed to be the mirror, and let Γ be the observer, Δ the visible object, and let E be a point on the mirror, falling on which the sight is bent towards the visible object, and let ΓE, EΔ be joined. I say that the angle ΑΕΓ is equal to the angle ΔΕB.***

* Different figures are given in different mss., with corresponding small variants in the text. With G. Schmidt, I have reproduced the figure in the Aldine edition.

---

1 καταλαμβάνουσα om. Ideler.
2 περιέχουσαi om. R. Schöne, περιέχουσαi Ideler, Stüve.
3 φέρονταi R. Schöne, φερομένas codd.
Εἰ γὰρ μὴ ἔστιν ίση, ἐστὶ δὲ ἐπερον σημεῖον τοῦ κατόπτρου, ἐν οἷς πρὸς ἀνίσους γωνίας ἀνακλᾶται, τὸ Ζ, καὶ ἐπεζεύξθω ἡ ΓΖ, ΖΔ. δὴ λοιπὸν ὅτι ἡ ὑπὸ ΓΖΑ γωνία μείζων ἐστὶ τῆς ὑπὸ ΔΖΕ γωνίας. λέγω ὅτι αἱ ΓΖ, ΖΔ εὐθείαι, αὖτινες τὰς ἀνίσους γωνίας περιέχουσι ύποκειμένης τῆς AB εὐθείας, μείζονες εἰσὶ τῶν GE, ED εὐθείων, αὖτινες τὰς ἴσας γωνίας περιέχουσι μετὰ τῆς AB. ἦχθω γὰρ κάθετος ἀπὸ τοῦ Δ ἐπὶ τὴν AB κατὰ τὸ Η σημεῖον καὶ ἐκβεβλήσθω ἐπὶ εὐθείας ὡς ἐπὶ τὸ Θ. φανερὸν δὴ ὅτι αἱ πρὸς τῷ Η γωνίαι ἴσαι εἰσίν· ὀρθὰ γὰρ εἰσὶ. καὶ ἐστὶν ἡ ΔΗ τῇ ΗΘ ἴση, καὶ ἐπεζεύξθω ἡ ΘΖ καὶ ἡ ΘΕ. αὐτὴ μὲν ἡ κατασκευή. ἐπεὶ οὖν ἴση ἔστιν ἡ ΔΗ τῇ ΗΘ, ἀλλὰ καὶ ἡ ὑπὸ ΔΗΕ γωνία τῇ ὑπὸ ΘΗΕ γωνία ἴση ἐστὶ, κοινὴ δὲ πλευρά τῶν δύο τριγώνων ἡ ΠΕ, [καὶ βάσις ἡ ΘΕ βάσει τῇ ED ἴσῃ ἐστὶ, καί] τὸ ΗΘΕ τρίγωνον τῷ ΔΗΕ τριγώνῳ ἴσον ἐστὶ, καὶ {αι} ὁμοίως γωνίαι ταῖς λοιπαῖς γωνίαις εἰσίν ἴσαι, ὅπ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσι. ἴση ἀρα ἡ ΘΕ τῇ ED. πάλιν ἐπειδὴ τῇ ΗΘ ἴση ἐστὶν ἡ ΗΔ καὶ γωνία ἡ ὑπὸ ΔΗΖ γωνία τῇ ὑπὸ ΘΗΖ ἴση ἐστὶ, κοινὴ δὲ ἡ ΗΖ τῶν δύο τριγώνων τῶν ΔΗΖ καὶ ΘΗΖ, [καὶ βάσις ἀρα ἡ ΘΖ βάσει τῇ ΖΔ ἴσῃ ἐστὶ, καί] τὸ ΖΗΔ τρίγωνον τῷ ΘΗΖ τριγώνῳ ἴσον ἐστίν. ἴση ἀρα ἐστὶν ἡ ΘΖ τῇ ΖΔ, καὶ ἐπεὶ ἴση ἐστὶν ἡ ΘΕ τῇ ED, κοινὴ προσκείσθω ἡ ΕΓ. δύο ἀρα αἱ GE, ED δυοί ταῖς GE, ΘΕ ἴσαι εἰσίν. ὡλη ἀρα ἡ ΓΘ δυοί ταῖς GE, ED ἴσῃ ἐστὶ. καὶ ἐπεὶ παντὸς τριγώνου ἀἱ δύο πλευραί.
For if it be not equal, let there be another point Z, on the mirror, falling on which the sight makes unequal angles, and let T'Z, ZΔ be joined. It is clear that the angle TZA is greater than the angle ΔZE. I say that the sum of the straight lines T'Z, ZΔ which make unequal angles with the base line AB, is greater than the sum of the straight lines ΓΕ, ΕΔ, which make equal angles with AB. For let a perpendicular be drawn from Δ to AB at the point H and let it be produced in a straight line to Θ. Then it is obvious that the angles at H are equal; for they are right angles. And let ΔH = HΘ, and let ΘZ and ΘE be joined. This is the construction. Then since ΔH = HΘ, and the angle ΔHE is equal to the angle ΘHE, while HE is a common side of the two triangles, the triangle HΘE is equal to the triangle ΔHE, and the remaining angles, subtended by the equal sides are severally equal one to the other [Eucl. i. 4]. Therefore ΘE = EΔ. Again, since HΔ = HΘ and angle ΔHZ = angle ΘHZ, while HZ is common to the two triangles ΔHZ and ΘHZ, the triangle ZHΔ is equal to the triangle ΘHZ [ibid.]. Therefore ΘZ = ZΔ. And since ΘE = EΔ, let EΓ be added to both. Then the sum of the two straight lines ΓΕ, EΔ is equal to the sum of the two straight lines ΓΕ, EΘ. Therefore the whole ΓΘ is equal to the sum of the two straight lines ΓΕ, EΔ. And since in any triangle the sum of two sides is always greater than

---

1 καὶ ... καὶ. These words are out of place here and superfluous.
2 αὐτ add. Schmidt. But possibly καὶ ... ἐποτείνουσιν, being superfluous, should be omitted.
3 καὶ ... καὶ. These words are out of place here and superfluous.
GREEK MATHEMATICS

tῆς λοιπῆς μειζόνες εἰςι πάντη μεταλαμβάνομεναι,
τριγώνοι ἂρα τοῦ ΘΖΓ αἱ δύο πλευραὶ αἱ ΘΖ, ΖΓ μᾶς ὑπὸ τῆς ΓΘ μειζόνες εἰσιν. ἀλλ’ ἡ ΓΘ ἴση ἐστὶ ταῖς ΓΕ, ΕΔ. αἱ ΘΖ, ΖΓ ἂρα μειζόνες εἰσι τῶν ΓΕ, ΕΔ. ἀλλ’ ἡ ΘΖ τὴν ΖΔ ἴσὴν ἀπὸ αἱ ΖΓ, ΖΔ ἂρα τῶν ΓΕ, ΕΔ μειζόνες εἰσι. καὶ εἰσιν αἱ ΓΖ, ΖΔ αἱ τὰς ἀνίσους γωνίας περιέχουσα. αἱ ἂρα τὰς ἀνίσους γωνίας περιέχουσα μειζόνες εἰσι τῶν τὰς ἰσας γωνίας περιεχουσῶν· ὅπερ ἔδει δεῖξαι.

(e) Quadratic Equations

Heron, Geom. 21. 9-10, ed. Heiberg (Heron iv.) 380. 15-31

Δοθέντων συναμφοτέρων τῶν ἀριθμῶν ἡγοι τῆς
diαμέτρου, τῆς περιμέτρου καὶ τοῦ ἐμβαθοῦ τοῦ
κύκλου ἐν ἀριθμῷ εἰνὶ διαστείλαι καὶ εὐρεῖν ἔκα-
στον ἀριθμὸν. πολει σύνω. ἔστιν ὁ δοθεῖς ἀριθμὸς
μονάδες σῖβ. τάστα ἀεὶ ἔπὶ τὰ ρνδ. γίνονται
μυρίαδες γαί καὶ ἐβχμη. τοῦτοις προστίθει καθ-
ολικῶς ωμᾶ. γίνονται μυρίαδες τρεῖς καὶ ἐνθ potion
πλευρά τετράγωνος γίνεται ῥπγ. ἀπὸ τοῦτων κοῦ-
φίσον κθ. λοιπὰ ρνδ. ὃν μέρος καὶ γίνεται ἵδ. τοσ-
οῦτοι ἡ διάμετρος τοῦ κύκλου. ἐαν δὲ θῆλης
καὶ τὴν περιφέρειαν εὐρεῖν, ὑφελον τὰ κθ ἀπὸ τῶν
ῥπγ. λοιπὰ ρνδ. ταύτα πούρον δή. γίνονται ῥπγ.
tοῦτων λάβῃ μέρος ᾣ. γίνονται μδ. τοσοῦτον ἐκ

* The proof here given appears to have been taken by
Olympiodorus from Heron’s Catoptrica, and it is substanti-
ally identical with the proof in De Speculis 4. This work
was formerly attributed to Ptolemy, but the discovery of
Ptolemy’s Optics in Arabic has encouraged the belief, now
502
the remaining side, in whatever way these may be taken [Eucl. i. 20], therefore in the triangle ΘΖΓ the sum of the two sides ΘΖ, ΖΓ is greater than the one side ΓΘ. But

\[ ΓΘ = ΓE + EΔ; \]

\[ \therefore ΘZ + ZΓ > ΓE + EΔ. \]

But \[ ΘZ = ZΔ; \]

\[ \therefore ZΓ + ZΔ > ΓE + EΔ. \]

And ΓΖ, ZΔ make unequal angles; therefore the sum of straight lines making unequal angles is greater than the sum of straight lines making equal angles; which was to be proved.⁴

(e) Quadratic Equations

Heron, Geometrica 21. 9-10, ed. Heiberg
(Heron iv.) 380. 15-31

Given the sum of the diameter, perimeter and area of a circle, to find each of them separately. It is done thus: Let the given sum be 212. Multiply this by 154; the result is 32648. To this add 841, making 33489, whose square root is 183. From this take away 29, leaving 154, whose eleventh part is 14; this will be the diameter of the circle. If you wish to find the circumference, take 29 from 183, leaving 154; double this, making 308, and take the seventh part, which is 44; this will be the perimeter. To

usually held, that it is a translation of Heron’s Catoptrica. The translation, made by William of Moerbeke in 1269, can be shown by internal evidence to have been made from the Greek original and not from an Arabic translation. It is published in the Teubner edition of Heron’s works, vol. ii. part i.
(f) Indeterminate Analysis

Heron, *Geom.* 24. 1, ed. Heiberg
(Heron iv.) 414. 28–415. 10

Εὑρεῖν δύο χωρία τετράγωνα, ὡς τὸ τοῦ πρώτου ἐμβαδὸν τοῦ τοῦ δεύτερου ἐμβαδὸν ἔσται τριπλάσιον. ποιώ οὗτως· τὰ γὺ κύποιον· γίνονται

"If *d* is the diameter of the circle, then the given relation is that

\[ d + \frac{22}{7} d + \frac{11}{14} d^2 = 212, \]

i.e.

\[ \frac{11}{14} d^2 + \frac{29}{7} d = 212. \]

To solve this quadratic equation, we should divide by \( 14 \) so as to make the first term a square; Heron makes the first term a square by multiplying by the lowest requisite factor, in this case 154, obtaining the equation

\[ 11^2 d^2 + 2 \cdot 29 \cdot 11d = 154 \cdot 212. \]

By adding 841 he completes the square on the left-hand side

\[ (11d + 29)^2 = 154 \cdot 212 + 841 \]
\[ = 32648 + 841 \]
\[ = 33489. \]

\[ \therefore \quad 11d + 29 = 183. \]

\[ \therefore \quad 11d = 154, \]

and \( d = 14. \)

The same equation is again solved in *Geom.* 24. 46 and a similar one in *Geom.* 24. 47. Another quadratic equation is solved in *Geom.* 24. 3 and the result of yet another is given in *Metr.* iii. 4.

504
find the area. It is done thus: Multiply the diameter, 14, by the perimeter, 44, making 616; take the fourth part of this, which is 154; this will be the area of the circle. The sum of the three numbers is 212.

(f) Indeterminate Analysis

Heron, Geometrica 24. 1, ed. Heiberg
(Heron iv.) 414. 28–415. 10

To find two rectangles such that the area of the first is three times the area of the second. I proceed thus:

* The Constantinople ms. in which Heron's Metrica was found in 1896 contains also a number of interesting problems in indeterminate analysis; and two were already extant in Heron's Geëponicus. The problems, thirteen in all, are now published by Heiberg in Heron iv. 414. 28–426. 29.

* It appears also to be a condition that the perimeter of the second should be three times the perimeter of the first. If we substitute any factor $n$ for 3 the general problem becomes: To solve the equations

$$u + v = n(x + y) \quad \cdots \cdots (1)$$

$$xy = nuv \quad \cdots \cdots (2)$$

The solution given is equivalent to

$$x = 2n^3 - 1, \quad y = 2n^3$$

$$u = n(4n^3 - 2), \quad v = n.$$}

Zeuthen (Bibliotheca mathematica, viii. (1907–1908), pp. 118–134) solves the problem thus: Let us start with the hypothesis that $v = n$. It follows from (1) that $u$ is a multiple of $n$, say $nz$. We have then

$$x + y = 1 + z,$$

while by (2)

$$xy = n^3z,$$

whence

$$xy = n^3(x + y) - n^3$$

or

$$(x - n^3)(y - n^3) = n^3(n^3 - 1).$$

An obvious solution of this equation is

$$x - n^3 = n^3 - 1, \quad y - n^3 = n^3,$$

which gives $z = 4n^3 - 2$, whence $u = n(4n^3 - 2)$. The other values follow.
GREEK MATHEMATICS

κέ· ταύτα δίς· γίνονται νῦ. νῦν ἄρον μονάδα ἀ· λοιπὸν γίνονται νῦ. ἔστω οὖν ἡ μὲν μία πλευρά ποδῶν νῦ, ἡ δὲ ἐτέρα πλευρά ποδῶν νῦ· καὶ τοῦ ἄλλου χωρίου οὕτως· θέσ όμοι τά νῦ καὶ τά νῦ· γίνονται πόδες πέ· ταύτα ποιεῖ ἐπὶ τά γ... λοιπὸν γίνονται πόδες τῆ. ἔστω οὖν ἡ τοῦ προτέρου πλευρᾶ ποδῶν τῆ, ἡ δὲ ἐτέρα πλευρᾶ ποδῶν γ· τά δὲ ἐμβαδά τοῦ εἴνος γίνεται ποδῶν ἄνδ καὶ τοῦ ἄλλου ποδῶν ἁμήβ.

Ibid. 24. 10, ed. Heiberg (Heron iv.) 422. 15-424. 5

Τριγώνου ὀρθογωνίου τὸ ἐμβαδόν μετὰ τῆς περιμέτρου ποδῶν σπ· ἀποδιαστείλαι τὰς πλευρὰς καὶ εὐρεῖν τὸ ἐμβαδόν. ποιῶ οὕτως· ἀεὶ ζήτει τοὺς ἀπαρτιζόμενας ἀριθμοὺς· ἀπαρτίζει δὲ τὸν σπ· ὁ δις τὸν ρμ· ὁ δ' τὸν ἵ, ὁ ἐ' τὸν υψ· ὁ ζ' τὸν μυ· ὁ η' τὸν λε· ὁ ι' τὸν κη· ὁ υδ' τὸν κ. ἐσκεφάλημα· ὅτι οὔ· ἢ καὶ ἰε ποιήσουι τὸ δοθὲν ἐπίταγμα. τῶν σπ· τὸ η'· γίνονται πόδες λε· διὰ παντὸς λάμβανε δυνάδα τῶν η'. λοιπὸν μένουσιν ζ· πόδες· τὰ οὖν λε καὶ τὰ ζ· ὁμοί γίνονται πόδες μα· ταύτα ποιεῖ ἐφ' ἑαυτά· γίνονται πόδες ἀκτί. τὰ λε ἐπὶ τὰ ζ· γίνονται πόδες δι· ταύτα ποιεῖ ἐπὶ τὰ η'· γίνονται πόδες ἀκτί· ταύτα ἄρον ἀπὸ τῶν ἀκτί. λοιπὸν μένει ἅ· ὃν πλευρά τετραγωνική γίνεται ἁ. ἀρτι θές τὰ μα καὶ ἄρον μονάδα ἅ· λοιπὸν μυ· ὃν λε' γίνεται κ. τοῦτο ἑστὶν ἡ κάθετος· ποδῶν ἅ· καὶ θές πάλιν τὰ μα καὶ πρόσθες ἅ· γίνονται πόδες μβ· ὃν λε' γίνεται πόδες κα· ἑστὶν ἡ βάσις· ποδῶν κα. καὶ θές τὰ λε καὶ ἄρον τὰ ζ· λοιπὸν μένουσι.
MENSURATION: HERON OF ALEXANDRIA

Take the cube of 3, making 27; double this, making 54. Now take away 1, leaving 53. Then let one side be 53 feet and the other 54 feet. As for the other rectangle, [I proceed] thus: Add together 53 and 54, making 107 feet: multiply this by 3, [making 321; take away 3], leaving 318. Then let one side be 318 feet and the other 3 feet. The area of the one will be 954 feet and of the other 2862 feet.a

Ibid. 24. 10, ed. Heiberg (Heron iv.) 422. 15–424. 5

In a right-angled triangle the sum of the area and the perimeter is 280 feet; to separate the sides and find the area. I proceed thus: Always look for the factors; now 280 can be factorized into 2.140, 4.70, 5.56, 7.40, 8.35, 10.28, 14.20. By inspection, we find 8 and 35 fulfil the requirements. For take one-eighth of 280, getting 35 feet. Take 2 from 8, leaving 6 feet. Then 35 and 6 together make 41 feet. Multiply this by itself, making 1681 feet. Now multiply 35 by 6, getting 210 feet. Multiply this by 8, getting 1680 feet. Take this away from the 1681, leaving 1, whose square root is 1. Now take the 41 and subtract 1, leaving 40, of which the half is 20; this is the perpendicular, 20 feet. And again take 41 and add 1, getting 42 feet, of which the half is 21; and let this be the base, 21 feet. And take 35 and subtract 6, leaving 29 feet. Now multiply

a The term "feet," πόδες, is used by Heron indiscriminately of lineal feet, square feet and the sum of numbers of lineal and square feet.

507
Heath (H.G.M. ii. 446-447) shows how this solution can be generalized. Let $a$, $b$ be the sides of the triangle containing the right angle, $c$ the hypotenuse, $S$ the area of the triangle, $r$ the radius of the inscribed circle; and let

$$s = \frac{1}{2}(a + b + c).$$

Then

$$S = rs = \frac{1}{2}ab, \quad r + s = a + b, \quad c = s - r.$$ 

Solving the first two equations, we have

$$\frac{a}{b} = \frac{1}{2} [r + s \mp \sqrt{(r + s)^2 - 8rs}],$$

and this formula is actually used in the problem. The
the perpendicular and the base together, [getting 420], of which the half is 210 feet; and the three sides comprising the perimeter amount to 70 feet; add them to the area, getting 280 feet.\textsuperscript{a}

method is to take the sum of the area and the perimeter \(S + 2s\), separated into its two obvious factors \(s(r + 2)\), to put \(s(r + 2) = A\) (the given number), and then to separate \(A\) into suitable factors to which \(s\) and \(r + 2\) may be equated. They must obviously be such that \(sr\), the area, is divisible by 6.

In the given problem \(A = 280\), and the suitable factors are \(r + 2 = 8, s = 35\), because \(r\) is then equal to 6 and \(rs\) is a multiple of 6. Then

\[
a = \frac{1}{2}[6 + 35 - \sqrt{(6 + 35)^2 - 8 \cdot 6 \cdot 35}] = \frac{1}{2}(41 - 1) = 20,
\]
\[
b = \frac{1}{2}(41 + 1) = 21,
\]
\[
c = 35 - 6 = 29.
\]

This problem is followed by three more of the same type.
XXIII. ALGEBRA: DIOPHANTUS
XXIII. ALGEBRA: DIOPHANTUS

(a) General

Anthol. Palat. xiv. 126, The Greek Anthology, ed. Paton (L.C.L.) v. 92-93

Oūtōs toı Διόφαντον ἔχει τάφος· ὃ μέγα θαύμα· καὶ τάφος ἐκ τέχνης μέτρα βίου λέγει. ἔκ την κουρίζειν βιότου θεὸς ὑπασε μοίρην. δωδεκάτην δ' ἐπιθεῖς, μηλα πόρεν χνοάειν· τῇ δ' ἄφ' ἐφ' ἐβδομάτῃ τὸ γαμήλιον ἠματο φέγγος, ἐκ δὲ γάμων πέμπτῳ παιδ' ἐπένευσεν ἔτει. αἰαὶ, τηλύγετον δειλὸν τέκος, ἡμιαυ πατρὸς τοῦδε καὶ ἡ κρυερὸς μέτρον ἐλῶν βιότον. πένθος δ' αὖ πισύρεσοι παρηγορεῶν ἐνιαυτοῖς τῆδε πόσου σοφίῃ τέρμ' ἐπέρησε βίου.

* There are in the Anthology 46 epigrams which are algebraical problems. Most of them (xiv. 116-146) were collected by Metrodorus, a grammarian who lived about A.D. 500, but their origin is obviously much earlier and many belong to a type described by Plato and the scholiast to the Charmides (v. vol. i. pp. 16, 20).

Problems in indeterminate analysis solved before the time of Diophantus include the Pythagorean and Platonic methods of finding numbers representing the sides of right-angled triangles (v. vol. i. pp. 90-95), the methods (also Pythagorean) of finding "side- and diameter-numbers" (vol. i. pp. 132-139), Archimedes' Cattle Problem (v. supra, pp. 202-205) and Heron's problems (v. supra, pp. 504-509).

512
This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father's life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.

Diophantus's surviving works and ancillary material are admirably edited by Tannery in two volumes of the Teubner series (Leipzig, 1895). There is a French translation by Paul Ver Eecke, Diophante d'Alexandre (Bruges, 1926). The history of Greek algebra as a whole is well treated by G. F. Nesselmann, Die Algebra der Griechen, and by T. L. Heath, Diophantus of Alexandria: A Study in the History of Greek Algebra, 2nd ed. 1910.

If \(x\) was his age at death, then
\[
\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4 = x,
\]
whence \(x = 84\).
GREEK MATHEMATICS

Theon Alex. in Ptol. Math. Syn. Comm. i. 10, ed. Rome, Studi e Testi, lxxii. (1936), 453. 4-6

Καθ' ἣ καὶ Διόφαντος φησιν: "τῆς γὰρ μονάδος ἀμεταθέτου οὔσης καὶ ἐστώσης πάντοτε, το πολλαπλασιαζόμενον εἶδος ἐπ' αὐτὴν αὐτὸ τὸ εἶδος ἔσται."

Dioph. De polyg. num. [5], Dioph. ed. Tannery i. 470. 27-472. 4

Καὶ ἀπεδείχθη τὸ παρὰ 'Ὑψικλεῖ ἐν ὅρῳ λεγόμενον, ὅτι, "ἐὰν ὅσιν ἀριθμοὶ ἀπὸ μονάδος ἐν ἴσῃ ύπεροχῇ ὑποσοιών, μονάδος μενούσης τῆς ύπεροχῆς, ο σύμπας ἐστίν (τρίγωνος, δυάδος δὲ), τετράγωνος, τριάδος δὲ, πεντάγωνος λέγεται δὲ τὸ πλήθος τῶν γωνιῶν κατὰ τὸν δυάδι μείζονα τῆς ύπεροχῆς, πλευρᾶί δὲ αὐτῶν τὸ πλῆθος τῶν ἐκτεθέντων συν τῇ μονάδι."


Περὶ δὲ τῆς Αἰγυπτιακῆς μεθόδου ταύτης Διόφαντος μὲν διελαβεῖν ἀκριβεστέρον, ὁ δὲ λογιστάτος Ἀνατόλιος τὰ συνεκτικώτατα μέρη τῆς κατ' τρίγωνος, δυάδος δὲ add. Bachet.

1 Cf. Dioph. ed. Tannery i. 8. 13-15. The word εἶδος, as will be seen in due course, is regularly used by Diophantus for a term of an equation.
ALGEBRA: DIOPHANTUS

Theon of Alexandria, Commentary on Ptolemy's Syntaxis i. 10, ed. Rome, Studi e Testi, lxxii. (1936), 453. 4-6

As Diophantus says: "The unit being without dimensions and everywhere the same, a term that is multiplied by it will remain the same term."*  

Diophantus, On Polygonal Numbers [5], Dioph. ed. Tannery i. 470. 27-472. 4

There has also been proved what was stated by Hypsicles in a definition, namely, that "if there be as many numbers as we please beginning from 1 and increasing by the same common difference, then, when the common difference is 1, the sum of all the numbers is a triangular number; when 2, a square number; when 3, a pentagonal number [; and so on]. The number of angles is called after the number which exceeds the common difference by 2, and the sides after the number of terms including 1." b

Michael Psellus, c A Letter, Dioph. ed. Tannery ii. 38. 22-39. 1

Diophantus dealt more accurately with this Egyptian method, but the most learned Anatolius collected the most essential parts of the theory as stated by

b i.e., the nth a-gonal number (1 being the first) is $\frac{1}{2}n^2 + (n-1)(a-2)$; v. vol. i. p. 98 n. a.

c Michael Psellus, "first of philosophers" in a barren age, flourished in the latter part of the eleventh century A.D. There has survived a book purporting to be by Psellus on arithmetic, music, geometry and astronomy, but it is clearly not all his own work. In the geometrical section it is observed that the most favoured method of finding the area of a circle is to take the mean between the inscribed and circumscribed squares, which would give $\pi = \sqrt{8} = 2.8284271$.  

515
The two passages cited before this one allow us to infer that Diophantus must have lived between Hypsicles and Theon, say 150 B.C. to A.D. 350. Before Tannery edited Michael Psellus's letter, there was no further evidence, but it is reasonable to infer from this letter that Diophantus was a contemporary of Anatolius, bishop of Laodicea about A.D. 280 (v. vol. i. pp. 2-3). For references by Plato and a scholiast to the Egyptian methods of reckoning, v. vol. i. pp. 16, 20.

Of these thirteen books in the *Arithmetica*, only six
ALGEBRA: DIOPHANTUS

him in a different way and in the most concise form, and dedicated his work to Diophantus.\(^a\)


Now let us tread the path to the propositions themselves, which contain a great mass of material compressed into the several species. As they are both numerous and very complex to express, they are only slowly grasped by those into whose hands they are put, and include things hard to remember; for this reason I have tried to divide them up according to their subject-matter, and especially to place, as is fitting, the elementary propositions at the beginning in order that passage may be made from the simpler to the more complex. For thus the way will be made easy for beginners and what they learn will be fixed in their memory; the treatise is divided into thirteen books.\(^b\)

*Ibid.* v. 3, Dioph. ed. Tannery i. 316. 6

We have it in the Porisms.\(^c\)

have survived. Tannery suggests that the commentary on it written by Hypatia, daughter of Theon of Alexandria, extended only to these first six books, and that consequently little notice was taken of the remaining seven. There would be a parallel in Eutocius's commentaries on Apollonius's _Conics_. Nesselmann argues that the lost books came in the middle, but Tannery (Dioph. ii. xix-xxi) gives strong reasons for thinking it is the last and most difficult books which have been lost.

\(^a\) Whether this collection of propositions in the Theory of Numbers, several times referred to in the _Arithmetica_, formed a separate treatise from, or was included in, that work is disputed; Hultsch and Heath take the former view, in my opinion judiciously, but Tannery takes the latter.
GREEK MATHEMATICS

(b) Notation

Ibid. l., Praef., Dioph. ed. Tannery 1. 2. 3–6. 21

Τὴν εὑρεσίν τῶν ἐν τοῖς ἀριθμοῖς προβλημάτων, τυμβωτατέ μοι Διονύσιος, γυνώσκων σε σπουδαίως ἔχοντα μαθεῖν, [ὁργανώσαι τὴν μέθοδον]¹ ἐπειραθήν, ἀρξάμενος ἀφ’ ὧν συνεστήκε τὰ πράγματα θεμελίων, ὑποστήσαι τὴν ἐν τοῖς ἀριθμοῖς φύσιν τε καὶ δύναμιν.

· Ἰσως μὲν οὖν δοκεῖ τὸ πράγμα δυσχερέστερον, ἐπειδὴ μήτω γυνώμον ἐστιν, δυσέλπιστοι γὰρ εἰς κατόρθωσιν εἰσιν αἱ τῶν ἁρχομένων ψυχαι, ὅμως δ’ εὐκατάληπτον σοι γενήσεται, διὰ τὴν σὴν προθυμίαν καὶ τὴν ἐμὴν ἀπόδειξιν· ταχεία γὰρ εἰς μάθησιν ἐπιθυμία προσλαβοῦσα διδαχή.

· Αλλὰ καὶ πρὸς τοῦσδε γυνώσκοντι σοι πάντας τοὺς ἀριθμοὺς συγκειμένους ἐκ μονάδων πλῆθους τινός, φανερὸν καθέστηκεν εἰς ἀπειρον ἐχειν τὴν ὑπαρξίν. τυγχανόντων δὴ οὖν ἐν τούτοις

· ὃν μὲν τετραγώνων, οἱ εἰσὶν ἐξ ἀριθμοῦ τινός ἐφ’ ἑαυτὸν πολυπλασιασθέντος· οὔτος δὲ ὁ ἀριθμὸς καλεῖται πλευρὰ τοῦ τετραγώνου.

· ὃν δὲ κύβων, οἱ εἰσὶν ἐκ τετραγώνων ἐπὶ τὰς αὐτῶν πλευρὰς πολυπλασιασθέντων,

· ὃν δὲ δυναμοδυνάμεως, οἱ εἰσὶν ἐκ τετραγώνων ἐφ’ ἑαυτοὺς πολυπλασιασθέντων,

· ὃν δὲ δυναμοκύβων, οἱ εἰσὶν ἐκ τετραγώνων ἐπὶ

¹ ὃργανωσαί τὴν μέθοδον om. Tannery, following the most ancient ms.

518
Knowing that you are anxious, my most esteemed Dionysius, to learn how to solve problems in numbers, I have tried, beginning from the foundations on which the subject is built, to set forth the nature and power in numbers.

Perhaps the subject will appear to you rather difficult, as it is not yet common knowledge, and the minds of beginners are apt to be discouraged by mistakes; but it will be easy for you to grasp, with your enthusiasm and my teaching; for keenness backed by teaching is a swift road to knowledge.

As you know, in addition to these things, that all numbers are made up of some multitude of units, it is clear that their formation has no limit. Among them are—

- squares, which are formed when any number is multiplied by itself; the number itself is called the side of the square;  
- cubes, which are formed when squares are multiplied by their sides,  
- square-squares, which are formed when squares are multiplied by themselves;  
- square-cubes, which are formed when squares are

* This subject is admirably treated, with two original contributions, by Heath, *Diophantus of Alexandria*, 2nd ed., pp. 34-53. Diophantus’s method of representing large numbers and fractions has already been discussed (vol. i. pp. 44-45). Among other abbreviations used by Diophantus are $\square^{59}$, declined throughout its cases, for τετράγωνος; and $\lambda\omega$ (apparently $\nu$ in the archetype) for the sign $=$, connecting two sides of an equation.

$^b$ Or “square root.”
GREEK MATHEMATICS

tous apò tis authis autois pleurás kúbous poluplasiasqentwn,

Ωn de kubokúbdwn, oi eisw en kubw ef' enautous poluplasiasqentwn,

ek te tis touton htoi syntheosys h yperechis
h poluplasiasmoi h logou tov prós allhlous h
kai ekástwn prós tas idias pleurás symbainei
plekesei plevista proplhmatata arithmhtika. lutei
de badiqontos sou tìn ypodeixhshoménhn odwn.

Eodokimásth h ou ekastos touw twn arithmwn
synptomwteran epwvymian ktpámenos stoixheion tis
arithmtikh thwria ejnai. kaleitai ou h men
tetragwnoi dynamis kai eswn autis smeion to
Δ epíshmon exon Υ, ΔΥ dynamis. 

O de kubos kal eswn autous smeion K epíshmon
exon Υ, KΥ kubos. 

O de ek tetragwnoi ef' enauton poluplasias-
sthontos dynamodýnamis kai eswn autous smeion
delta dúo epíshmon exonta Υ, ΔΥΔ dynamodýnamis.

O de ek tetragwnoi epì tov apò tis autis autow
pleurás kubw poluplasiasqentos dynamokubos
kai eswn autous smeion ta ΔΚ epíshmon exonta
Υ, ΔΚΥ dynamokubos. 

O de ek kubw enauton poluplasiasqantas kubó-
kubos kal eswn autous smeion dúo kappia epíshmon
exonta Υ, KΥK kubokubos. 

520
ALGEBRA: DIOPHANTUS

multiplied by the cubes formed from the same side;

cube-cubes, which are formed when cubes are multiplied by themselves;

and it is from the addition, subtraction, or multiplication of these numbers or from the ratio which they bear one to another or to their own sides that most arithmetical problems are formed; you will be able to solve them if you follow the method shown below.

Now each of these numbers, which have been given abbreviated names, is recognized as an element in arithmetical science; the square [of the unknown quantity]\(^a\) is called dynamis and its sign is \(\Delta\) with the index \(Y\), that is \(\Delta^Y\);

the cube is called cubus and has for its sign \(K\) with the index \(Y\), that is \(K^Y\);

the square multiplied by itself is called dynamodynamis and its sign is two deltas with the index \(Y\), that is \(\Delta^Y\Delta\);

the square multiplied by the cube formed from the same root is called dynamocubus and its sign is \(\Delta K\) with the index \(Y\), that is \(\Delta K^Y\);

the cube multiplied by itself is called cubocubus and its sign is two kappas with the index \(Y\), \(K^YK\).

\(^a\) It is not here stated in so many words, but becomes obvious as the argument proceeds that \(\delta\nu\alpha\mu\nu\) and its abbreviation are restricted to the square of the unknown quantity; the square of a determinate number is \(\tau\epsilon\tau\rho\acute{\alpha}\gamma\omega\nu\omicron\omicron\). There is only one term, \(\kappa\upsilon\beta\omicron\sigma\), for the cube both of a determinate and of the unknown quantity. The higher terms, when written in full as \(\delta\nu\alpha\mu\delta\nu\alpha\mu\nu\), \(\delta\nu\alpha\mu\delta\nu\alpha\kappa\upsilon\beta\omicron\sigma\) and \(\kappa\upsilon\beta\omicron\cdot\kappa\upsilon\beta\omicron\sigma\), are used respectively for the fourth, fifth and sixth powers both of determinate quantities and of the unknown, but their abbreviations, and that for \(\kappa\upsilon\beta\omicron\sigma\), are used to denote powers of the unknown only.
GREEK MATHEMATICS

"Ο δὲ μηδὲν τούτων τῶν ἴδιωμάτων κτησάμενος, ἐχθὲν δὲ ἐν ἑαυτῷ πλήθος μονάδων ἀόριστον, ἀριθμὸς καλεῖται καὶ ἐστιν αὐτὸς σημεῖον τὸ ἦ.

"Εστὶ δὲ καὶ ἑτέρων σημείων τὸ ἀμετάθετον τῶν ὀρισμένων, ἡ μονάς, καὶ ἐστιν αὐτὴς σημείον τὸ Μ ἐπίσημον ἐχθὸν τὸ Ο, Μ.

"Ωσπερ δὲ τῶν ἀριθμῶν τὰ ὀμόνυμα μόρια παραμοιώσεις καλεῖται τοῖς ἀριθμοῖς, τοῦ μὲν τριῶν τὸ τρίτον, τοῦ δὲ τέσσαρα τὸ τέταρτον, οὕτως καὶ τῶν νῦν ἐπονομαθέντων ἀριθμῶν τὰ ὀμόνυμα μόρια κληθήσεται παραμοιώς τοῖς ἀριθμοῖς:

tοῦ μὲν ἀριθμοῦ
tῆς δὲ δυνάμεως
tοῦ δὲ κύβου
tῆς δὲ δυναμοδυνάμεως
tοῦ δὲ δυναμοκύβου
tοῦ δὲ κυβοκύβου

ἐξεὶ δὲ ἐκαστον αὐτῶν ἔπει τὸ τοῦ ὀμονύμου ἀριθμοῦ σημείον γράμμην Χ διαστελλούσαν τὸ εἶδος.

*I am entirely convinced by Heath’s argument, based on the Bodleian ms. of Diophantus and general considerations, that this symbol is really the first two letters of ἀριθμός; this suggestion brings the symbol into line with Diophantus’s abbreviations for δύναμις, κύβος, and so on. It may be declined throughout its cases, e.g., ἕνων for the genitive plural, infra p. 552, line 5.

Diophantus has only one symbol for an unknown quantity, but his problems often lead to subsidiary equations involving other unknowns. He shows great ingenuity in isolating these subsidiary unknowns. In the translation I shall use 522
ALGEBRA: DIOPHANTUS

The number which has none of these characteristics, but merely has in it an undetermined multitude of units, is called *arithmos*, and its sign is $\mathcal{S} [x]$.\(^a\)

There is also another sign denoting the invariable element in determinate numbers, the unit, and its sign is $\mathcal{M}$ with the index O, that is $\mathcal{M}$.

As in the case of numbers the corresponding fractions are called after the numbers, a *third* being called after 3 and a *fourth* after 4, so the functions named above will have reciprocals called after them:

\[
\begin{align*}
\text{arithmos} [x] & \quad \text{arithmoston} \left[ \frac{1}{x} \right], \\
\text{dynamis} [x^2] & \quad \text{dynamoston} \left[ \frac{1}{x^2} \right], \\
\text{cubus} [x^3] & \quad \text{cuboston} \left[ \frac{1}{x^3} \right], \\
\text{dynamodynamis} [x^4] & \quad \text{dynamodynamoston} \left[ \frac{1}{x^4} \right], \\
\text{dynamocubus} [x^5] & \quad \text{dynamocuboston} \left[ \frac{1}{x^5} \right], \\
\text{cubocubus} [x^6] & \quad \text{cubocuboston} \left[ \frac{1}{x^6} \right].
\end{align*}
\]

And each of these will have the same sign as the corresponding process, but with the mark $\chi$ to distinguish its nature.\(^b\)

different letters for the different unknowns as they occur, for example, $x, z, m$.

Diophantus does not admit negative or zero values of the unknown, but positive fractional values are admitted.

\(^b\) So the symbol is printed by Tannery, but there are many variants in the mss.
GREEK MATHEMATICS

Ibid. i., Praef., Dioph. ed. Tannery i. 12, 19-21

\[ \text{Leûψis épì leûψiv polllaprassiaothèisα poieî ὑπαρξεῖν, leûψis dé épì ὑπαρξεῖν poieî leûψiv, kai τῆς leûψews σημεῖον Ψ ἐλλιτεὶς κάτω νεῦον, Λ.} \]

(c) DETERMINATE EQUATIONS

(i.) Pure Determinate Equations

Ibid. i., Praef., Dioph. ed. Tannery i. 14. 11-20

\[ \text{Metà dé tauta éaîn ἀπὸ προβλήματος των γενηται εἶδη τινὰ ἵσα εἰδεῖν τοῖς αὐτοῖς, μὴ ὀμοπληθὴ δὲ, ἀπὸ ἐκατέρων τῶν μερῶν δεῖσει ἀφαιρεῖν τὰ ὁμοία ἀπὸ τῶν ὁμοίων, ἦσο ἄν ἐν εἴδος ἐν εἰδεὶ ἵṣον γενηται. ἦάν δὲ πως ἐν ὀποτέρῳ ἐνυπάρχῃ ἢ ἐν ἀμφοτέροις ἐν ἐλλείψει τινα εἶδη, δεῖσει προσθεῖναι τὰ λείποντα εἰδῆ ἐν ἀμφοτέροις τοῖς μέρεσιν, ἦσο ἄν ἐκατέρων τῶν μερῶν τὰ εἴδη ἐνυπάρχοντα γενηται, καὶ πάλιν ἀφελεῖν τὰ ὁμοία ἀπὸ τῶν ὁμοίων, ἦσο ἄν ἐκατέρω τῶν μερῶν ἐν εἴδος καταλειψθῇ.} \]

---

\[ ^{a} \text{Lit. "a deficiency multiplied by a deficiency makes a forthcoming."} \]

\[ ^{b} \text{The sign has nothing to do with Ψ, but I see no reason why Diophantus should not have described it by means of Ψ, 524} \]
A minus multiplied by a minus makes a plus, a minus multiplied by a plus makes a minus, and the sign of a minus is a truncated Ψ turned upside down, that is Λ.

(c) Determinate Equations

(i.) Pure \textit{c} Determinate Equations

Next, if there result from a problem an equation in which certain terms are equal to terms of the same species, but with different coefficients, it will be necessary to subtract like from like on both sides until one term is found equal to one term. If perchance there be on either side or on both sides any negative terms, it will be necessary to add the negative terms on both sides, until the terms on both sides become positive, and again to subtract like from like until on each side one term only is left.

and cannot agree with Heath (\textit{H.G.M.} ii. 459) that "the description is evidently interpolated." But Heath seems right in his conjecture, first made in 1885, that the sign Λ is a compendium for the root of the verb λείπειν, and is, in fact, a Λ with an I placed in the middle. When the sign is resolved in the manuscripts into a word, the dative λείπει is generally used, but there is no conclusive proof that Diophantus himself used this non-classical form.

\textit{c} A \textit{pure} equation is one containing only one power of the unknown, whatever its degree; a \textit{mixed} equation contains more than one power of the unknown.

\textit{d} In modern notation, Diophantus manipulates the equation until it is of the form $Ax^n = B$; as he recognizes only one value of $x$ satisfying this equation, it is then considered solved.
(ii.) Quadratic Equations

Ibid. iv. 39, Dioph. ed. Tannery i. 298. 7–306. 8

Εὐρείν τρεῖς ἀριθμοὺς ὡς ἡ ὑπεροχή τοῦ μείζονος καὶ τοῦ μέσου πρὸς τὴν ὑπεροχήν τοῦ μέσου καὶ τοῦ ἐλάσσονος λόγων ἔχῃ δεδομένον, ἔτι δὲ καὶ σὺν δύο λαμβανόμενοι, ποιῶσι τετράγωνον.

'Επιτετάχθω δὴ τὴν ὑπεροχὴν του μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλάχιστου εἶναι γ''''λ'''' .

'Επεὶ δὲ συναμφότερος ὁ μέσος καὶ ὁ ἐλάσσων ποιεῖ □ον, ποιεῖτω Ἄ. ὁ ἁρα μέσος μείζων ἐστὶ δύαδος· ἐστώ ζ Ἄ. ὁ ἁρα ἐλάχιστος ἐσται ζ Ἄ. Λ ἡ ἁ.

Καὶ ἐπειδὴ ἡ ὑπεροχή τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχιστοῦ γ''''λ'''' . ἐστί, 1 καὶ ἡ ὑπεροχή τοῦ μέσου καὶ τοῦ ἐλάχιστου ζ Ἄ., ἡ ἁρα ὑπεροχή τοῦ μείζονος καὶ τοῦ μέσου ἐσται ζ Ἄ, καὶ ὁ μείζων ἁρα ἐσται ζ Ἄ ζ Ἄ. Μ. Β. Μ. Β.

Λοιπὸν ἐστὶ δύο ἐπιτάγματα, τὸ τε συναμφότερον τὸν μείζονα καὶ τὸν ἐλάχιστον ποιεῖν □ον, καὶ τὸ τὸν μείζονα 2 καὶ τὸν μέσον ποιεῖν □ον. καὶ γίνεται μοι διπλῇ ἡ ἰσότης:

ζ ζ Ἄ. Μ. Δ. Ι. Σ. □, καὶ ζ ζ Ἄ. Μ. Δ. Ι. Σ. □.

καὶ διὰ τὸ τὰς Μ. εἶναι τετράγωνικάς, εὐχερῆς ἐστὶν ἡ ἰσόωσις.

1 ἐστί add. Bachet.
2 τὸν μείζονα . . . τὸν μείζονα add. Tannery.
To find three numbers such that the difference of the greatest and the middle has to the difference of the middle and the least a given ratio, and further such that the sum of any two is a square.

Let it be laid down that the difference of the greatest and the middle has to the difference of the middle and the least the ratio 3:1.

Since the sum of the middle term and the least makes a square, let it be 4. Then the middle term > 2. Let it be $x + 2$. Then the least term = $2 - x$.

And since the difference of the greatest and the middle has to the difference of the middle and the least the ratio 3:1, and the difference of the middle and the least is $2x$, therefore the difference of the greatest and the middle is $6x$, and therefore the greatest will be $7x + 2$.

There remain two conditions, that the sum of the greatest and the least make a square and the sum of the greatest and the middle make a square. And I am left with the double equation:

\[ 8x + 4 = \text{a square}, \]
\[ 6x + 4 = \text{a square}. \]

And as the units are squares, the equation is convenient to solve.

The quadratic equation takes up only a small part of this problem, but the whole problem will give an excellent illustration of Diophantus's methods, and especially of his ingenuity in passing from one unknown to another. The geometrical solution of quadratic equations by the application of areas is treated in vol. i. pp. 192-215, and Heron's algebraical formula for solving quadratics, supra, pp. 502-505.

For double equations, v. infra p. 543 n. b.
Πλάσσω ἀριθμοὺς δύο ἑνα ὁ υπ’ αὐτῶν ἤ ἢ ἢ β, καθὼς Ἰομεν διπλὴν ἰσότητα· ἔστω οὖν ἢ ζ καὶ ῥ ἢ β. ἔλθων ἐπ’ τὰς ὑπο-
στάσεις, οὐ δύναμαι ἀφελεῖν ἀπὸ ῥ ἢ β τὸν ἢ α τοὺς ῥ ἢ β· θέλω οὖν τὸν ἢ εὐρεθήναι ἐλάττυνα ῥ ἢ β, ὡστε καὶ ἢ θ ἢ τ ἰ ἀ ἐλάσσωνες ἐσονται ῥ ἢ τ. ἐὰν γὰρ ἡ δυάς ἐπὶ ἢ τ γενηται καὶ προσ-
lάβῃ ῥ ἢ δ, ποιεῖ ῥ ἢ τ. 

'Επεὶ οὖν ζητῶ ἢ θ ἢ τ ἰ ἤ. ὅ, καὶ ἢ τ ἰ ἢ τ ῥ ἢ τ ἰ θ, ἀλλὰ καὶ ἢ ἀπὸ τῆς δυάδος, τούτουτο ῥ ἢ τ, ἢ ἑς ἐστι, γεγόνασι τρεῖς ὅ, ἢ θ ἢ δ, καὶ ἢ τ ἰ ἢ τ ἰ, καὶ ἢ ὑπεροχή τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχιστου γ' μέρος ἐστίν. ἀπήκται οὖν μοι εἰς τὸ εὑρεῖν τρεῖς1 τετραγώνους, ὅπως ἡ ὑπεροχή τοῦ μείζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχιστου γ' μέρος ἢ, ἐτὶ δὲ ὁ μὲν ἐλάχιστος ἢ θ ἢ δ, ὁ δὲ μέσος ἐλάσσων ῥ ἢ τ. 

1 τρεῖς add. Bachet.

---

* If we put

\[ 8x + 4 = (p + q)^2, \]
\[ 6x + 4 = (p - q)^2, \]
on subtracting,

\[ 2x = 4pq. \]

Substituting \( 2p = \frac{1}{2}x, \) \( 2q = 4 \) (i.e., \( p = \frac{1}{4}x, \) \( q = 2 \)) in the first equation we get

\[ 8x + 4 = \left(\frac{1}{2}x + 2\right)^2, \]
or

\[ 112x = x^2, \]
whence

\[ x = 112. \]

528
I form two numbers whose product is $2x$, according to what we know about a double equation; let them be $\frac{1}{2}x$ and $4$; and therefore $x = 112$.

But, returning to the conditions, I cannot subtract $x$, that is $112$, from $2$; I desire, then, that $x$ be found <2, so that $6x + 4 < 16$. For $2 \cdot 6 + 4 = 16$.

Then since I seek to make $8x + 4 = a$ square, and $6x + 4 = a$ square, while $2 \cdot 2 = 4$ is a square, there are three squares, $8x + 4$, $6x + 4$, and $4$, and the difference of the greatest and the middle is one-third of the difference of the middle and least. My problem therefore resolves itself into finding three squares such that the difference of the greatest and the middle is one-third of the difference of the middle and least, and further such that the least $= 4$ and the middle $< 16$.

This method of solving such equations is explicitly given by Diophantus in ii. 11, Dioph. ed. Tannery i. 96. 8-14:

"The equations will then be $x + 2 = a$ square, $x + 3 = a$ square; and this species is called a double equation. It is solved in this manner: observe the difference, and seek two [suitable] numbers whose product is equal to the difference; they are 4 and $\frac{1}{2}$. Then, either the square of half the difference of these numbers is equated to the lesser, or the square of half the sum to the greater."
GREEK MATHEMATICS

Τετάγθω ο μὲν ἐλάχιστος Ὄδ, ἡ δὲ τοῦ μέσου πλ. ἐὰν ἌΜΒ. αὐτὸς ἀρα ἐσται ὁ ὁ, ΔΥ Ἄζδ Ὄδ.

'Επει οὖν ἡ ὑπεροχή τοῦ μεῖζονος καὶ τοῦ μέσου τῆς ὑπεροχῆς τοῦ μέσου καὶ τοῦ ἐλαχίστου γ' μέρος ἐστίν, καὶ ἔστιν ἡ ὑπεροχή τοῦ μέσου καὶ τοῦ ἐλαχίστου ΔΥ Ἄζδ, ὥστε ἡ ὑπεροχή τοῦ μεγίστου καὶ τοῦ μέσου ἐσται ΔΥ γ' Ἄαγ'. καὶ ἔστιν ὁ μέσος ΔΥ Ἄζδ Ὄδ. ὁ ἀρα μέγιστος ἐσται ΔΥ Ἄαγ' = ἔγ' Ὄδ. ισ. ὁ. πάντα θ' κα. ΔΥ ἁρα ἐβ ἐμή Ὄδλ. ἀσ. ὁ. καὶ τὸ δ' αὐτῶν. ΔΥ γ' ἐβ Ὄδ. ισ. ὁ.

'Ετι δὲ θέλω τὸν μέσον τετράγωνον ἑλάσσονα εἶναι Ὄδ ἀπό τον πλ. δηλαδή ἑλάσσωνον Ὄδ. ἡ δὲ πλευρὰ τοῦ μέσου ἐστίν Ἐ ἌΜΒ. ἑλάττονες εἰσὶ Ὄδ. καὶ κοινῶν ἀφαίρεθεισῶν τῶν ἉΜ, ὁ ἔσται ἑλάσσωνον ἌΜΒ.

Γέγονεν οὖν μοι ΔΥ γ' ἐβ Ὄδ. ισ. ποιῆσαι ὁ. πλάσσω ὁ. τινα ἀπὸ Ὄδ γ' λειποῦσών ἕ τινας: καὶ γίνεται ὁ ἐκ τινὸς ἀριθμοῦ σ' γενομένου καὶ προσλαβόντος τὸν ἂβ, τούτεστι τῆς ἱσώσεως τῆς ἐβ, καὶ μερισθέντος εἰς τὴν ὑπεροχήν ἡ ὑπερέχει ὁ ἀπὸ τοῦ ἀριθμοῦ ὁ. τῶν ΔΥ τῶν ἐν τῇ ἱσώσει γ'. ἀπήκτηκαν οὖν μοι εἰς τὸ εὑρεῖν τινα ἀριθμόν, ὅσ τ' ἱσώμενος καὶ προσλαβῶν Ὄδ θ' καὶ μεριζόμενος εἰς τὴν ὑπεροχήν ἡ ὑπερέχει ὁ ἀπὸ τοῦ αὐτοῦ ὁ. τριάδος, ποιεῖ τὴν παραβολὴν ἑλάσσωνον Ὄδ. θ'.
ALGEBRA: DIOPHANTUS

Let the least be taken as $4$, and the side of the middle as $z + 2$; then the square is $z^2 + 4z + 4$.

Then since the difference of the greatest and the middle is one-third of the difference of the middle and the least, and the difference of the middle and the least is $z^2 + 4z$, so that the difference of the greatest and the least is $\frac{1}{3}z^2 + \frac{1}{3}z$, while the middle term is $z^2 + 4z + 4$, therefore the greatest term $= \frac{1}{3}z^2 + \frac{5}{3}z + 4 = a$ square. Multiply throughout by 9:

$$12z^2 + 48z + 36 = a$$

and take the fourth part:

$$3z^2 + 12z + 9 = a$$

Further, I desire that the middle square <16, whence clearly its side <4. But the side of the middle square is $z + 2$, and so $z + 2 < 4$. Take away 2 from each side, and $z < 2$.

My equation is now

$$3z^2 + 12z + 9 = a$$

$$=(mz - 3)^2$$, say.\(a\)

Then

$$z = \frac{6m + 12}{m^2 - 3}$$

and the equation to which my problem is now resolved is

$$\frac{6m + 12}{m^2 - 3} < 2$$

\(i.e.,\)

$$\frac{2}{1} < 1$$

\(a\) As a literal translation of the Greek at this point would be intolerably prolix, I have made free use of modern notation.
Esta y ζητούμενος Σ Æ. οὖτως εκείνον γενόμενον καὶ προσλαβῶν Μ ᾑ, ποιεῖ Σ Ἐ Σ Μ ᾑ. ó δὲ ἀπὸ αὐτοῦ ὧν, Λ Μ ᾑ, ποιεῖ Δγ ᾑ Λ Μ ᾑ. θέλω οὖν Ἐ Σ Μ ᾑ μερίζομαι εἰς Δγ ᾑ Λ Μ ᾑ καὶ ποιεῖν τὴν παραβολὴν ἐλάσσονος Μ ᾑ. ἀλλὰ καὶ ὁ μεριζόμενος εἰς Μ ᾑ, ποιεῖ τὴν παραβολὴν ἑ᾽ ὥστε Ἐ Σ Μ ᾑ πρὸς Δγ ᾑ Λ Μ ᾑ ἐλάσσονα λόγου ἔχουσιν ἕπερ ᾑ πρὸς ᾑ.

Καὶ χωρίων χωρίῳ ἄνισον. ὁ ἀρα ὑπὸ Ἐ Σ Μ ᾑ καὶ Μ ᾑ ἐλάσσων ἐστὶν τοῦ ὑπὸ δυάδος καὶ Δγ ᾑ Λ Μ ᾑ, τουτέστων Ἐ Σ Μ ᾑ ἐλάσσονός εἰς, Δγ ᾑ Λ Μ ᾑ. καὶ κοινὰ προσκείσθωσαν αἱ Ἐ Σ. Ἐ Σ Μ ᾑ ἐλάσσονες Δγ ᾑ.

"Οταν δὲ τοιαύτην ἤσωσιν ἤσώσωμεν, ποιοῦμεν τῶν Σ τὸ Λ ἐφ’ ἐαυτό, γίνεται θ, καὶ τὰς Δγ ᾑ ἐπὶ τὰς Μ ᾑ, γίνονται Ἐ ὁ πρόσθες τοῖς θ, γίνονται ἐν, ὅλ’ ὁκ ἐλαττόν ἐστὶ Μ Ἐ. πρόσθες τὸ ἡμίσευμα τῶν Σ, γίνεται ὁκ ἐλαττὸν Μ ᾑ. καὶ κέρισον εἰς τὰς Δγ ᾑ γίνεται ὁκ ἐλαττὸν Μ ᾑ.

Γέγονεν οὖν μοι Δγ ᾑ Ε ᾑ Μ ᾑ ἐσ. ὧν τῷ ἀπὸ πλ. Μ ᾑ Λ Ἐ ᾑ, καὶ γίνεται ἐς Μ ᾑ κβ τουτέστων ἔτει.

Τέταχα δὲ τὴν τοῦ μέσου ὧν πλ. Ἐ ᾑ Μ ᾑ. γίνεται . . . τὰς Δγ ἐπὶ add. Tannery.

1 γίνεται . . . τὰς Δγ a This is not strictly true. But since \( \sqrt{45} \) lies between 6 and 7, no smaller integral value than 7 will satisfy the conditions of the problem.
ALGEBRA: DIOPHANTUS

The inequality will be preserved when the term are cross-multiplied,

\[ (6m + 12) \cdot 1 < 2 \cdot (m^2 - 3); \]

i.e., \[ 6m + 12 < 2m^2 - 6. \]

By adding 6 to both sides,

\[ 6m + 18 < 2m^2. \]

When we solve such an equation, we multiply half the coefficient of \( x \) [or \( m \)] into itself—getting 9; then multiply the coefficient of \( x^2 \) into the units \(-2 \cdot 18 = 36\); add this last number to the 9—getting 45; take the square root—which is \( \sqrt{45} \); add half the coefficient of \( x \)—making a number \( \frac{a}{2} \); and divide the result by the coefficient of \( x^2 \)—getting a number \( \frac{b}{2} \).

My equation is therefore

\[ 3z^2 + 12z + 9 = a \text{ square on side } (3 - 5z), \]

and

\[ z = \frac{42}{22} = \frac{21}{11}. \]

I have made the side of the middle square to be

\[ \text{This shows that Diophantus had a perfectly general formula for solving the equation} \]

\[ ax^2 = bx + c, \]

namely

\[ x = \frac{\frac{b}{2} + \sqrt{\frac{b^2}{4} + ac}}{a}. \]

From vi. 6 it becomes clear that he had a similar general formula for solving

\[ ax^2 + bx = c, \]

and from v. 10 and vi. 22 it may be inferred that he had a general solution for

\[ 2x^2 + c = bx. \]
Εσται ἡ τοῦ □οῦ π. Μ. ἁγ. αὐτὸς δὲ ὁ □οῦ Μ. ρακ ᾿αομβή.

Ερχομαι οὖν ἐπὶ τὸ ἐξ ἀρχῆς καὶ τάσσων Μ. ρακ ᾿αομβήν, ὄντα □οῦ, ἵστ. τοῖς Ξ Μ. δ. καὶ πάντα εἰς ρακ. καὶ γίνεται ὁ ᾿ψκε, ἀτζ', καὶ ἐστὶν ἐλάσσων δυνάδος.

Ἐπὶ τὰς ὑποστάσεις τοῦ προβλήματος τοῦ ἐξ ἀρχῆς. ὑπέστημεν δὴ τὸν μὲν μέσον Ξ α.Μ. β, τὸν δὲ ἐλάχιστον Μ. β λεία, τὸν δὲ μέγιστον Ξ Ζ Μ. β. ἔσται ὁ μὲν μέγιστος α. αζ', ὁ δὲ βος ᾿βωίζ', ὁ δὲ ἐλάχιστος ὁ γος πζ'. καὶ ἐπεί τὸ μόριον, ἐστὶ τὸ ψκεοῦν, οὐκ ἔστιν □οῦ, Εονν δὲ ἔστιν αὐτοῦ, ἔαν λάβωμεν ρακ. ὁ ἐστὶ □οῦ, πάντων οὖν τὸ Εονν, καὶ ὁμοίως ἔσται ὁ μὲν αος ρακων ᾿αολδ ζ', ὁ δὲ βος υζζ' ζ', ὁ δὲ γος ιδ ζ'.

Καὶ ἐὰν ἐν ὀλοκλήροις θέλησιν μὴ τὸ ζ', ἐπι- τρέξῃ, εἰς δο εἴμβαλε. καὶ ἐσται ὁ αος ὑπδ 'ξτλη', ὁ δὲ βος ὑπδ ᾿αον', ὁ δὲ γος νη'. καὶ ἡ ἀπόδειξις φανερά. 534
ALGEBRA: DIOPHANTUS

$x + 2$; therefore the side will be $\frac{43}{11}$ and the square itself $\frac{1849}{121}$.

I return now to the original problem and make $\frac{1849}{121}$, which is a square, $= 6x + 4$. Multiplying by $121$ throughout, I get $x = \frac{1365}{726}$, which is $< 2$.

In the conditions of the original problem we made the middle term $= x + 2$, the least $= 2 - x$, and the greatest $7x + 2$.

Therefore

\[
\text{the greatest} = \frac{11007}{726}, \\
\text{the middle} = \frac{2817}{726}, \\
\text{the least} = \frac{87}{726}.
\]

Since the denominator, $726$, is not a square, but its sixth part is, if we take $121$, which is a square, and divide throughout by $6$, then similarly the numbers are

\[
\frac{1834\frac{1}{2}}{121}, \ \frac{469\frac{1}{2}}{121}, \ \frac{14\frac{1}{2}}{121}.
\]

And if you prefer to use integers only, avoiding the $\frac{1}{2}$, multiply throughout by $4$. Then the numbers will be

\[
\frac{7338}{484}, \ \frac{1878}{484}, \ \frac{58}{484}.
\]

And the proof is obvious.
(iii.) Simultaneous Equations Leading to a Quadratic

*Ibid.* i. 28, Dioph. ed. Tannery i. 62. 20–64. 10

Εὐρεῖν δύο ἀριθμοὺς ὡς καὶ ἡ σύνθεσις αὐτῶν καὶ ἡ σύνθεσις τῶν ἀπ’ αὐτῶν τετραγώνων ποιή
dοθέντας ἀριθμοὺς.

Δεῖ δὴ τοὺς δίς ἀπ’ αὐτῶν τετραγώνων τοὺ ἀπὸ
συναμφοτέρου αὐτῶν τετραγώνου ὑπερέχειν τετρα-
γώνων. ἔστι δὲ καὶ τούτο πλασματικὸν.

'Επιτετάχθω δὴ τὴν μὲν σύνθεσιν αὐτῶν ποιεῖν
Μṅ, τὴν δὲ σύνθεσιν τῶν ἀπ’ αὐτῶν τετραγώνων
ποιεῖν Μارة.

Τετάχθω δὴ ἡ ὑπεροχὴ αὐτῶν 5β. καὶ ἔστω
ὁ μεῖζων ξα καὶ Μĩ, τῶν ἡμίσιως πάλιν τοῦ
συνθέματος, ὁ δὲ ἔλασσων Μὶ λαζ. καὶ μένει
πάλιν τὸ μὲν σύνθεμα αὐτῶν Μṅ, ἡ δὲ ὑπεροχή
ζβ.

Λοιπὸν ἔστι καὶ τὸ σύνθεμα τῶν ἀπ’ αὐτῶν
τετραγώνων ποιεῖν Μارة. ἀλλὰ τὸ σύνθεμα τῶν
ἀπ’ αὐτῶν τετραγώνων ποιεῖ Δβ Μσ. ταῦτα
ίσα Μارة, καὶ γίνεται ὁ ζβ.

'Επὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν μεῖζων Μβ,
ὁ δὲ ἔλασσων Μη. καὶ ποιοῦσι τὰ τῆς προτάσεως.

* In general terms, Diophantus's problem is to solve the simultaneous equations

\[ \xi + \eta = 2a \]
\[ \xi^2 + \eta^2 = A. \]

He says, in effect, let \[ \xi - \eta = 2x; \]
then \[ \xi = a + x, \eta = a - x, \]
536
(iii.) Simultaneous Equations Leading to a Quadratic

To find two numbers such that their sum and the sum of their squares are given numbers.\(^a\)

It is a necessary condition that double the sum of their squares exceed the square of their sum by a square. This is of the nature of a formula.\(^b\)

Let it be required to make their sum 20 and the sum of their squares 208.

Let their difference be \(2x\), and let the greater \(= x + 10\) (again adding half the sum) and the lesser \(= 10 - x\).

Then again their sum is 20 and their difference \(2x\).

It remains to make the sum of their squares 208. But the sum of their squares is \(2x^2 + 200\).

Therefore \(2x^2 + 200 = 208\),

and \(x = 2\).

To return to the hypotheses—the greater = 12 and the lesser = 8. And these satisfy the conditions of the problem.

\[ (a + x)^2 + (a - x)^2 = \Lambda, \]
\[ i.e., \quad 2(a^2 + x^2) = \Lambda. \]

A procedure equivalent to the solution of the pair of simultaneous equations \(\xi + \eta = 2\alpha, \xi\eta = \Lambda\), is given in i. 27, and a procedure equivalent to the solution of \(\xi - \eta = 2\alpha, \xi\eta = \Lambda\), in i. 30.

\(^b\) In other words, \(2(\xi^2 + \eta^2) - (\xi + \eta)^2 = \) a square; it is, in fact, \((\xi - \eta)^2\). I have followed Heath in translating \(\varepsilon\sigma\tau\iota \delta\iota \kappa\alpha \iota \tau\omega\nu\tau\iota\kappa\iota\kappa\iota\nu\kappa\iota\nu\) as “this is of the nature of a formula.” Tannery evades the difficulty by translating “est et hoc formativum,” but Bachet came nearer the mark with his “effictum aliunde.” The meaning of \(\pi\lambda\alpha\sigma\mu\alpha\tau\iota\kappa\iota\nu\) should be “easy to form a mould,” i.e. the formula is easy to discover.

537
GREEK MATHEMATICS

(iv.) Cubic Equation

Ibid. vi. 17, Dioph. ed. Tannery i. 432. 19–434. 22

Εὐρείν τρίγωνον ὀρθογώνιον ὅπως ὅ ἐν τῷ ἐμβαδῷ αὐτοῦ, προσλαβὼν τὸν ἐν τῇ ὑποτευνοῦσῃ, πολὺ τετράγωνον, ὅ δὲ ἐν τῇ περιμέτρῳ αὐτοῦ ἦ κύβος.

Τετάχθω ὅ ἐν τῷ ἐμβαδῷ αὐτοῦ 5 ἀ, ὅ δὲ ἐν τῇ ὑποτευνοῦσῃ αὐτοῦ Μ τινῶν τετραγωνικῶν Λ 5 ἀ, ἔστω Μ ἱπν ἔλ ἀ.

'Αλλ' ἐπει ὑπεθέμεθα τὸν ἐν τῷ ἐμβαδῷ αὐτοῦ εἶναι 5 ἀ, ὅ ἁρα ὑπὸ τῶν περὶ τὴν ὀρθὴν αὐτοῦ γίνεται 5 β. ἀλλά 5 β περιέχονται ὑπὸ 5 ἀ καὶ Μ β. ἐὰν οὖν τάξιμεν μίαν τῶν ὀρθῶν Μ β, ἔσται ἡ ἐτέρα 5 ἀ.

Καὶ γίνεται ἡ περιμέτρος Μ ἱῆ καὶ οὐκ ἔστι κύβος· ὅ δὲ τῇ γέγονεν ἐκ τινὸς ποὺ καὶ Μ β· δεῖσει ἁρα εὐρεῖν ποὺ τινα, ὅσ, προσλαβὼν Μ β, ποιεῖ κύβον, ὡστε κύβον ποὺ ὑπερέχειν Μ β.

Τετάχθω οὖν ἡ μὲν τοῦ ποὺ ἱπν ἱα Ἄ Μ ἀ, ἡ δὲ τοῦ κύβου 5 ἀ Ἄ Μ ἀ. γίνεται ὃ μὲν ποὺ ποὐ, Ε ν ἀ β Μ ἀ, ὅ δὲ κύβος, Κ ν ἃ ᾱ γ λ Ε ν γ Ἄ Μ ἀ. θέλω οὖν τὸν κύβον τὸν ποὺ ὑπερέχειν δυάδι. ὁ ἁρα ποὺ μετά δυάδος, τούτεστιν Ε ν ἀ β Μ γ, ἔστων ἱοσ Κ ν ἃ ᾱ γ λ Ε ν γ Μ ἀ, ὅθεν ὅ ἐφισκεται Μ δ.

'Εσται οὖν ἡ μὲν τοῦ ποὺ ποὺ Μ ἐ, ἡ δὲ τοῦ 538
(iv.) *Cubic Equation*

*Ibid. vi. 17, Dioph. ed. Tannery i. 432. 19-434. 22*

To find a right-angled triangle such that its area, added to one of the perpendiculars, makes a square, while its perimeter is a cube.

Let its area = \(x\), and let its hypotenuse be some square number *minus* \(x\), say \(16-x\).

But since we supposed the area = \(x\), therefore the product of the sides about the right angle = \(2x\). But \(2x\) can be factorized into \(x\) and 2; if, then, we make one of the sides about the right angle = 2, the other = \(x\).

The perimeter then becomes 18, which is not a cube; but 18 is made up of a square \([16]+2\). It shall be required, therefore, to find a square number which, when 2 is added, shall make a cube, so that the cube shall exceed the square by 2.

Let the side of the square = \(m+1\) and that of the cube \(m-1\). Then the square = \(m^2+2m+1\) and the cube = \(m^3+3m-3m^2-1\). Now I want the cube to exceed the square by 2. Therefore, by adding 2 to the square,

\[ m^2 + 2m + 3 = m^3 + 3m - 3m^2 - 1, \]

whence \(m = 4\).

Therefore the side of the square = 5 and that of

* This is the only example of a cubic equation solved by Diophantus. For Archimedes' geometrical solution of a cubic equation, *v. supra*, pp. 126-163.
GREEK MATHEMATICS

κύβοι Ἄνυ. αὐτοῖς ἀρα ὁ μὲν ὅ τὸῦ Ἑκ, ὁ δὲ κύβος Ἄνκ.

Μεθυφίσταμαι οὖν τὸ ὀρθογώνιον, καὶ τάξας αὐτοῖς τὸ ἐμβαδὸν Ἐν, τάσσω τὴν ὑποτεύνουσαν Ἄνκ Ἐν Ἐὶ Ἐν. μένει δὲ καὶ ἡ βάσις Ἄνβ, ἡ δὲ κάθετος Ἐν.

Λοιπὸν ἔστω τὸν ἀπὸ τῆς ὑποτευνούσης ἑσον εἰναι τοῖς ἀπὸ τῶν περὶ τὴν ὀρθὴν' γίνεται δὲ . Δυ Ἄνχ Χκ Ἐν Ἐν. ἔσται ἕση Δυ Ἄνδ. ὅθεν ὁ Ἐν Χκα.

Ἐπὶ τὰς ὑποστάσεις καὶ μένει.

(d) INDETERMINATE EQUATIONS

(i.) Indeterminate Equations of the Second Degree

(a) Single Equations

Ibid. ii. 20, Dioph. ed. Tannery i. 114. 11-22

Εὑρεῖν δύο ἀριθμοὺς ὅπως ὁ ἀπὸ τοῦ ἐκατέρου αὐτῶν τετράγωνον, προσλαβὼν τὸν λοιπὸν, ποὺ τετράγωνον.

Τετάχθω ὁ ἂς Ἐν, ὁ δὲ βς Ἄν β, ἵνα ὁ ἀπὸ τοῦ ἂον ὅ, προσλαβὼν τὸν βον, ποὺ ὅ ὅν. λοιπὸν ἔστι καὶ τὸν ἀπὸ τοῦ βον ὅ, προσλαβόντα τὸν ἂον, ποιεῖν ὅν. ἀλλ' ὁ ἀπὸ τοῦ βον ὅ ὅν, προσ-

α. Diophantus makes no mention of indeterminate equations of the first degree, presumably because he admits 540
the cube = 3; and hence the square is 25 and the cube 27.

I now transform the right-angled [triangle], and, assuming its area to be \( x \), I make the hypotenuse = 25 - \( x \); the base remains = 2 and the perpendicular = \( x \).

The condition is still left that the square on the hypotenuse is equal to the sum of the squares on the sides about the right angle;

\[ x^2 + 625 - 50x = x^2 + 4, \]

whence \( x = \frac{621}{50} \).

This satisfies the conditions.

(d) Indeterminate Equations

(i.) Indeterminate Equations of the Second Degree

(a) Single Equations

\textit{Ibid.} ii. 20, Dioph. ed. Tannery i. 114. 11-22

To find two numbers such that the square of either, added to the other, shall make a square.

Let the first be \( x \), and the second \( 2x + 1 \), in order that the square on the first, added to the second, may make a square. There remains to be satisfied the condition that the square on the second, added to the first, shall make a square. But the square on the second, added to the first, is \( 4x^2 + 5x + 1 \); and therefore this must be a square.

rational \textit{fractional} solutions, and the whole point of solving an indeterminate equation of the first degree is to get a solution in integers.
GREEK MATHEMATICS

Πλάσω τὸν □ον ἀπὸ ἤβ ΛMOVED H Characters Μ β. αὐτὸς ἄρα ἔσται ΔΥδ Λζ η. καὶ γίνεται ὁ δὲ γ. Ἐσται ὁ μὲν αος γ, ὁ δὲ βος ιθ, καὶ ποιοῦν τὸ πρόβλημα.

(β) Double Equations

Ibid. iv. 32, Dioph. ed. Tannery 268. 18–272. 15

Δοθέντα ἄριθμον διελείψ εἰς τρεῖς ἁριθμούς ὀπως ὁ ὑπὸ τοῦ πρώτου καὶ τοῦ δευτέρου, εάν τε προσλάβη τὸν τρίτον, εάν τε λειψῃ, ποιῇ τετράγωνων. "Εστῶ ὁ δοθεῖς ὁ δ. Τετάχθω ὁ γος ε ἀ, καὶ ὁ βος Μ ἐλασσόνων τοῦ δ. ἔστω Μ β. ὁ ἄρα αος ἐσται Μδ Λζ δ. καὶ λοιπά ἔστι δύο ἐπιτάγματα, τὸν ὑπὸ αον καὶ βον, εάν τε προσλάβη τὸν γον, εάν τε λειψῃ, ποιεῖν □ον. καὶ γίνεται διπλὴ ἡ ἱσότης. Μη Λζ δ ἱσ. □ον καὶ Μη Λζ γ ἱσ. □ον καὶ οὐ ρητον

* The problem, in its most general terms, is to solve the equation

\[ Ax^2 + Bx + C = y^2. \]

Diophantus does not give a general solution, but takes a number of special cases. In this case \( A \) is a square number \((= a^2, \text{say})\), and in the equation

\[ a^2x^2 + Bx + C = y^2 \]

he apparently puts \( y^2 = (ax - m)^2 \),

where \( m \) is some integer,

whence

\[ x = \frac{m^2 - C}{2am + B}. \]
ALGEBRA: DIOPHANTUS

I form the square from $2x - 2$; it will be $4x^2 + 4 - 8x$; and $x = \frac{3}{13}$.

The first number will be $\frac{3}{13}$, the second $\frac{19}{13}$, and they satisfy the conditions of the problem.«

(β) Double Equations b

Ibid. iv. 32, Dioph. ed. Tannery 268. 18–272. 15

To divide a given number into three parts such that the product of the first and second ± the third shall make a square.

Let the given number be 6.

Let the third part be $x$, and the second part any number <6, say 2; then the first part = $4 - x$; and the two remaining conditions are that the product of the first and second ± the third = a square. There results the double equation

\[ 8 - x = \text{a square}, \]
\[ 8 - 3x = \text{a square}. \]

And this does not give a rational result since the ratio

b Diophantus's term for a double equation is διπλοίοντας, διπλή ἴσοντας or διπλή ἴσως. It always means with him that two different functions of the unknown have to be made simultaneously equal to two squares. The general equations are therefore

\[ A_1 x^2 + B_1 x + C_1 = u_1^2, \]
\[ A_2 x^2 + B_2 x + C_2 = u_2^2. \]

Diophantus solves several examples in which the terms in $x^2$ are missing, and also several forms of the general equation.
GREEK MATHEMATICS

'esti dia tò μή εἶναι τοὺς 5 πρὸς ἀλλήλους λόγον ἔχοντας ὁν □ος ἄριθμος πρὸς □ου ἄριθμον.
'Αλλὰ ὁ 5 ὁ ἄ μονάδι ἐλάσσων τοῦ β, οἱ δὲ 5 γ' ὑμοίων μεῖζονς Μ' τοῦ β. ἀπῆκται οὖν μοι εἰς τὸ εὑρεῖν ἄριθμον τινα, ὡς τὸν β, ἵνα ὁ Μ' αὐτοῦ μεῖζον, πρὸς τὸν Μ' (αὐτοῦ ἐλάσσονα, λόγον ἔχει ὁν □ος ἄριθμος πρὸς)1 □ου ἄριθμον.

'Εστω ἡ ζητουμένος 5 α, καὶ ὁ Μ' αὐτοῦ μεῖζον ἔσται 5 α Μ α, ὁ δὲ Μ' αὐτοῦ ἐλάσσων 5 α Λ Μ α. θέλων σὺν αὐτοῦ πρὸς ἀλλήλους λόγων ἔχειν ὁν □ος ἄριθμος πρὸς □ου ἄριθμον. ἔστω δὲ πρὸς α'. ἀφετε 5 α Λ Μ α ἐπὶ Μ δ γίνονται 5 δ Λ Μ δ. καὶ 5 α Μ α ἐπὶ τὴν Μ α (γίνονται 5 α Μ α).3 καὶ εἴσω οὖτοι οἱ ἐκκείμενοι ἄριθμοι λόγων ἔχοντες πρὸς ἀλλήλους ὁν ἔχει □ος ἄριθμος πρὸς □ου ἄριθμον. νῦν 5 δ Λ Μ δ ἰσ. 5 α Μ α, καὶ γίνεται ὁ 5 Μ γ.

Τάσσω οὖν τὸν β' 5 Μ γ. ὁ γὰρ γ' ἐστὶν 5 α. ὁ ἄρα α' ἐσται Μ γ Λ 5 α.

Δοιτὸν δεὶ εἶναι τὸ ἐπίταγμα, ἐστω τὸν ὑπὸ α' καὶ β', προσλαβόντα τὸν γ', ποιεῖν □ου, καὶ λεῖψαντα τὸν γ', ποιεῖν □ου. ἀλλ' ὁ ὑπὸ α' καὶ β', προσλαβὼν τὸν γ', ποιεῖ Μ θ Λ 5 ω' ἰσ. □ου. Λ δὲ τοῦ γ', ποιεῖ Μ θ Λ 5 β ω' ἰσ. □ου. καὶ 544
of the coefficients of \( x \) is not the ratio of a square to a square.

But the coefficient 1 of \( x \) is \( 2-1 \) and the coefficient 3 of \( x \) likewise is \( 2+1 \); therefore my problem resolves itself into finding a number to take the place of 2 such that (the number +1) bears to (the number –1) the same ratio as a square to a square.

Let the number sought be \( y \); then (the number +1)=\( y+1 \), and (the number –1)=\( y-1 \). We require these to have the ratio of a square to a square, say \( 4:1 \). Now \( (y-1)\cdot4 = 4y-4 \) and \( (y+1)\cdot1 = y+1 \). And these are the numbers having the ratio of a square to a square. Now I put

\[
4y-4 = y+1,
\]

giving

\[
y = \frac{5}{3}.
\]

Therefore I make the second part \( \frac{5}{3} \), for the third=\( x \); and therefore the first=\( \frac{13}{3} - x \).

There remains the condition, that the product of the first and second ± the third=a square. But the product of the first and second + the third=

\[
\frac{65}{9} - \frac{2}{3}x = \text{a square},
\]

and the product of the first and second – the third=

\[
\frac{65}{9} + \frac{2\frac{2}{3}}{2}x = \text{a square}.
\]

---

1 \( \alphaυρου \ldots \piρος \) add. Bachet.
2 \( \gammaινονται \beta \alpha \alpha \beta \) add. Tannery.
These are a pair of equations of the form
\[ am^2x + a = u^2, \]
\[ an^2x + b = v^2. \]

Multiply by \( n^2, m^2 \) respectively, getting, say
\[ am^2n^2x + an^2 = u'^2, \]
\[ am^2n^2x + bm^2 = v'^2. \]

Let
\[ an^2 - bm^2 = u'^2 - v'^2, \]
and put
\[ u' + v' = p, \]
\[ u' - v' = q; \]

\[ u'^2 = \frac{1}{4}(p + q)^2, \quad v'^2 = \frac{1}{4}(p - q)^2, \]
and so
\[ am^2n^2x + an^2 = \frac{1}{4}(p + q)^2, \]
\[ am^2n^2x + bm^2 = \frac{1}{4}(p - q)^2; \]

whence, from either,
\[ x = \frac{\frac{1}{4}(p^2 + q^2) - \frac{1}{2}(an^2 + bm^2)}{am^2n^2}. \]
ALGEBRA: DIOPHANTUS

Multiply throughout by 9, getting

\[65 - 6x = \text{a square}\]

and

\[65 - 24x = \text{a square}.\]

Equating the coefficients of \(x\) by multiplying the first equation by 4, I get

\[260 - 24x = \text{a square}\]

and

\[65 - 24x = \text{a square}\]

Now I take their difference, which is 195, and split it into the two factors 15 and 13. Squaring the half of their difference, and equating the result to the lesser square, I get \(x = \frac{8}{3}\).

Returning to the conditions—the first part will be \(\frac{5}{3}\), the second \(\frac{5}{3}\), and the third \(\frac{8}{3}\). And the proof is obvious.

This is the procedure indicated by Diophantus. In his example,

\[p = 15, \quad q = 13,\]

and

\[\left\{\frac{1}{2}(15 - 13)^2 = 65 - 24x,\right\]

whence

\[24x = 64, \quad \text{and} \quad x = \frac{8}{3}\]
GREEK MATHEMATICS

(ii.) Indeterminate Equations of Higher Degree

Ibid. iv. 18, Dioph. ed. Tannery. i. 226. 2-228. §

Εὑρεῖν δύο ἀριθμούς, ὥσπερ ὦ ἀπὸ τοῦ πρώτου κύβος προσλαβὼν τὸν δεύτερον ποιῇ κύβον, ὦ ὁ δὲ ἀπὸ τοῦ δεύτερου τετράγωνος προσλαβὼν τὸν πρώτον ποιῇ τετράγωνον.

Τετάχθω ὦ a os 5 ἀ. ὦ ἀρα βος ἕσται Ὠ κυβικαὶ ἡ ἦ Κ ἤ. καὶ γίνεται ὦ ἀπὸ τοῦ αων κύβος, προσ-

λαβῶν τὸν βον, κύβος.

Λοιπὸν ἔστι καὶ τὸν ἀπὸ τοῦ βον δον, προσλαβόντα τὸν αον, ποιεῖν δον. ἀλλά ὦ ἀπὸ τοῦ βον δον, προσλαβῶν τὸν αον, ποιεῖ

Κ ν Κ ἅ ἂ ὦ Ὁ ξδ Ό Κ ν ἅ. (ταῦτα ἐσθα δον τὸ ἀπὸ π. Κ ν Α ὕ ἦ, τούτων τὸν 

καὶ κοινῶν προστιθεμένων τῶν λειτομενῶν καὶ ἀφαιρουμένων τῶν ὁμοίων ἀπὸ ὁμοίων, λοιποὶ

Κ ν ἅ ἔσυ ἂ. καὶ πάντα παρᾶ ἃ. Δ ν ἅ ἔσαι ὢ ἂ.

Καὶ ἕστιν ὦ ὢ δον, καὶ Δ ν ἅ ὦ ἔσαι δον, λειτο-

μένῃ ἀν μοι ἦ ἔν ἔν ἔσαις. ἀλλα ἂ δι 

tων δις Κ ν ἅ. οἱ δὲ Κ ν ἅ ἔσαι ἐπὸ τῶν δις ὢ ἦ

1 ταῦτα ... ὤ ξδ οὐδ. Bachet.

As with equations of the second degree, these may be single or double. Single equations always take the form that an expression in \(x\), of a degree not exceeding the sixth, is to be made equal to a square or cube. The general form is therefore

\[ A_0x^6 + A_1x^5 + \ldots + A_6 = y^2 \text{ or } y^3. \]

Diophantus solves a number of special cases of different degrees.

In double equations, one expression is made equal to a
ALGEBRA: DIOPHANTUS

(ii.) Indeterminate Equations of Higher Degree

Ibid. iv. 18, Dioph. ed. Tannery i. 226. 2–228. 5

To find two numbers such that the cube of the first added to the second shall make a cube, and the square of the second added to the first shall make a square.

Let the first number be \( x \). Then the second will be a cube number less \( x^3 \), say \( 8 - x^3 \). And the cube of the first, added to the second, makes a cube.

There remains the condition that the square on the second, added to the first, shall make a square. But the square on the second, added to the first, is \( x^6 + x + 64 - 16x^3 \). Let this be equal to \((x^3 + 8)^2\), that is to \( x^6 + 16x^3 + 64 \). Then, by adding or subtracting like terms,

\[
32x^3 = x;
\]

and, after dividing by \( x \),

\[
32x^2 = 1.
\]

Now 1 is a square, and if \( 32x^2 \) were a square, my equation would be soluble. But \( 32x^2 \) is formed from \( 2 \cdot 16x^3 \), and \( 16x^3 \) is \((2 \cdot 8)(x^3)\), that is, it is formed cube and the other to a square, but only a few simple cases are solved by Diophantus.

The general type of the equation is

\[
x^6 - Ax^3 + Bx + c^2 = y^2.
\]

Put \( y = x^3 + c \), then

\[
x^2 = \frac{B}{A + 2c},
\]

and if the right-hand expression is a square, there is a rational solution.

In the case of the equation \( x^6 - 16x^3 + x + 64 = y^2 \) it is not a square, and Diophantus replaces the equation by another, \( x^6 - 128x^3 + x + 4096 = y^2 \), in which it is a square.

549
GREEK MATHEMATICS

καὶ τοῦ $K^\gamma \alpha$, τοπεστὶ δὶς τῶν $\tilde{M} \eta$. ὥστε αἰ $\lambda \beta \Delta^\gamma$
ἐκ δὲ τῶν $\tilde{M} \eta$. γέγονεν οὖν μοι εὐρείων κύβον ὁς
dὲ γενόμενος ποιεὶ $\square$"ν.

"Εστώ ὁ γεωμεστὸς $K^\gamma \alpha$. οὖτος δὲ γενόμενος
ποιεὶ $K^\gamma \delta$ ἴσο. $\square$"ν. ἐστὼ $\Delta^\gamma \tau"$ καὶ γίνεται ὁ $\tilde{M} \delta$. ἐπὶ τὰς ὑποστάσεις. ἐσται ὁ $K^\gamma \tilde{M} \xi$δ.

Τάσσω ἀρα τὸν $\beta^{\nu} \tilde{M} \xi \delta \Lambda K^\gamma \bar{\alpha}$. καὶ λοιπὸν ἐστὶ
tὸν ἀπὸ τοῦ $\beta^{\nu} \square$"ν προσλαβόντα τὸν $\alpha^{\nu}$ ποιεῖν
$\square$"ν. ἀλλὰ ὁ ἀπὸ τοῦ $\beta^{\nu}$ προσλαβὼν τὸν $\alpha^{\nu}$ ποιεὶ
$K^\gamma \tilde{M}, \delta \xi \gamma \delta \Lambda K^\gamma \rho \kappa \eta$ ἴσο. $\square$"ν τῶ ἀπὸ $\pi$. $K^\gamma \tilde{M} \xi \delta \delta$, καὶ γίνεται ὁ $\square^{\nu}$ $K^\gamma K \tilde{M}, \delta \xi \gamma \delta \xi Κ^\gamma \rho \kappa \eta$.
καὶ γίνονται λοιπῶ $K^\gamma \sigma \nu$ ἴσο $\delta \alpha$. καὶ γίνεται
ὁ $\tilde{M}$ ἐνὸς ἴσο"ν.

Ἐπὶ τὰς ὑποστάσεις. ἐσται ὁ $\alpha^{\nu}$ ἐνὸς ἴσο"ν, ὁ $\delta$ μὲ
$\beta^{\nu} \delta \xi \gamma \delta βρυγ$.

(e) Theory of Numbers: Sums of Squares

Ibid. ii. 8, Dioph. ed. Tannery i. 90. 9-21

Τὸν ἐπιταχθέντα τετράγωνον διελεῖν εἰς δύο
tετραγώνους.

— It was on this proposition that Fermat wrote a famous
note: "On the other hand, it is impossible to separate a
cube into two cubes, or a biquadrate into two biquadrates, or
generally any power except a square into two powers with
the same exponent. I have discovered a truly marvellous
proof of this, which, however, the margin is not large enough
550
from 2.8. Therefore $32x^2$ is formed from 4.8. My problem therefore becomes to find a cube which, when multiplied by 4, makes a square.

Let the number sought be $y^3$. Then $4y^3 = a$ square $= 16y^2$ say; whence $y = 4$. Returning to the conditions—the cube will be 64.

I therefore take the second number as $64 - x^3$. There remains the condition that the square on the second added to the first shall make a square. But the square on the second added to the first =

$$x^6 + 4096 + x - 128x^3 = a \text{ square}$$

$$= (x^3 + 64)^2, \text{ say},$$

$$= x^6 + 4096 + 128x^3.$$

On taking away the common terms,

$$256x^3 = x,$$

and

$$x = \frac{1}{16}.$$

Returning to the conditions—

first number $= \frac{1}{16}$, second number $= \frac{262143}{4096}$.

(e) Theory of Numbers: Sums of Squares

Ibid. ii. 8, Dioph. ed. Tannery i. 90. 9-21

To divide a given square number into two squares:*

to contain.” Fermat claimed, in other words, to have proved that $x^m + y^m = z^m$ cannot be solved in rational numbers if $m > 2$. Despite the efforts of many great mathematicians, a proof of this general theorem is still lacking.

Fermat’s notes, which established the modern Theory of Numbers, were published in 1670 in Bachet’s second edition of the works of Diophantus.
GREEK MATHEMATICS

'Επιτετάχθω δὴ τὸν ἰθα διελείν εἰς δύο τετραγώνους.

Καὶ τετάχθω ὁ αὐτὸς Δ-border-άλλως, ὁ ἄρα ἄλλως ἐσται 

Μ ἰθαὶ ἀνεφορεῖ ἄρα Μ ἰθαὶ Δ-άλλως ἀκόμα εἰναι ὁ ἐπίπεδον τὸν ἱκὼν ἵνα 

τοὺς ἱκὼν Μ ἰθαὶ ἐστὶ τῶν ἰθαὶ Μ πλευράν ἐστὶν ὁ ἰθαὶ 

διαγόρευσι ἀνεφορεῖ ἄρα Μ ἰθαὶ Δ-άλλως, αὐτὸς ἄρα ὁ ὁ ἐστι 

Δ-άλλως ἤκατε ἰκὼν ἵνα ἀνεφορεῖ ἄρα Μ ἰθαὶ Δ-άλλως, κοινὴ 

προσκείσθω ἓν λείψως καὶ ἀπὸ ὁμοίων ὁμοῖα.

Δ-άλλως ἤκατε ἰκὼν ἵνα ἀνεφορεῖ ἄρα Μ ἰθαὶ Δ-άλλως, καὶ γίνεται ὁ ἰκὼν 

πέμπτων. Ἑσται ὁ μεν' ἐκείνου ἐκεῖνος, ὁ δὲ ἐκεῖνος ἐκεῖνος, καὶ ὁ ὁ ἐκεῖνος 

θέντες ποιοῦσιν ἐκεῖνος ὁ, ἡτοῖ Μ ἰθαὶ ἐστὶν ἐκατερος 

tetragōnous.

Ibid. v. 11, Dioph. ed. Tannery 1. 342. 13–346. 12

Μονάδα διελείν εἰς τρεῖς ἀριθμοὺς καὶ προσθέειναι ἐκάστῳ αὐτῶν πρότερον τὸν αὐτὸν διαγόρευσι καὶ 

ποιεῖν ἐκατερος τετράγωνων.

Δεῖ δὴ τὸν διδομένου ἀριθμοῦ μήτε διαδόθω δεῖν 

μήτε τινὰ τῶν ἀπὸ διάδοσι σκόπωσιν παραπεφαμένων.

'Επιτετάχθω δὴ τὴν Μ διελείν εἰς τρεῖς ἀριθμοὺς καὶ προσθέειναι ἐκάστῳ Μ γ καὶ ποιεῖν ἐκατερος 

τετράγωνων

---

⁶ Lit. “I take the square from any number of ἀριθμοὶ minus as many units as there are in the side of 16.”
⁷ i.e., a number of the form 3(8n + 2) + 1 or 24n + 7 cannot be the sum of three squares. In fact, a number of the form 8n + 7 cannot be the sum of three squares, but there are other
Let it be required to divide 16 into two squares. And let the first square $= x^2$; then the other will be $16 - x^2$; it shall be required therefore to make $16 - x^2 = a$ square.

I take a square of the form $a (mx - 4)^2$, $m$ being any integer and 4 the root of 16; for example, let the side be $2x - 4$, and the square itself $4x^2 + 16 - 16x$. Then

$$4x^2 + 16 - 16x = 16 - x^2.$$ 

Add to both sides the negative terms and take like from like. Then

$$5x^2 = 16x,$$

and

$$x = \frac{16}{5}.$$

One number will therefore be $\frac{256}{25}$, the other $\frac{144}{25}$, and their sum is $\frac{400}{25}$ or 16, and each is a square.

_Ibid_. v. 11, Dioph. ed. Tannery i. 342. 13–346. 12

To divide unity into three parts such that, if we add the same number to each of the parts, the results shall all be squares.

It is necessary that the given number be neither 2 nor any multiple of 8 increased by 2.

Let it be required to divide unity into three parts such that, when 3 is added to each, the results shall all be squares. Numbers not of this form which also are not the sum of three squares. Fermat showed that, if $3a + 1$ is the sum of three squares, then it cannot be of the form $4^n (24k + 7)$ or $4^n (8k + 7)$, where $k = 0$ or any integer.
Πάλιν δεί τὸν ἰ διελείν εἰς τρεῖς ὁφτὸς ὁπως ἐκαστὸς αὐτῶν μεῖζων ἢ Ῥ. ἦν οὖν πάλιν τὸν ἰ διελώμεν εἰς τρεῖς ὁφτ. τῇ τῆς παρισσότητος ἀγωγῇ, ἐσται ἐκαστὸς αὐτῶν μεῖζων τριάδος καὶ δυνησόμεθα, ἂφ ἐκάστον αὐτῶν ἀφελόντες Ῥ, ἔχεω εἰς οὖς ἢ Ῥ διαιρεῖται.

Λαμβάνομεν ἀρτί τοῦ ἰ τὸ γ ὁφτ. γ. τῇ, καὶ ξητούμεν τῷ προστιθέντες μόριον τετραγωνικὸν ταῖς Ῥ γ. κ. ποιήσουμεν ὁφτ. τάντα θαίσ. δὲ καὶ τῷ λ προσθείναι τῷ μόριον τετραγωνικὸν καὶ ποιεῖν τὸν ἀλοῦν ὁφτ.

"Εστω τὸ προστιθέμενον μόριον Δγ. ἀ. καὶ πάντα ἐπὶ Δγ. γίνονται Δγ. Ῥ ἣ Ῥ ἀ. ὁφτ. τῷ ἀπὸ πλευρᾶς εἰ. γίνεται ο ὁφτ. Δγ. εἰ. Δγ. Ῥ ἢ. ὁθεὶν ὁ Π. Ῥ ἢ. Ῥ ἢ. ἢ Δγ. Ῥ θ., τὸ Δγ. Ῥ θ.

Εἰ οὖν ταῖς Ῥ λ προστίθεται Ῥ ἢ. ταῖς Ῥ γ. κ. προστεθήςσαι λς καὶ γίνεται Ῥκα. δὲ καὶ οὖν τὸν ἰ διελεῖν εἰς τρεῖς ὁφτ. ὁπως ἐκάστον ὁφτ. ἀ. πλευρὰ πάρισος ἢ Ῥ ἢ.

'Αλλὰ καὶ ὅ τι σύγκειναι ἐκ δύο ὁφτ. τοῦ τῇ τῇ καὶ τῆς Ῥ. διαιροῦμεν τῇ τὴν Ῥ εἰς δύο ὁφτ. τὰ τῇ καὶ τὰ καὶ ὡστε τὸν ἰ σύγκειον ἐκ τριῶν ὁφτ.,

* The method has been explained in v. 19, where it is proposed to divide 13 into two squares each > 6. It will be sufficiently obvious from this example. The method is also used in v. 10, 12, 13, 14.

554
Then it is required to divide 10 into three squares such that each of them > 3. If then we divide 10 into three squares, according to the method of approximation, each of them will be > 3 and, by taking 3 from each, we shall be able to obtain the parts into which unity is to be divided.

We take, therefore, the third part of 10, which is \( \frac{3}{3} \), and try by adding some square part to \( \frac{3}{3} \) to make a square. On multiplying throughout by 9, it is required to add to 30 some square part which will make the whole a square.

Let the added part be \( \frac{1}{x^2} \); multiply throughout by \( x^2 \); then

\[
30x^2 + 1 = \text{a square.}
\]

Let the root be \( 5x + 1 \); then, squaring,

\[
25x^2 + 10x + 1 = 30x^2 + 1;
\]

whence

\[
x = 2, \quad x^2 = 4, \quad \frac{1}{x^2} = \frac{1}{4}.
\]

If, then, to 30 there be added \( \frac{1}{4} \), to \( \frac{3}{3} \) there is added \( \frac{1}{36} \), and the result is \( \frac{121}{36} \). It is therefore required to divide 10 into three squares such that the side of each shall approximate to \( \frac{11}{6} \).

But 10 is composed of two squares, 9 and 1. We divide 1 into two squares, \( \frac{9}{25} \) and \( \frac{16}{25} \), so that 10 is composed of three squares, 9, \( \frac{9}{25} \) and \( \frac{16}{25} \). It is there-
GREEK MATHEMATICS

\[ \varepsilon k \; te \; to\nu\; \theta \; kai \; to\nu\; \kappa \; kai \; to\nu\; \kappa \; \delta \; de\; o\nu \; \varepsilon k\; a\; stnh \; tw\nu\; p\; \lambda \; to\uvtwn \; p\; raskekuv\; asai \; p\; rmisov \; \eta. \]

'Alla \; kai \; ai \; p\; \lambda \; a\uvtw\nu\; e\is\nu\; M\; \gamma \; kai \; M\; \delta \; kai \; M\; \xi \; kai \; M\; \kappa \; kai \; M\; \eta. \; ta\; de\; id\; 5\; a\; g\; inon\; ta\; M\; \nu\; e\; de\; o\nu\; \varepsilon k\; a\; stnh \; p\; \lambda \; kataskekuv\; asai \; \nu\; e.\]

Pla\sosomen\nu\; enos\; pl\vnu\nu\; M\; \gamma \; \Lambda\; \delta\; \lambda\; \eta, \; et\vnu\ou\; \d\; \varepsilon\nu\; \mu\vnu\; \theta\; \xi\nu\nu, \; to\nu\; de\; et\vnu\ou\; \xi\; \lambda\; \xi\; \gamma \; e\; \nu\nu. \; g\; inon\; ta\; o\; ap\; tw\nu\; e\is\nu\; me\nu\nu\; \square\; \ou, \; \Delta\; \gamma\; \nu\nu\; \eta\; \Lambda\; \delta\; \rho\; \tau\; tauta\; id\; a\; M\; \iota. \; ou\nu\; e\is\nu\nu\; keta\; \d\; \xi\; \gamma\; \nu\nu\; \rho\; \tau.\]

'Eni\; tas\; upo\; st\; ae\; \nu\; kai\; g\; inon\; ta\; ai\; pl\vnu\nu\; tw\nu\; tet\nu\nu\; w\; dou\; ex\; a, \; ws\nu\; kai\; ai\; a\; utoi. \; ta\; lo\; uta\; di\; hla.\]

\[ I\; b\; i\; di. \; iv. \; 29, \; Dioph. \; ed. \; Tannery \; i. \; 258. \; 19-260. \; 16 \]

E\ufriv\nu\; t\; s\; sar\; as\; a\; ri\; th\mu\nu\; (tet\nu\nu\nu\nu\nu\nu\nu),\; o\; i\; su\; nteb\; ete\; \nu\nu\; kai\; prosl\; ab\; o\; nt\; e\; t\; s\; i\; di\; as\; pl\; 
u\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\nu\n
ALGEBRA: DIOPHANTUS

fore required to make each of the sides approximate to $\frac{11}{6}$.

But their sides are $3, \frac{4}{5}$ and $\frac{3}{5}$. Multiply throughout by 30, getting 90, 24 and 18; and $\frac{11}{6}$ [when multiplied by 30] becomes 55. It is therefore required to make each side approximate to 55.

[Now $3\frac{55}{30}$ by $\frac{35}{30}, \frac{4}{5} < \frac{55}{30}$ by $\frac{31}{30}$, and $\frac{3}{5} < \frac{55}{30}$ by $\frac{37}{30}$.]

If, then, we took the sides of the squares as $3 - \frac{35}{30}$, $\frac{4}{5} + \frac{31}{30}, \frac{3}{5} + \frac{37}{30}$, the sum of the squares would be $3 \cdot (\frac{11}{6})^2$ or $\frac{363}{36}$, which $> 10$.

Therefore] we take the side of the first square as $3 - 35x$, of the second as $\frac{4}{5} + 31x$, and of the third as $\frac{3}{5} + 37x$. The sum of the aforesaid squares

$3555x^2 + 10 - 116x = 10$;

whence

$x = \frac{116}{3555}$.

Returning to the conditions—as the sides of the squares are given, the squares themselves are also given. The rest is obvious.\footnote{Ibid. iv. 29, Dioph. ed. Tannery i. 258. 19–260. 16}

To find four square numbers such that their sum added to the sum of their sides shall make a given number.
"Εστω δή τὸν ἴβ.
'Επεί πᾶς □ος προσλαβών τὴν ἴδιαν πλ. καὶ Μδξ, ποιεῖ □ον, οὐ δὲ πλ. ΛΜ ζ' ποιεῖ ἀριθμόν τινα, ὡς ἔστι τοῦ ἐξ ἀρχῆς □ον πλευρά, οἱ τέσσαρες ἀριθμοὶ ἀρα, προσλαβόντες μὲν τὰς ἴδιας πλ. ποιοῦσι Μίβ, προσλαβόντες δὲ καὶ δδ, ποιοῦσι τέσσαρας □ονς. εἰςὶ δὲ καὶ αἱ Μίβ μετὰ δδ, ὡς ἔστι Μά, Μίγ. τὰς ίγ ἄρα Μ διαιρεῖν δεῖ εἰς τέσσαρας □ονς, καὶ ἀπὸ τῶν πλευρῶν, ἀφελῶν ἀπὸ ἑκάστης πλ. Μζ', ἔξω τῶν δ □ον τὰς πλ.
Διαιρεῖται δὲ ὁ ίγ εἰς δύο □ονς, τὸν τε δ καὶ θ. καὶ πάλιν ἑκάτερος τούτων διαιρεῖται εἰς δύο □ονς, εἰς κε καὶ κε, καὶ ρμδ καὶ πα. λαβῶν τοῖνυν ἑκάστου τὴν πλευρὰν, ε, ε, ε, ε, καὶ αἱρώ ἀπὸ ἑκάστου τούτων πλευρᾶς Μζ', καὶ ἔσονται αἱ πλ. τῶν ζητουμένων □ονς, ια', ζ', ιθ', ιγ'. αὐτοὶ ἄρα οἱ □ονς, ὡς μὲν ρκα, ὡς δὲ ρθ, ὡς δὲ τξα, ὡς δὲ ρζ'.
Let it be 12.

Since any square added to its own side and \( \frac{1}{4} \) makes a square, whose side minus \( \frac{1}{2} \) is the number which is the side of the original square, and the four numbers added to their own sides make 12, then if we add 4 \( \times \frac{1}{4} \) they will make four squares. But

\[
12 + 4 \times \frac{1}{4} \text{ (or 1)} = 13.
\]

Therefore it is required to divide 13 into four squares, and then, if I subtract \( \frac{1}{2} \) from each of their sides, I shall have the sides of the four squares.

Now 13 may be divided into two squares, 4 and 9. And again, each of these may be divided into two squares, \( \frac{64}{25} \) and \( \frac{36}{25} \), and \( \frac{144}{25} \) and \( \frac{81}{25} \). I take the side of each \( \frac{8}{5}, \frac{6}{5}, \frac{12}{5}, \frac{9}{5} \) and subtract half from each side, and the sides of the required squares will be

\[
\begin{align*}
11 & \quad 7 & \quad 19 & \quad 13 \\
10' & \quad 10' & \quad 10' & \quad 10'.
\end{align*}
\]

The squares themselves are therefore respectively

\[
\begin{align*}
121 & \quad 49 & \quad 361 & \quad 169 \quad b \\
100' & \quad 100' & \quad 100' & \quad 100' \\
\end{align*}
\]

\* i.e., \( x^2 + x + \frac{1}{4} = (x + \frac{1}{2})^2 \).

\( b \) In iv. 30 and v. 14 it is also required to divide a number into four squares. As every number is either a square or the sum of two, three or four squares (a theorem stated by Fermat and proved by Lagrange), and a square can always be divided into two squares, it follows that any number can be divided into four squares. It is not known whether Diophantus was aware of this.
GREEK MATHEMATICS

(f) POLYGONAL NUMBERS

Dioph. De polyg. num., Praef., Dioph. ed. Tannery
i. 450. 3-19

"Εκαστος τῶν ἀπὸ τῆς τριάδος ἀριθμῶν αὐξομένων μονάδι, πολύγωνός ἐστι πρῶτος" ἀπὸ τῆς μονάδος, καὶ ἔχει γωνίας τοσαύτας ὡςον ἐστὶν τὸ πλῆθος τῶν ἐν αὐτῷ μονάδων· πλευρά τε αὐτοῦ ἐστὶν ὁ ἐξῆς τῆς μονάδος ἀριθμὸς, ὁ β. ἐσται δὲ ὁ μὲν ᾗ τρίγωνος, ὁ δὲ ὁ τετράγωνος, ὁ δὲ ἐ πεντάγωνος, καὶ τοῦτο ἐξῆς.

Τῶν δὴ τετράγωνων προδήλων ὄντων ὃτι καθεστήκασι τετράγωνοι διὰ τὸ γεγονέναι αὐτοὺς ἐξ ἀριθμοῦ των ἐφ' ἐαυτὸν πολλαπλασιασθέντος, ἐδοκιμάσθη ἐκαστὸν τῶν πολυγώνων, πολυπλασιαζόμενον ἐπὶ τινα ἀριθμὸν κατὰ τὴν ἀναλογίαν τοῦ πλῆθους τῶν γωνιῶν αὐτοῦ, καὶ προσλαβόντα τετράγωνον των πάλιν κατὰ τὴν ἀναλογίαν τοῦ πλῆθους τῶν γωνιῶν αὐτῶν, φαίνεσθαι τετράγωνον· ὁ δὲ παραστήσομεν ύποδείξαντες πῶς ἀπὸ δοθείσης πλευρᾶς ὁ ἐπιταχθεὶς πολύγωνος εὐρίσκεται, καὶ πῶς δοθέντι πολυγώνῳ ἡ πλευρὰ λαμβάνεται.

1 πρῶτος Bachet, πρῶτον codd.

a A fragment of the tract On Polygonal Numbers is the only work by Diophantus to have survived with the Arithmetica. The main fact established in it is that stated in Hypsicles’ definition, that the a-gonal number of side n is

560
ALGEBRA: DIOPHANTUS

(f) POLYGONAL NUMBERS

Diophantus, On Polygonal Numbers, Preface, Dioph.
ed. Tannery i. 450. 3-19

From 3 onwards, every member of the series of natural numbers increasing by unity is the first (after unity) of a particular species of polygon, and it has as many angles as there are units in it; its side is the number next in order after the unit, that is, 2. Thus 3 will be a triangle, 4 a square, 5 a pentagon, and so on in order.\(^b\)

In the case of squares, it is clear that they are squares because they are formed by the multiplication of a number into itself. Similarly it was thought that any polygon, when multiplied by a certain number depending on the number of its angles, with the addition of a certain square also depending on the number of its angles, would also be a square. This we shall establish, showing how any assigned polygonal number may be found from a given side, and how the side may be calculated from a given polygonal number.

\[
\frac{1}{2}n\left(2 + (n - 1)(a - 2)\right) \quad (v. \ supra, \ p. 396 \ n. \ a, \ and \ vol. \ i. \ p. \ 98 \ n. \ a). \quad \text{The method of proof contrasts with that of the Arithmetica in being geometrical. For polygonal numbers, v. vol. i. pp. 86-99.}
\]

\(^b\) The meaning is explained in vol. i. p. 86 n. a, especially in the diagram on p. 89. In the example there given, 5 is the first (after unity) of the series of pentagonal numbers 1, 5, 12, 22 . . . . It has 5 angles, and each side joins 2 units.
XXIV. REVIVAL OF GEOMETRY:
PAPPUS OF ALEXANDRIA
XXIV. REVIVAL OF GEOMETRY: PAPPUS OF ALEXANDRIA

(a) General

Suidas, s.v. Πάππος

Πάππος, Ἀλεξανδρεύς, φιλόσοφος, γεγονός κατὰ τὸν πρεσβύτερον Θεοδόσιον τὸν βασιλέα, ὥστε καὶ Θέων ὁ φιλόσοφος ἡκμαζεν, ὁ γράψας εἰς τὸν Πτολεμαίον Κανόνα. βιβλία δὲ αὐτοῦ Χωρογραφία οἰκουμενική, Εἰς τὰ ὅ βιβλία τῆς Πτολεμαίου

* Theodosius I reigned from A.D. 379 to 395, but Suidas may have made a mistake over the date. A marginal note opposite the entry Diocletian in a Leyden ms. of chronological tables by Theon of Alexandria says, "In his time Pappus wrote"; Diocletian reigned from A.D. 284 to 305. In Rome's edition of Pappus's commentary on Ptolemy's Syntaxis (Studi e Testi, liv. pp. x-xiii), a cogent argument is given for believing that Pappus actually wrote his Collection about A.D. 320.

Suidas obviously had a most imperfect knowledge of Pappus, as he does not mention his greatest work, the Synagoge or Collection. It is a handbook to the whole of Greek geometry, and is now our sole source for much of the history of that science. The first book and half of the second are missing. The remainder of the second book gives an account of Apollonius's method of working with large numbers (v. supra, pp. 352-357). The nature of the remaining books to the eighth will be indicated by the passages here cited. There is some evidence (v. infra, p. 607 n. a) that the work was originally in twelve books.

The edition of the Collection with ancillary material published in three volumes by Friedrich Hultsch (Berlin, 1876–564)
XXIV. REVIVAL OF GEOMETRY: PAPPUS OF ALEXANDRIA

(a) General

Suidas, s.v. Pappus

Pappus, an Alexandrian, a philosopher, born in the time of the Emperor Theodosius I, when Theon the philosopher also flourished, who commented on Ptolemy's Table. His works include a Universal Geography, a Commentary on the Four Books of Ptolemy (1878) was a notable event in the revival of Greek mathematical studies. The editor's only major fault is one which he shares with his generation, a tendency to condemn on slender grounds passages as interpolated.

Pappus also wrote a commentary on Euclid's Elements; fragments on Book x. are believed to survive in Arabic (v. vol. i. p. 456 n. a). A commentary by Pappus on Euclid's Data is referred to in Marinus's commentary on that work. Pappus (v. vol. i. p. 301) himself refers to his commentary on the Analemma of Diodorus. The Arabic Fihrist says that he commented on Ptolemy's Planisphaerium.

The separate books of the Collection were divided by Pappus himself into numbered sections, generally preceded by a preface, and the editors have also divided the books into chapters. References to the Collection in the selections here given (e.g., Coll. iii. 11. 28, ed. Hultsch 68. 17–70. 8) are first to the book, then to the number or preface in Pappus's division, then to the chapter in the editors' division, and finally to the page and line of Hultsch's edition. In the selections from Book vii. Pappus's own divisions are omitted as they are too complicated, but in the collection of lemmas the numbers of the propositions in Hultsch's edition are added as these are often cited.

VOL. II 1 565
GREEK MATHEMATICS

Мегάλης συντάξεως ὑπόμνημα, Ποταμίων τοὺς εἰν Λίβυῃ, ὁ Οὐειροκριτικά.

(b) PROBLEMS AND THEOREMS

Papp. Coll. iii., Praef. 1, ed. Hultsch 30. 8-32. 3

Οἱ τὰ ἐν γεωμετρίᾳ ζητούμενα βουλόμενοι τεχνικώτερον διακρίνειν, ὡς κράτιστο Πανδροσίου, πρόβλημα μὲν ἀξίουσι καλεῖν ἐφ' οὐ προβάλλεται τι ποιήσαι καὶ κατασκευάσαι, θεώρημα δὲ ἐν ὧν τινῶν ὑποκειμένων τὸ ἐπόμενον αὐτοῖς καὶ πάντως ἐπισυμβάινων θεωρεῖται, τῶν παλαιῶν τῶν μὲν προβλήματα πάντα, τῶν δὲ θεώρηματα εἰναι φασκόντων. ὡ μὲν οὖν τὸ θεώρημα προτεινών, συνιδῶν ὄντων τρόπον, τὸ ἀκόλουθον τούτῳ ἀξιοὶ ζητεῖν καὶ οὐκ ἀν ἄλλως υγίως προτεινοί, δὲ τὸ πρόβλημα προτεινῶν [ἂν μὲν ἀμαθῆς ἢ καὶ παντάπασων ἴδιώτης], κἂν ἀδύνατόν πως κατασκευασθήναι προστάξῃ, σύγγνωστός ἄστιν καὶ ἀνυπεύθυνος. τοὺ γὰρ ζητοῦντος ἑργον καὶ τούτῳ διορίσαι, τὸ τε δυνατὸν καὶ τὸ ἀδύνατον, κἂν ἢ δυνατόν, πότε καὶ πῶς καὶ ποσαχῶς δυνατὸν. ἔὰν δὲ προσποιούμενος ἢ τὰ μαθήματα πως ἀπείρως προβάλλων, οὐκ ἐστιν αἰτίας ἔξω. πρώην γοῦν τινὲς τῶν τὰ μαθήματα προσποιούμενων εἰδέναι διὰ σοῦ τὰς τῶν προβλημάτων προτάσεις ἀμαθῶς ἡμῖν ἄρισταν. περὶ ὧν ἐδει καὶ τῶν

1 ἄν . . . ἴδιώτης ὁμ. Hultsch.

* Suidas seems to be confusing Ptolemy's Μαθηματικὴ τετράβιβλος σύνταξις (Tetrabiblos or Quadripartitum) which was in four books but on which Pappus did not comment, with the Μαθηματικὴ σύνταξις (Syntaxis or Almagest), which was the subject of a commentary by Pappus but extended to 566
REVIVAL OF GEOMETRY: PAPPUS


(b) PROBLEMS AND THEOREMS

Pappus, Collection iii., Preface 1, ed. Hultsch 30. 3–32. 3

Those who favour a more exact terminology in the subjects studied in geometry, most excellent Pandrosion, use the term problem to mean an inquiry in which it is proposed to do or to construct something, and the term theorem an inquiry in which the consequences and necessary implications of certain hypotheses are investigated, but among the ancients some described them all as problems, some as theorems. Therefore he who propounds a theorem, no matter how he has become aware of it, must set for investigation the conclusion inherent in the premises, and in no other way would he correctly propound the theorem; but he who propounds a problem, even though he may require us to construct something which is in some way impossible, is free from blame and criticism. For it is part of the investigator’s task to determine the conditions under which a problem is possible and impossible, and, if possible, when, how and in how many ways it is possible. But when a man professing to know mathematics sets an investigation wrongly he is not free from censure. For example, some persons professing to have learnt mathematics from you lately gave me a wrong enunciation of problems. It is desirable that I should state some of the proofs of thirteen books. Pappus’s commentary now survives only for Books v. and vi., which have been edited by A. Rome, Studi e Testi, liv., but it certainly covered the first six books and possibly all thirteen.
GREEK MATHEMATICS

παραπλησίων αὐτοῖς ἀποδείξεις τινὰς ἡμᾶς εἶπεῖν εἰς ὀφέλειαν σήν τε καὶ τῶν φιλομαθῶντων ἐν τῷ τρίτῳ τούτῳ τῆς Συναγωγῆς βιβλίω. τὸ μὲν οὖν πρῶτον τῶν προβλημάτων μέγας τις γεωμέτρης εἶναι δοκῶν ὃρισεν ἁμαθῶς. τὸ γὰρ δύο δοθεισῶν εὐθειῶν δύο μέσας ἄναλογον ἐν συνεχεί ἀναλογία λαβεῖν ἐφασκεν εἰδέναι δι’ ἑπιπέδου θεωρίας, ήξίου δὲ καὶ ἡμᾶς ὁ ἀνήρ ἑπισκεψαμένους ἀποκρίνασθαι περὶ τῆς ὑπ’ αὐτοῦ γεννηθείσης κατασκευῆς, ἦτις ἔχει τὸν τρόπον τούτον.

(c) The Theory of Means

Ibid. iii. 11. 28, ed. Hultsch 68. 17–70. 8

Τὸ δὲ δεύτερον τῶν προβλημάτων ἢν τόδε·

Ἐν ἡμικυκλίῳ τὰς τρεῖς μεσότητας λαβεῖν ἄλλος τις ἐφασκεν, καὶ ἡμικύκλιον τὸ ΑΒΓ ἐκθέμενος, οὐ κέντρον τὸ Ε, καὶ τυχὸν σημεῖον ἐπὶ τῆς ΑΓ λαβῶν τὸ Δ, καὶ ἀπ’ αὐτοῦ πρὸς ορθὰς ἁγαγῶν τῇ ΕΓ τὴν ΔΒ, καὶ ἐπιζεύξας τὴν ΕΒ, καὶ αὐτῇ κάθετον ἁγαγῶν ἀπὸ τοῦ Δ τῆς ΔΖ, τὰς τρεῖς μεσότητας ἐλεγεν ἀπλῶς εἰς τῷ ἡμικυκλίῳ ἐκτεθεῖσαι, τὴν μὲν ΕΓ μέσην ἀριθμητικὴν, τὴν δὲ ΔΒ μέσην γεωμετρικὴν, τὴν δὲ ΒΖ ἁρμονικὴν.

"Ὅτι μὲν οὖν ἡ ΒΔ μέσῃ ἐστὶ τῶν ΑΔ, ΔΓ ἐν

*The method, as described by Pappus, but not reproduced here, does not actually solve the problem, but it does furnish a series of successive approximations to the solution, and deserves more kindly treatment than it receives from him.

568
these and of matters akin to them, for the benefit both of yourself and of other lovers of this science, in this third book of the Collection. Now the first of these problems was set wrongly by a person who was thought to be a great geometer. For, given two straight lines, he claimed to know how to find by plane methods two means in continuous proportion, and he even asked that I should look into the matter and comment on his construction, which is after this manner.  

\[\text{(c) The Theory of Means} \]

\textit{Ibid.} iii. 11. 28, ed. Hultsch 68. 17–70. 8

The second of the problems was this:

A certain other [geometer] set the problem of exhibiting the three means in a semicircle. Describing a semicircle \( \Delta \Gamma \), with centre \( E \), and taking any point \( \Delta \) on \( \Delta \Gamma \), and from it drawing \( \Delta B \) perpendicular to \( \Gamma \), and joining \( EB \), and from \( \Delta \) drawing \( \Delta Z \) perpendicular to it, he claimed simply that the three means had been set out in the semicircle, \( \Delta \Gamma \) being the arithmetic mean, \( \Delta B \) the geometric mean and \( BZ \) the harmonic mean.

That \( B\Delta \) is a mean between \( \Delta \Delta, \Delta \Gamma \) in geometrical

569


(d) The Paradoxes of Erycinus


Τὸ δὲ τρίτον τῶν προβλημάτων ἢν τὸδε.

"Εστω τρίγωνον ὀρθογώνιον τὸ ἈΒΓ ὀρθὴν

---

570
REVIVAL OF GEOMETRY: PAPPUS

proportion, and $\Delta \Gamma \propto \propto \propto \propto$ in arithmetical proportion, is clear. For

$$A \Delta : \Delta B = \Delta B : \Delta \Gamma, \quad [\text{Eucl. iii. 31, vi. 8 Por.}]$$

and

$$A \Delta : A \Delta = (A \Delta - AE) : (E \Gamma - \Gamma \Delta) = (A \Delta - E \Gamma) : (E \Gamma - \Gamma \Delta).$$

But how $ZB$ is a harmonic mean, or between what kind of lines, he did not say, but only that it is a third proportional to $EB$, $B \Delta$, not knowing that from $EB$, $B \Delta$, $BZ$, which are in geometrical proportion, the harmonic mean is formed. For it will be proved by me later that a harmonic proportion can thus be formed—

greater extreme $= 2EB + 3\Delta B + BZ$,
mean term $= 2B \Delta + BZ$,
lesser extreme $= B \Delta + BZ$.

(d) THE PARADOXES OF ERYCINUS

Ibid. iii. 24. 58, ed. Hultsch 104. 14–106. 9

The third of the problems was this:
Let $A \Gamma \Gamma \propto \propto \propto \propto$ be a right-angled triangle having the

* It is Pappus, in fact, who seems to have erred, for $BZ$ is a harmonic mean between $A \Delta$, $\Delta \Gamma$, as can thus be proved:
Since $B \Delta E$ is a right-angled triangle in which $\Delta Z$ is perpendicular to $BE$,

$$BZ : B \Delta = B \Delta : BE,$$

i.e.,

$$BZ \cdot BE = B \Delta^2 = A \Delta \cdot \Delta \Gamma.$$

But

$$BE = \frac{1}{2}(A \Delta + \Delta \Gamma);$$

$$BZ(A \Delta + \Delta \Gamma) = 2A \Delta \cdot \Delta \Gamma.$$

\therefore

$$A \Delta(BZ - \Delta \Gamma) = \Delta \Gamma(A \Delta - BZ),$$

i.e.,

$$A \Delta : \Delta \Gamma = (A \Delta - BZ) : (BZ - \Delta \Gamma),$$

and $\therefore$ $BZ$ is a harmonic mean between $A \Delta$, $\Delta \Gamma$.

The three means and the several extremes have thus been
GREEK MATHEMATICS

εὖχον τὴν Β γωνίαν, καὶ διήχωθω τις ἡ ΑΔ, καὶ κείσθω τῇ ΑΒ ἵση ἡ ΔΕ, καὶ δίχα τιμήθεισι τῇς ΕΑ κατὰ τὸ Ζ, καὶ ἐπιζευγθείσης τῆς ΖΓ δεῖξαι συναμφοτέρας τὰς ΔΖΓ δύο πλευρὰς ἐντὸς τοῦ τριγώνου μείζονας τῶν ἐκτὸς συναμφοτέρων τῶν ΒΑΓ πλευρῶν.

Καὶ ἔστι δήλον. ἐπεὶ γὰρ αἱ ΓΖΑ, τούτεστιν αἱ ΓΖΕ, τῆς ΓΑ μείζονες εἰσιν, ἵσῃ δὲ ἡ ΔΕ τῇ ΑΒ, αἱ ΓΖΔ ἄρα δύο τῶν ΓΑΒ μείζονες εἰσιν . . .

'Αλλ' ὁτι τοῦτο μὲν, ὅπως ἂν τις έθέλοι προτείνειν, ἀπειραχώς δεῖκνυται δήλον, οὔκ ἀκαίρον δὲ καθολικώτερον περὶ τῶν τοιούτων προβλημάτων διαλαβεῖν ἀπὸ τῶν φερομένων παραδόξων Ἑρυκίνου προτείνοντας οὕτως.

(e) THE REGULAR SOLIDS

Ibid. iii. 40. 75, ed. Hultsch 132. 1-11

Εἰς τὴν δοθείσαν σφαῖραν ἐγγράψαι τὰ πέντε πολύεδρα, προγράφεται δὲ τάδε.

'Εστω ἐν σφαίρᾳ κύκλος ὁ ΑΒΓ, οὗ διάμετρος ἡ ΑΓ καὶ κέντρον τὸ Δ, καὶ προκεῖσθω εἰς τὸν

represented by five straight lines (ΕΒ, ΒΖ, ΑΔ, ΔΓ, ΒΔ). Pappus takes six lines to solve the problem. He proceeds to define the seven other means and to form all ten means as linear functions of three terms in geometrical progression (v. vol. i. pp. 124-129).

572
angle B right, and let $A\Delta$ be drawn, and let $\Delta E$ be placed equal to $AB$, then if $EA$ be bisected at $Z$, and $Z\Gamma$ be joined, to show that the sum of the two sides $\Delta Z$, $Z\Gamma$ within the triangle, is greater than the sum of the two sides $BA$, $A\Gamma$ without the triangle.

And it is obvious. For

since $\Gamma Z + ZA > \Gamma A$, \hspace{1cm} [Eucl. i. 20

i.e., $\Gamma Z + ZE > \Gamma A$,

while $\Delta E = AB$,

\therefore $[\Gamma Z + ZE + E\Delta = \Gamma A + A$

i.e.,] $\Gamma Z + Z\Delta > \Gamma A + AB$. \ldots

But it is clear that this type of proposition, according to the different ways in which one might wish to propound it, can take an infinite number of forms, and it is not out of place to discuss such problems more generally and [first] to propound this from the so-called paradoxes of Erycinus.\footnote{Nothing further is known of Erycinus. The propositions next investigated are more elaborate than the one just solved.}

(e) The Regular Solids \footnote{This is the fourth subject dealt with in Coll. iii. For the treatment of the subject by earlier geometers, v. vol. i. pp. 216-225, 466-479.}

\textit{Ibid.} iii. 40. 75, ed. Hultsch 132. 1-11

In order to inscribe the five polyhedra in a sphere, these things are premised.

Let $AB\Gamma$ be a circle in a sphere, with diameter $A\Gamma$ and centre $\Delta$, and let it be proposed to insert in the
GREEK MATHEMATICS

κύκλου ἐμβαλεῖν εὐθείαν παράλληλον μὲν τῇ ΑΓ διαμέτρῳ, ἵσυν δὲ τῇ δοθείᾳ μὴ μείζον οὐσι τῆς ΑΓ διαμέτρου.

Κείσθω τῇ ἴμουσεῖα τῆς δοθείσης ἴση ἡ ΕΔ, καὶ τῇ ΑΓ διαμέτρῳ ἴχθῳ πρὸς ὄρθας ἡ ΕΒ, τῇ δὲ ΑΓ παράλληλος ἡ ΒΖ, ἵτις ἴση ἐσται τῇ δοθείῃ. διπλή γάρ ἐστὶν τῆς ΕΔ, ἐπει καὶ ἴση τῇ ΕΗ, παράλληλον ἀχθείσης τῆς ΖΗ τῇ ΒΕ.

(f) EXTENSION OF PYTHAGORAS'S THEOREM

Ibid. iv. 1. 1, ed. Hultsch 176. 9–178. 13

Εὖν ἡ τρίγωνον τὸ ΑΒΓ, καὶ ἀπὸ τῶν ΑΒ, ΒΓ ἀναγραφῇ τυχόντα παράλληλόγραμμα τὰ ΑΒΔΕ, ΒΓΖΗ, καὶ αἱ ΔΕ, ΖΗ ἐκβληθῶσιν ἐπὶ τὸ Θ, καὶ ἐπιζευγθή ἡ ΘΒ, γίνεται τὰ ΑΒΔΕ, 574
circle a chord parallel to the diameter $\Delta \Gamma$ and equal to a given straight line not greater than the diameter $\Delta \Gamma$.

Let $E\Delta$ be placed equal to half of the given straight line, and let $EB$ be drawn perpendicular to the diameter $\Delta \Gamma$, and let $BZ$ be drawn parallel to $\Delta \Gamma$; then shall this line be equal to the given straight line. For it is double of $E\Delta$, inasmuch as $ZH$, when drawn, is parallel to $BE$, and it is therefore equal to $EH$.*

\( \text{(f) Extension of Pythagoras's Theorem} \)

*Ibid. iv. 1. 1, ed. Hultsch 176. 9-178. 13

If $AB\Gamma$ be a triangle, and on $AB$, $B\Gamma$ there be described any parallelograms $AB\Delta E$, $B\Gamma ZH$, and $\Delta E$, $ZH$ be produced to $\Theta$, and $\Theta B$ be joined, then the

* This lemma gives the key to Pappus's method of inscribing the regular solids, which is to find in the case of each solid certain parallel circular sections of the sphere. In the case of the cube, for example, he finds two equal and parallel circular sections, the square on whose diameter is two-thirds of the square on the diameter of the sphere. The squares inscribed in these circles are then opposite faces of the cube. In each case the method of analysis and synthesis is followed. The treatment is quite different from Euclid's.
GREEK MATHEMATICS

ΒΓΖΗ παραλληλόγραμμα ἵσα τῷ ὑπὸ τῶν ΑΓ, ΘΒ περιεχομένω παραλληλογράμμῳ ἐν γωνίᾳ ἡ ἕστιν ἴση συναμφοτέρῳ τῇ ὑπὸ ΒΑΓ, ΔΘΒ.

'Εκβεβλήσθω γὰρ ἡ ΘΒ ἐπὶ τὸ Κ, καὶ διὰ τῶν Α, Γ τῇ ΘΚ παράλληλοι ἡχθωσαν αἱ ΑΛ, ΓΜ, καὶ ἐπεζεῦχθω ἡ ΛΜ. ἔπει παραλληλόγραμμὸν ἕστιν τὸ ΑΛΘΒ, αἱ ΑΛ, ΘΒ ἵσαι τέ εἰσιν καὶ παράλληλοι. ὀμοίως καὶ αἱ ΜΓ, ΘΒ ἵσαι τέ εἰσιν καὶ παράλληλοι, ὡστε καὶ αἱ ΛΑ, ΜΓ ἵσαι τέ εἰσιν καὶ παράλληλοι. καὶ αἱ ΛΜ, ΑΓ ἀρα ἵσαι τε καὶ παράλληλοι εἰσιν. παραλληλόγραμμὸν ἀρα ἕστιν τὸ ΑΛΜΓ ἐν γωνίᾳ τῇ ὑπὸ ΔΑΓ, τούτου ἕστιν συναμφοτέρῳ τῇ τε ὑπὸ ΒΑΓ καὶ ὑπὸ ΔΘΒ. ἵση γὰρ ἕστιν ἡ ὑπὸ ΔΘΒ τῇ ὑπὸ ΛΑΒ. καὶ ἔπει τὸ ΔΑΒΕ παραλληλόγραμμὸν τῷ ΛΑΒΘ ἵσον ἕστιν (ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως ἕστιν 576.
REVIVAL OF GEOMETRY: PAPPUS

parallelograms $AB\Delta E$, $B\Gamma ZH$ are together equal to the parallelogram contained by $\Delta \Gamma$, $\Theta B$ in an angle which is equal to the sum of the angles $BA\Gamma$, $\Delta \Theta B$.

For let $\Theta B$ be produced to $K$, and through $A$, $\Gamma$ let $\Lambda \Lambda$, $\Gamma \Lambda$ be drawn parallel to $\Theta K$, and let $\Lambda M$ be joined. Since $\Lambda \Lambda \Theta B$ is a parallelogram, $\Lambda \Lambda$, $\Theta B$ are equal and parallel. Similarly $M\Gamma$, $\Theta B$ are equal and parallel, so that $\Lambda \Lambda$, $M\Gamma$ are equal and parallel. And therefore $\Lambda M$, $\Lambda \Gamma$ are equal and parallel; therefore $\Lambda \Lambda M\Gamma$ is a parallelogram in the angle $\Lambda A\Gamma$, that is an angle equal to the sum of the angles $BA\Gamma$ and $\Delta \Theta B$; for the angle $\Delta \Theta B$ = angle $\Lambda A \Theta B$. And since the parallelogram $\Delta ABE$ is equal to the parallelogram $\Delta A \Theta B$ (for they are upon the same base $AB$ and in the
τῆς ΑΒ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΑΒ, 
ΔΘ), ἀλλὰ τὸ ΛΑΒΘ τῷ ΛΑΚΝ ἴσον ἐστὶν (ἐπὶ 
τε γὰρ τῆς αὐτῆς βάσεως ἐστὶν τῆς ΛΑ καὶ ἐν 
ταῖς αὐταῖς παραλλήλοις ταῖς ΛΑ, ΘΚ), καὶ τὸ 
ΑΔΕΒ ἄρα τῷ ΛΑΚΝ ἴσον ἐστὶν. διὰ τὰ αὐτὰ 
καὶ τὸ ΒΗΖΓ τῷ ΝΚΓΜ ἴσον ἐστὶν. τὰ ἄρα 
ΔΑΒΕ, ΒΗΖΓ παραλληλόγραμμα τῷ ΛΑΓΜ ἴσα 
ἐστὶν, τούτεστιν τῷ ὑπὸ ΑΓ, ΘΒ ἐν γωνίᾳ τῇ ὑπὸ 
ΛΑΓ, ἡ ἐστὶν ἴση συναμφοτέραις ταῖς ὑπὸ ΒΑΓ, 
ΒΘΔ. καὶ ἐστὶ τούτῳ καθολικῶτερον πολλῷ τοῦ 
ἐν τοῖς ὁρθογωνίοις ἐπὶ τῶν τετραγώνων ἐν τοῖς 
Στοιχείοις δεδειγμένου.

(g) Circles Inscribed in the ἅρβηλος

Ibid. iv. 14. 19, ed. Hultsch 208. 9-21

Φέρεται ἐν τισιν ἄρχαια πρότασις τοιαύτης 
ὑποκείσθω τρία ἡμικύκλια ἐφαπτόμενα ἀλλήλων 

τὰ ΑΒΓ, ΑΔΕ, ΕΖΓ, καὶ εἰς τὸ μεταξὺ τῶν 
περιφερείων αὐτῶν χωρίον, ὅ δὲ καλοῦσιν ἅρβηλον, 
578
same parallels $AB, \Delta \Theta$, while $\Lambda \Lambda \Theta = \Lambda \Lambda K N$ (for they are upon the same base $\Lambda \Lambda$ and in the same parallels $\Lambda \Lambda, \Theta K$), therefore $\Lambda \Delta E B = \Lambda \Lambda K N$. By the same reasoning $\Lambda H Z \Gamma = \Lambda K \Gamma M$; therefore the parallelograms $\Lambda \Lambda B E, \Lambda H Z \Gamma$ are together equal to $\Lambda \Lambda \Gamma M$, that is, to the parallelogram contained by $\Lambda \Gamma, \Theta B$ in the angle $\Lambda \Lambda \Gamma$, which is equal to the sum of the angles $\Lambda \Lambda \Gamma, \Theta \Delta$. And this is much more general than the theorem proved in the Elements about the squares on right-angled triangles.\textsuperscript{a}

\textbf{(g) Circles Inscribed in the $\alpha \rho \beta \gamma \lambda \sigma$}

\textit{Ibid.} iv. 14. 19, ed. Hultsch 208. 9-21

There is found in certain [books] an ancient proposition to this effect: Let $\Lambda \Lambda \Gamma, \Lambda \Delta E, \Lambda \Delta Z \Gamma$ be supposed to be three semicircles touching each other, and in the space between their circumferences, which

\textsuperscript{a} Eucl. i. 47, v. vol. i. pp. 178-185. In the case taken by Pappus, the first two parallelograms are drawn outwards and the third, equal to their sum, is drawn inwards. If the areas of parallelograms drawn outwards be regarded as of opposite sign to the areas of those drawn inwards, the theorem may be still further generalized, for the algebraic sum of the three parallelograms is equal to zero.
GREEK MATHEMATICS

διγγεγράφθωσαν κύκλοι ἐφαπτόμενοι τῶν τε ἡμικυκλίων καὶ ἄλλῃ λων ὀσοιδηποτοῦν, ὡς οἱ περὶ κέντρα τὰ Ἥ, Θ, Κ, Λ· δεῖξαι τὴν μὲν ἀπὸ τοῦ Η κέντρου κάθετον ἐπὶ τὴν ΑΓ ἱσθην τῆς διαμέτρου τοῦ περὶ τὸ Η κύκλου, τὴν δ' ἀπὸ τοῦ Θ κάθετον διπλασίαν τῆς διαμέτρου τοῦ περὶ τὸ Θ κύκλου, τὴν δ' ἀπὸ τοῦ Κ κάθετον τριπλασίαν, καὶ τὰς ἐξῆς καθέτους τῶν οἰκείων διαμέτρων πολλαπλασίας κατὰ τοὺς ἐξῆς μονάδι ἄλλῃ λων ὑπερέχοντας ἀριθμοὺς ἐν ἀπειρον γνωμένης τῆς τῶν κύκλων ἐγγραφῆς.

(h) SPIRAL ON A SPHERE

Ibid. iv. 35. 53-56, ed. Hultsch 264. 3-268. 21

"Ωσπερ ἐν ἐπιπέδῳ νοείται γνωμένη τις ἐλξι φερομένου σημείου κατ' εὐθείας κύκλου περιγραφοῦσης, καὶ ἐπὶ στερεῶν φερομένου σημείου κατὰ μιᾶς πλευρᾶς των ἐπιφάνειαν περιγραφοῦσης, ούτως ὅτι καὶ ἐπὶ σφαῖρας ἐλικα νοεῖ ἀκόλουθον ἐστι γραφομένην τὸν τρόπον τοῦτον.

"Εστω ἐν σφαῖρα μέγιστοσ κύκλος ὁ ΚΛΜ περὶ πόλον τὸ Θ σημείον, καὶ ἀπὸ τοῦ Θ μεγίστου

a Three propositions (Nos. 4, 5 and 6) about the figure known as the ἄρβηλος from its resemblance to a leatherworker's knife are contained in Archimedes' Liber Assumptorum, which has survived in Arabic. They are included as particular cases in Pappus's exposition, which is unfortunately too long for reproduction here. Professor D'Arcy W. Thompson (The Classical Review, lvii. (1942), pp. 75-76) gives reasons for thinking that the ἄρβηλος was a saddler's knife rather than a shoemaker's knife, as usually translated.
REVIVAL OF GEOMETRY: PAPPUS

is called the "leather-worker's knife," let there be inscribed any number whatever of circles touching both the semicircles and one another, as those about the centres H, Θ, K, Λ; to prove that the perpendicular from the centre H to ΛΓ is equal to the diameter of the circle about H, the perpendicular from Θ is double of the diameter of the circle about Θ, the perpendicular from K is triple, and the [remaining] perpendiculars in order are so many times the diameters of the proper circles according to the numbers in a series increasing by unity, the inscription of the circles proceeding without limit.a

(h) Spiral on a Sphere b

Ibid. iv. 35. 53-56, ed. Hultsch 264. 3-268. 21

Just as in a plane a spiral is conceived to be generated by the motion of a point along a straight line revolving in a circle, and in solids [such as the cylinder or cone.,]c by the motion of a point along one straight line describing a certain surface, so also a corresponding spiral can be conceived as described on the sphere after this manner.

Let ΚΑΜ be a great circle in a sphere with pole Θ, and from Θ let the quadrant of a great circle ΘΝΚ be

b After leaving the ἄρβηγλος, Pappus devotes the remainder of Book iv. to solutions of the problems of doubling the cube, squaring the circle and trisecting an angle. This part has been frequently cited already (v. vol. i. pp. 298-309, 336-363). His treatment of the spiral is noteworthy because his method of proof is often markedly different from that of Archimedes; and in the course of it he makes this interesting digression.

c Some such addition is necessary, as Commandinus, Chasles and Hultsch realized.
GREEK MATHEMATICS

κύκλου τεταρτημόριον γεγράφθω τὸ ὈΝΚ, καὶ ἤ μὲν ὈΝΚ περιφέρεια, περὶ τὸ Ὁ μένον φερομένη κατὰ τῆς ἐπιφανείας ὡς ἐπὶ τὰ Δ, Μ μέρη.

ἀποκαθιστάσθω πάλιν ἐπὶ τὸ αὐτὸ, σημειοῦν δὲ τι φερόμενον ἐπὶ αὐτῆς ἀπὸ τοῦ Ὁ ἐπὶ τὸ Κ παραγινέσθω. γράφει δὴ τινα ἐπὶ τῆς ἐπιφανείας ἐλικα, οὐα ἐστὶν ἡ ΘΟΙΚ, καὶ ἦτις ἀν ἀπὸ τοῦ Ὁ γραφὴ μεγίστον κύκλου περιφέρεια, πρὸς τὴν ΚΛ περιφέρειαν λόγον ἔχει ὃν ἡ ΛΘ πρὸς τὴν ΘΟ. λέγω δὴ ὅτι, ἀν ἐκτεθῇ τεταρτημόριον τοῦ μεγίστου ἐν τῇ σφαῖρᾳ κύκλου τὸ ΑΒΓ περὶ κέντρου τὸ Δ, καὶ ἐπιζευγθῇ ἡ ΓΔ, γίνεται ὡς ἡ τοῦ ἡμισφαιρίου ἐπιφάνεια πρὸς τὴν μεταξὺ τῆς ΘΟΙΚ ἐλικός καὶ τῆς ΚΝΘ περιφερείας ἀπολαμβανομένη ἐπιφάνειαν, οὕτως ὁ ΑΒΓΔ τομεὺς πρὸς τὸ ΑΒΓ τμῆμα.

"Ἡχθω γὰρ ἐφαπτομένη τῆς περιφερείας ἡ ΓΖ, καὶ περὶ κέντρον τὸ Γ διὰ τοῦ Α γεγράφθω περιφέρεια ἡ ΑΕΖ· ἴσος ἀρὰ ὁ ΑΒΓΔ τομεὺς τῷ 582
described, and, $\Theta$ remaining stationary, let the arc $\Theta N K$ revolve about the surface in the direction $A, M$

and again return to the same place, and [in the same time] let a point on it move from $\Theta$ to $K$; then it will describe on the surface a certain spiral, such as $\Theta O I K$, and if any arc of a great circle be drawn from $\Theta$ [cutting the circle $K \Lambda M$ first in $\Lambda$ and the spiral first in $O$], its circumference will bear to the arc $K \Lambda$ the same ratio as $\Lambda \Theta$ bears to $\Theta O$. I say then that if a quadrant $A B \Gamma$ of a great circle in the sphere be set out about centre $\Delta$, and $\Gamma A$ be joined, the surface of the hemisphere will bear to the portion of the surface intercepted between the spiral $\Theta O I K$ and the arc $K N \Theta$ the same ratio as the sector $A B \Gamma \Delta$ bears to the segment $A B \Gamma$.

For let $\Gamma Z$ be drawn to touch the circumference, and with centre $\Gamma$ let there be described through $\Lambda$ the arc $A E Z$; then the sector $A B \Gamma \Delta$ is equal to the

* Or, of course, the circumference of the circle $K \Lambda M$ to which it is equal.
Pappus's method of proof is, in the Archimedean manner, to circumscribe about the surface to be measured a figure consisting of sectors on the sphere, and to circumscribe about the segment $AB\Gamma$ a figure consisting of sectors of circles; in the same way figures can be inscribed. The divisions need, therefore, to be as numerous as possible. The conclusion can then be reached by the method of exhaustion.
REVIVAL OF GEOMETRY: PAPPUS

sector $AEZ\Gamma$ (for $\angle A\Delta \Gamma = 2 \cdot \angle A\Gamma Z$, and $\Delta A^2 = \frac{1}{2} \Delta \Gamma^2$); I say, then, that the ratio of the aforesaid surfaces one towards the other is the same as the ratio of the sector $AEZ\Gamma$ to the segment $AB\Gamma$.

Let $ZE$ be the same [small] $^a$ part of $ZA$ as $K\Lambda$ is of the whole circumference of the circle, and let $EI\Gamma$ be joined; then the arc $B\Gamma$ will be the same part of the arc $AB\Gamma$. $^b$ But $\Theta O$ is the same part of $\Theta O\Lambda$ as $K\Lambda$ is of the whole circumference [by the property of the spiral]. And $\text{arc } \Theta O\Lambda = \text{arc } AB\Gamma$ [ex constructione]. Therefore $\Theta O = B\Gamma$. Let there be described through $O$ about the pole $\theta$ the arc $ON$, and through $B$ about centre $\Gamma$ the arc $BH$. Then since the [sector of the] spherical surface $\Delta K\Theta$ bears to the [sector] $O\Theta N$ the same ratio as the whole surface of the hemisphere bears to the surface of the segment with pole $\theta$ and circular base $ON$, $^c$ while the surface of the hemisphere bears to the surface of the segment the same ratio as $\Theta \Lambda^2$ to $\Theta O^2$, $^d$ or $EI^2$ to $B\Gamma^2$, therefore the sector $K\Lambda \Theta$ on the surface [of the sphere] bears to $O\Theta N$ the same ratio as the sector $EZ\Gamma$ [in the plane] bears to the sector $B\Gamma \Theta$. Similarly we may show that all the sectors [on the surface of] the hemi-

$^a$ For arc $ZA$ : arc $ZE$ = angle $Z\Gamma A$ : angle $Z\Gamma E$. But angle $Z\Gamma A = \frac{1}{2} \cdot \angle A\Delta \Gamma$, and angle $Z\Gamma E = \frac{1}{2} \cdot \angle B\Delta \Gamma$ [Eucl. iii. 32, 20]. $^b$ arc $ZA$ : arc $ZE$ = arc $AB\Gamma$ : arc $B\Gamma$.

$^c$ Because the arc $\Delta K$ is the same part of the circumference $K\Lambda M$ as the arc $ON$ is of its circumference.

$^d$ The square on $\Theta \Lambda$ is double the square on the radius of the hemisphere, and therefore half the surface of the hemisphere is equal to a circle of radius $\Theta \Lambda$ [Archim. De sph. et cyl. i. 33]; and the surface of the segment is equal to a circle of radius $\Theta O$ [ibid. i. 42]; and as circles are to one another as the squares on their radii [Eucl. xii. 2], the surface of the hemisphere bears to the surface of the segment the ratio $\Theta \Lambda^2$ : $\Theta O^2$.

585
This would be proved by the method of exhaustion. It is proof of the great part played by this method in Greek geometry that Pappus can take its validity for granted.

For the surface of the hemisphere is double of the circle of radius $\Delta$ [Archim. De sph. et cyl. i. 33] and the sector $\Delta \Gamma \Delta$ is one-quarter of the circle of radius $\Delta$.

For the surface between the spiral and the base of the hemisphere is equal to the surface of the hemisphere less the surface cut off from the spiral in the direction $\Theta N K$,

**i.e.** Surface in question = surface of hemisphere −

$8 \text{ segment } ABG$

$= 8 \text{ sector } ABG \Delta - 8 \text{ segment } ABG$
sphere equal to $K\Lambda \Theta$, together making up the whole surface of the hemisphere, bear to the sectors described about the spiral similar to $O\Theta N$ the same ratio as the sectors in $AZ\Gamma$ equal to $EZ\Gamma$, that is the whole sector $AZ\Gamma$, bear to the sectors described about the segment $AB\Gamma$ similar to $\Gamma B\Pi$. In the same manner it may be shown that the surface of the hemisphere bears to the [sum of the] sectors inscribed in the spiral the same ratio as the sector $AZ\Gamma$ bears to the [sum of the] sectors inscribed in the segment $AB\Gamma$, so that the surface of the hemisphere bears to the surface cut off by the spiral the same ratio as the sector $AZ\Gamma$, that is the quadrant $AB\Gamma\Delta$, bears to the segment $AB\Gamma$. From this it may be deduced that the surface cut off from the spiral in the direction of the arc $\Theta N\Pi$ is eight times the segment $AB\Gamma$ (since the surface of the hemisphere is eight times the sector $AB\Gamma\Pi\Delta$), while the surface between the spiral and the base of the hemisphere is eight times the triangle $AB\Gamma\Delta$, that is, it is equal to the square on the diameter of the sphere.

Heath (H.G.M. ii. 384-385) gives for this elegant proposition an analytical equivalent, which I have adapted to the Greek lettering. If $\rho$, $\omega$ are the spherical co-ordinates of $O$ with reference to $\Theta$ as pole and the arc $\Theta N\Pi$ as polar axis, the equation of the spiral is $\omega = 4\rho$. If $A$ is the area of the spiral to be measured, and the radius of the sphere is taken as unity, we have as the element of area

$$dA = d\omega(1 - \cos \rho) = 4\rho(1 - \cos \rho).$$

and $2A\Delta$ is the diameter of the sphere.
GREEK MATHEMATICS

(ii) Isoperimetric Figures

Ibid. v., Fracf. 1-3, ed. Hultsch 304. 5-308. 5

Σοφίας καὶ μαθημάτων ἐννοιαν ἀρίστην μὲν καὶ τελειοτάτην ἀνθρώποις θεὸς ἐδωκεν, ὡς κράτιοτε Μεγεθιόν, ἐκ μέρους δὲ που καὶ τῶν ἀλόγων ζώων μοῖραν ἀπένειμεν τισιν. ἀνθρώποις μὲν οὖν ἄτε λογικοῖς οὐσὶ τὸ μετὰ λόγου καὶ ἀποδείξεως παρέσχεν ἑκάστα ποιεῖν, τοῖς δὲ λοιποῖς ζώοις ἀνευ λόγου τὸ χρήσιμον καὶ βιωφελές αὐτὸ μόνον κατὰ τινα φυσικὴν πρόνοιαν ἑκάστοις ἐχεῖν ἐδωρη- σατο. τούτῳ δὲ μάθοι τις ἃν ὑπάρχον καὶ ἐν ἑτέροις μὲν πλείστοις γένεσιν τῶν ζώων, οὐχ ἥκιστα ἀλλὰ καὶ ταῖς μελίσσαις. ἦ τε γὰρ εὐταξία καὶ πρὸς τὰς ἡγομένας τῆς ἐν αὐταῖς πολιτείας εὐπείθεια θαυμαστὴ τις, ἦ τε φιλοσοφία καὶ καθαριότης ἦ περὶ τῆν τοῦ μέλιτος συναγωγήν καὶ ἦ περὶ τῆν φυλακήν αὐτοῦ πρόνοια καὶ οἰκονομία πολὺ μᾶλλον θαυμασιωτέρα. πεπιστευμέναι γὰρ, ὡς εἰκός, παρὰ θεῶν κομίζειν τοῖς τῶν ἀνθρώπων μονοκοίς

\[
A = \int_{0}^{\frac{1}{2}\pi} 4\, \rho (1 - \cos \rho) \, d\rho
\]

\[
= 2\pi - 4.
\]

\[
\frac{A}{\text{surface of hemisphere}} = \frac{2\pi - 4}{2\pi}
\]

\[
= \frac{1}{2}\pi - \frac{1}{2}
\]

\[
= \text{segment } ABΓ
\]

\[
= \text{sector } ABΓΔ.
\]

The whole of Book v. in Pappus's Collection is devoted to isoperimetry. The first section follows closely the exposition of Zenodorus as given by Theon (v. supra, pp. 386-395), 588
REVIVAL OF GEOMETRY: PAPPUS

(i) ISOPERIMETRIC FIGURES

Ibid. v., Preface 1-3, ed. Hultsch 304. 5-308. 5

Though God has given to men, most excellent Megethion, the best and most perfect understanding of wisdom and mathematics, He has allotted a partial share to some of the unreasoning creatures as well. To men, as being endowed with reason, He granted that they should do everything in the light of reason and demonstration, but to the other unreasoning creatures He gave only this gift, that each of them should, in accordance with a certain natural forethought, obtain so much as is needful for supporting life. This instinct may be observed to exist in many other species of creatures, but it is specially marked among bees. Their good order and their obedience to the queens who rule in their commonwealths are truly admirable, but much more admirable still is their emulation, their cleanliness in the gathering of honey, and the forethought and domestic care they give to its protection. Believing themselves, no doubt, to be entrusted with the task of bringing from the gods to the more cultured part of mankind a share of

except that Pappus includes the proposition that of all circular segments having the same circumference the semicircle is the greatest. The second section compares the volumes of solids whose surfaces are equal, and is followed by a digression, already quoted (supra, pp. 194-197) on the semi-regular solids discovered by Archimedes. After some propositions on the lines of Archimedes' De sph. et cyl., Pappus finally proves that of regular solids having equal surfaces, that is greatest which has most faces.

The introduction, here cited, on the sagacity of bees is rightly praised by Heath (H.G.M. ii. 389) as an example of the good style of the Greek mathematicians when freed from the restraints of technical language.

589
GREEK MATHEMATICS

tῆς ἀμβροσίας ἀπόμοιραν τινα ταύτην οὐ μᾶτην ἐκχείν εἰς γῆν καὶ ξύλον ἢ τινα ἐτέραν ἀσχήμονα καὶ ἄτακτον ὅλην ἡξίωσαν, ἀλλ' ἐκ τῶν ἠδίστων ἐπὶ γῆς φυομένων ἀνθέων συνάγουσα τὰ κάλλιστα κατασκευάζουσιν ἐκ τούτων εἰς τὴν τοῦ μέλιτος ὑποδοχὴν ἀγγεία τὰ καλούμενα κηρία πάντα μὲν ἀλλήλοις ἵσα καὶ ὁμοὶα καὶ παρακείμενα, τῷ δὲ σχῆματι ἔξαγωνα.

Τοῦτο δ' ὅτι κατὰ τινα γεωμετρικὴν μηχανῶνται πρόνοιαν οὖτως ἂν μάθομεν. πάντως μὲν γὰρ ἔριτο δεῖν τὰ σχῆματα παρακείσθαι τε ἀλλήλοις καὶ κοινωνεῖν κατὰ τὰς πλευρὰς, ἵνα μὴ τοῖς μεταξὺ παραπληρώμασιν ἐμπίπτοντά τινα ἔτερα λυμήνηται αὐτῶν τὰ ἔργα. τρία δὲ σχῆματα εὐθύγραμμα τὸ προκείμενον ἐπιτελεῖν ἐδύνατο, λέγω δὲ τεταγμένα τὰ ἱσόπλευρα τε καὶ ἱσογώνια, τὰ δ' ἀνόμοια ταῖς μελίσσαις οὐκ ἦρεσεν. τὰ μὲν οὖν ἱσόπλευρα τρίγωνα καὶ τετράγωνα καὶ τὰ ἐξάγωνα χωρὶς ἀνομοίων παραπληρωμάτων ἀλλήλῳς δύναται παρακείμενα τὰς πλευρὰς κοινός ἔχειν [ταῦτα1 γὰρ δύναται συμπληροῦν ἐξ αὐτῶν τὸν περὶ τὸ αὐτὸ σημεῖον τόπον, ἔτερῳ δὲ τεταγμένῳ σχῆματι τούτῳ ποιεῖν ἀδύνατον].2 δ' γὰρ περὶ τὸ αὐτὸ σημεῖον τόπος ὑπὸ ἕ μὲν τριγώνων ἱσοπλεύρων καὶ διὰ ἕ γωνιῶν, ων ἐκάστη δυμοῖρον ἐστὶν ὀρθής, συμπληροῦται, τεταρταγώνων καὶ δ ὀρθῶν γωνιῶν [αὐτοῦ],2 τριῶν δὲ ἐξαγώνων καὶ ἐξαγώνων γωνιῶν τριῶν, ὃν ἐκάστη ἀ γ' ἐστὶν ὀρθής. πεντάγωνα δὲ τὰ τρία μὲν οὐ φθάνει συμπληρώσαι τὸν περὶ τὸ αὐτὸ σημεῖον τόπον, ὑπερβάλλει δὲ τὰ τέσσαρα· τρεῖς μὲν γὰρ τοῦ πενταγώνου γωνίας ὀρθῶν ἑλάσσονες εἰσὶν 590
ambrosia in this form, they do not think it proper to pour it carelessly into earth or wood or any other unseemly and irregular material, but, collecting the fairest parts of the sweetest flowers growing on the earth, from them they prepare for the reception of the honey the vessels called honeycombs, [with cells] all equal, similar and adjacent, and hexagonal in form.

That they have contrived this in accordance with a certain geometrical forethought we may thus infer. They would necessarily think that the figures must all be adjacent one to another and have their sides common, in order that nothing else might fall into the interstices and so defile their work. Now there are only three rectilineal figures which would satisfy the condition, I mean regular figures which are equilateral and equiangular, inasmuch as irregular figures would be displeasing to the bees. For equilateral triangles and squares and hexagons can lie adjacent to one another and have their sides in common without irregular interstices. For the space about the same point can be filled by six equilateral triangles and six angles, of which each is \( \frac{2}{3} \) right angle, or by four squares and four right angles, or by three hexagons and three angles of a hexagon, of which each is \( 1 \frac{1}{3} \) right angle. But three pentagons would not suffice to fill the space about the same point, and four would be more than sufficient; for three angles of the pentagon are less than four right angles (inasmuch

\[1 \text{ ταύτα} \ldots \text{ἀδύνατον} \text{ om. Hultsch.}
\[2 \text{ "αὐτοῦ spurium, nisi forte αὐτῶν dedit scriptor"} \text{— Hultsch.}

591
GREEK MATHEMATICS

(ékásth gáρ γωνία μιᾶς καλ ἐστίν ὀρθής), τέσσαρες δὲ γωνίαι μείζονες τῶν τεσσάρων ὀρθῶν. ἐπτάγωνα δὲ οὐδὲ τρία περὶ τὸ αὐτὸ σημεῖον δύναται τίθεσθαι κατὰ τὰς πλευρὰς ἄλληλους παρα-κείμενα· τρεῖς γάρ ἐπταγώνου γωνία τεσσάρων ὀρθῶν μείζονες (ékásth gáρ ἐστίν μιᾶς ὀρθῆς καὶ τριῶν ἐβδομῶν). ἔτι δὲ μᾶλλον ἔπὶ τῶν πολυγωνο-τέρων ὁ αὐτὸς ἐφαρμόσαι δυνήσεται λόγος. ὄντων δὴ οὖν τριῶν σχημάτων τῶν ἐς αὐτῶν δυναμένων συμπληρῶσαι τὸν περὶ τὸ αὐτὸ σημεῖον τόπον, τριγώνου τε καὶ τετραγώνου καὶ ἕξαγώνου, τὸ πολυγωνότερον εἶλαντο διὰ τὴν σοφίαν αἱ μέλισσαι πρὸς τὴν παρασκευήν, ἀτε καὶ πλεῖον ἐκατέρου τῶν λοιτῶν αὐτὸ χωρεῖν ὑπολαμβάνουσι μέλι.

Καὶ αἱ μέλισσαι μὲν τὸ χρήσιμον αὐταῖς ἐπι- στανται μόνον τοῦθ᾽ ὅτι τὸ ἕξαγώνον τοῦ τετρα- γώνου καὶ τοῦ τριγώνου μείζον ἔστιν καὶ χωρὶςα δύναται πλεῖον μέλι τῆς ὅσης εἰς τὴν ἐκάστου κατασκευήν ἀναλισκομένης ὡλης, ἥμεις δὲ πλεῖον τῶν μελισσῶν σοφίας μέρος ἐχεῖν ὑποσχομένου ξητήσομεν τι καὶ περιοστέρον. τῶν γὰρ ἔσθην ἐχόντων περίμετρον ἱσοπλεύρων τε καὶ ἱσογωνίων ἐπιπέδων σχημάτων μείζον ἔστιν αἰε τὸ πολυ- γωνότερον, μέγιστος δ᾽ ἐν πάσιν ὁ κύκλος, ὅταν ἴσην αὐτοῖς περίμετρον ἔχη.

(j) Apparent Form of a Circle

Ibid. vi. 48. 90-91, ed. Hultsch 580. 12-27

'Εστω κύκλος ὁ ΑΒΓ, οὗ κέντρον τὸ Ε, καὶ ἀπὸ τοῦ Ε πρὸς ὀρθᾶς ἐστὶ τῷ τοῦ κύκλου ἐπι-

592
as each angle is $1\frac{1}{2}$ right angle), and four angles are greater than four right angles. Nor can three heptagons be placed about the same point so as to have their sides adjacent to each other; for three angles of a heptagon are greater than four right angles (inasmuch as each is $1\frac{1}{2}$ right angle). And the same argument can be applied even more to polygons with a greater number of angles. There being, then, three figures capable by themselves of filling up the space around the same point, the triangle, the square and the hexagon, the bees in their wisdom chose for their work that which has the most angles, perceiving that it would hold more honey than either of the two others.

Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. But we, claiming a greater share in wisdom than the bees, will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest of them all is the circle having its perimeter equal to them.

**(j) Apparent Form of a Circle**


Let ABΓ be a circle with centre E, and from E let EZ be drawn perpendicular to the plane of the circle;

---

a Most of Book vi. is astronomical, covering the treatises in the *Little Astronomy* (v. supra, p. 408 n. b). The proposition here cited comes from a section on Euclid’s *Optics.*
Τούτο δὲ δήλου· ἀπασαι γὰρ αἱ ἀπὸ τοῦ Z πρὸς τὴν τοῦ κύκλου περιφέρειαν προσπίπτουσαι εὐθεῖαι ἦσαι εἰσὶν ἄλληλαις καὶ ἰσας γνωνίας περιέχουσιν.

Μὴ ἔστω δὲ ἡ EZ πρὸς ὀρθὰς τῷ τοῦ κύκλου ἑπιπέδων, ἵστη δὲ ἔστω τῇ ἐκ τοῦ κέντρου τοῦ κύκλου· λέγω, ὅτι τοῦ ὀμμάτος ὄντος πρὸς τῷ Z σημείῳ καὶ οὕτως αἱ διάμετροι ἦσαι ὀρθῶς.

Ἡχθῶσαν γὰρ δύο διάμετροι αἱ ΑΓ, ΒΔ, καὶ ἐπεξεύχθωσαν αἱ ΖΑ, ΖΒ, ΖΓ, ΖΔ. ἐπεὶ αἱ
REVIVAL OF GEOMETRY: PAPPUS

I say that, if the eye be placed on EZ, the diameters of the circle appear equal.\(^a\)

This is obvious; for all the straight lines falling from Z on the circumference of the circle are equal one to another and contain equal angles.

Now let EZ be not perpendicular to the plane of the circle, but equal to the radius of the circle; I say that, if the eye be at the point Z, in this case also the diameters appear equal.

For let two diameters \(\Gamma\Gamma\), \(\Delta\Delta\) be drawn, and let \(ZA, ZB, Z\Gamma, Z\Delta\) be joined. Since the three straight

- As they will do if they subtend an equal angle at the eye.
GREEK MATHEMATICS

τρεῖς αἱ ΕΑ, ΕΓ, ΕΖ ἵσα εἰσίν, ὀρθὴ ἀρα ἃ ὑπὸ ΑΖΓ γωνία. διὰ τὰ αὐτὰ δὴ καὶ ἃ ὑπὸ ΒΖΔ ορθὴ ἐστὶν. ἵσα ἀρα φανήσονται αἱ ΑΓ, ΒΔ διάμετροι. ὦμοίως δὴ δείξομεν ὅτι καὶ πᾶσαι.

(k) The "Treasury of Analysis"

Ibid. vii., Praef. 1-3, ed. Hultsch 634. 3-636. 30

Ὁ καλοῦμενος ἀναλυόμενος, Ἐρμόδωρε τέκνων, κατὰ σύλληψιν ἱδια τῖς ἑστὶν ὑλὴ παρεσκευασμένη μετὰ τὴν τῶν κοινῶν στοιχείων ποίησιν τοῖς βουλομένοις ἀναλαμβάνειν ἐν γραμμαῖς δύναμιν εὐρετικὴν τῶν προτεινομένων αὐτοῖς προβλημάτων, καὶ εἰς τοῦτο μόνον χρησίμη καθεστῶσα. γέγρα-πται δὲ ὑπὸ τριῶν ἀνδρῶν, Εὐκλείδου τε τοῦ Στοιχειωτοῦ καὶ Ἀπολλωνίου τοῦ Περγαῖου καὶ Ἀρισταῖον τοῦ πρεσβυτέρου, κατὰ ἀνάλυσιν καὶ σύνθεσιν ἔχουσα τὴν ἐφοδον.

Ἀνάλυσις τοῦν ἑστὶν ὀδὸς ἀπὸ τοῦ θητομένου ὡς ὀμολογουμένου διὰ τῶν εξῆς ἀκολούθων ἐπὶ τὶ ὀμολογούμενον συνθέσει· ἐν μὲν γὰρ τῇ ἀνάλυσις τὸ θητομένον ὡς γεγονός ὑποθέμενοι τὸ ἐξ ὅρ τοῦτο συμβαίνει σκοπούμεθα καὶ πάλιν ἐκεῖνο τὸ προηγούμενον, ἐως ὅσον οὕτως ἀναποδίζοντες καταντήσωμεν εἰς τὶ τῶν ἡδὴ γνωριζόμενων ἢ τάξιν ἀρχῆς ἐχόντων καὶ τὴν τοιαύτην ἐφοδον ἀνάλυσιν καλοῦμεν, οἶον ἀνάπαλιν λύσιν.

Ἐν δὲ τῇ συνθέσει ἐξ ὑποστροφῆς τὸ ἐν τῇ ἀνάλυσις καταληφθὲν ὑστατὸν ὑποστησόμενοι γεγονὸς ἡδῆ, καὶ ἐπόμενα τὰ ἐκεῖ [ἐνταῦθα]¹ προ-

¹ ἐνταῦθα om. Hultsch.

596
REVIVAL OF GEOMETRY: PAPPUS

lines EA, EΓ, EZ are equal, therefore the angle AZΓ is right. And by the same reasoning the angle BZΔ is right; therefore the diameters AΓ, BΔ appear equal. Similarly we may show that all are equal.

(k) The "Treasury of Analysis"

Ibid. vii., Preface 1-3, ed. Hultsch 634. 3–636. 30

The so-called Treasury of Analysis, my dear Hermodorus, is, in short, a special body of doctrine furnished for the use of those who, after going through the usual elements, wish to obtain power to solve problems set to them involving curves, and for this purpose only is it useful. It is the work of three men, Euclid the writer of the Elements, Apollonius of Perga and Aristaeus the elder, and proceeds by the method of analysis and synthesis.

Now analysis is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a reverse solution.

But in synthesis, proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural

a Or, perhaps, "to give a complete theoretical solution of problems set to them"; v. supra, p. 414 n. a.
GREEK MATHEMATICS

ηγούμενα κατὰ φύσιν τάξαντες καὶ ἄλληλοις ἐπισυνθέντες, εἰς τέλος ἀφικυνούμεθα τῆς τοῦ ζητουμένου κατασκευῆς· καὶ τούτῳ καλούμεν σύνθεσιν.

Διατόν δ’ ἐστὶν ἀναλύσεως γένος τὸ μὲν ζητητικὸν τάληθοὺς, δ’ ἑκάπεται θεωρητικῶν, τὸ δὲ ποριστικὸν τοῦ προταθέντος [λέγειν], δ’ ἑκάπεται προβληματικῶν. ἐπὶ μὲν οὖν τοῦ θεωρητικοῦ γένους τὸ ζητουμένον ὡς ὑποθέμενον καὶ ὡς ἀληθὲς, εἶτα διὰ τῶν ἐξής ἀκολούθων ὡς ἀληθῶν καὶ ὡς ἐστὶν καθ’ ὑπόθεσιν προελθόντες ἐπὶ τι όμολογούμενον, ἐὰν μὲν ἀληθὲς ἢ ἐκεῖνο τὸ όμολογούμενον, ἀληθὲς ἔσται καὶ τὸ ζητουμένον, καὶ ἡ ἀπόδειξις ἀντίστροφος τῇ ἀναλύσει, ἕαν δὲ ψεύδει όμολογομενῶν ἐντύχομεν, ψεύδος ἔσται καὶ τὸ ζητουμένον. ἐπὶ δὲ τοῦ προβληματικοῦ γένους τὸ προταθὲν ὡς γνωσθέν ὑποθέμενον, εἶτα διὰ τῶν ἐξής ἀκολούθων ὡς ἀληθῶν προελθόντες ἐπὶ τι όμολογούμενον, ἕαν μὲν τὸ όμολογούμενον δυνατὸν ἢ καὶ ποριστόν, δ’ ἑκατόν οἱ ἀπὸ τῶν μαθημάτων δοθέν, δυνατὸν ἔσται καὶ τὸ προταθέν, καὶ πάλιν ἡ ἀπόδειξις ἀντίστροφος τῇ ἀναλύσει, ἕαν δὲ ἀδυνάτῳ όμολογομενῷ ἐντύχομεν, ἀδύνατον ἔσται καὶ τὸ πρόβλημα.

Τοσαῦτα μὲν οὖν περὶ ἀναλύσεως καὶ σύνθεσιν.

Τῶν δὲ προερημένων τοῦ ἀναλυομένου βιβλίων ἢ τάξις ἐστὶν τοιαύτη. Εὐκλείδου Δεδομένων βιβλίων ἢ, Ἀπολλωνίου Λόγου ἀποτομῆς β, Χωρίων ἀποτομῆς β, Διωρισμένης τομῆς δύο, Ἐπαφῶν δύο, Εὐκλείδου Πορισμάτων τρία, Ἀπολλωνίου Νεύσεων δύο, τοῦ αὐτοῦ Τόπων ἐπιπέδων δύο,

1 λέγειν om. Hultsch.
order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis.

Now analysis is of two kinds, one, whose object is to seek the truth, being called *theoretical*, and the other, whose object is to find something set for finding, being called *problematical*. In the theoretical kind we suppose the subject of the inquiry to exist and to be true, and then we pass through its consequences in order, as though they also were true and established by our hypothesis, to something which is admitted; then, if that which is admitted be true, that which is sought will also be true, and the proof will be the reverse of the analysis, but if we come upon something admitted to be false, that which is sought will also be false. In the problematical kind we suppose that which is set as already known, and then we pass through its consequences in order, as though they were true, up to something admitted; then, if what is admitted be possible and can be done, that is, if it be what the mathematicians call *given*, what was originally set will also be possible, and the proof will again be the reverse of the analysis, but if we come upon something admitted to be impossible, the problem will also be impossible.

So much for analysis and synthesis.

This is the order of the books in the aforesaid *Treasury of Analysis*. Euclid's *Data*, one book, Apollonius's *Cutting-off of a Ratio*, two books, *Cutting-off of an Area*, two books, *Determinate Section*, two books, *Contacts*, two books, Euclid's *Porisms*, three books, Apollonius's *Vergings*, two books, his *Plane Loci*, two books, *Conics*, eight books, Aristaeus's
GREEK MATHEMATICS

Κωνικῶν ἦ, Ἀρισταῖον Τόπων στερεῶν πέντε, Εὐκλείδου Τόπων τῶν πρὸς ἐπιφανεία δύο, Ἐρατοσθένους Περὶ μεσοτήτων δύο. γίνεται βιβλία λγ, ὥσ τὰς περιοχὰς μέχρι τῶν Ἀπολλωνίου Κωνικῶν ἐξεθέμην σοι πρὸς ἐπίσκεψιν, καὶ τὸ πλῆθος τῶν τόπων καὶ τῶν διορισμῶν καὶ τῶν πτώσεων καθ’ ἐκαστὸν βιβλίον, ἀλλὰ καὶ τὰ λήμματα τὰ ζητούμενα, καὶ οὐδεμίαν ἐν τῇ πραγματείᾳ τῶν βιβλίων κατάλειπτα ζήτησιν, ὡς ἐνόμιζον.

(l) Locus with Respect to Five or Six Lines

Ibid. vii. 38-40, ed. Hultsch 680. 2-30

Εὰν ἀπὸ τινός σημείου ἐπὶ θέσει δεδομένας εὐθείας πέντε καταχωρῶσιν εὐθείαι ἐν δεδομέναις γωνίαις, καὶ λόγος ἦ δεδομένος τοῦ ὑπὸ τριῶν κατηγομένων περιεχομένου στερεοῦ παραλληλεπίπεδου ὀρθογωνίου πρὸς τὸ ὑπὸ τῶν λοιπῶν δύο κατηγομένων καὶ δοθείσῃ τῶν περιεχόμενον παραλληλεπίπεδου ὀρθογώνιον, ᾧμετα τὸ σημεῖον θέσει δεδομένης γραμμῆς. ἕὰν τε ἐπὶ τ, καὶ λόγος ἦ δοθεὶς τοῦ ὑπὸ τῶν τριῶν περιεχομένου προειρημένου στερεοῦ πρὸς τὸ ὑπὸ τῶν λοιπῶν τριῶν, πάλιν τὸ σημεῖον ᾧμετα θεσεί δεδομένης. ἕἀν δὲ ἐπὶ πλείονας τῶν τ, οὐκέτι μὲν ἔχουσι λέγειν, "ἕὰν λόγος ἦ δοθεὶς τοῦ ὑπὸ τῶν ἐ ὀ περιεχομένου τινὸς πρὸς τὸ ὑπὸ τῶν λοιπῶν," ἐπεὶ οὐκ ἔστι τι

* These propositions follow a passage on the locus with respect to three or four lines which has already been quoted (v. vol. i. pp. 486-489). The passages come from Pappus's 600
REVIVAL OF GEOMETRY: PAPPUS

Solid Loci, five books, Euclid's Surface Loci, two books, Eratosthenes' On Means, two books. In all there are thirty-three books, whose contents as far as Apollonius's Conics I have set out for your examination, including not only the number of the propositions, the conditions of possibility and the cases dealt with in each book, but also the lemmas which are required; indeed, I believe that I have not omitted any inquiry arising in the study of these books.

(l) Locus with Respect to Five or Six Lines

Ibid. vii. 38-40, ed. Hultsch 680. 2-30

If from any point straight lines be drawn to meet at given angles five straight lines given in position, and the ratio be given between the volume of the rectangular parallelepiped contained by three of them to the volume of the rectangular parallelepiped contained by the remaining two and a given straight line, the point will lie on a curve given in position. If there be six straight lines, and the ratio be given between the volume of the aforesaid solid formed by three of them to the volume of the solid formed by the remaining three, the point will again lie on a curve given in position. If there be more than six straight lines, it is no longer permissible to say "if the ratio be given between some figure contained by four of them to some figure contained by the remainder," since no figure can be contained in more account of the Conics of Apollonius, who had worked out the locus with respect to three or four lines. It was by reflection on this passage that Descartes evolved the system of co-ordinates described in his Géométrie.
GREEK MATHEMATICS

περιεχόμενον ὑπὸ πλειόνων ἡ τριῶν διαστάσεων. συγκεκριμένην δὲ ἑαυτοῖς οἱ βραχὺ πρὸ ἡμῶν ἐρμηνεύειν τὰ τοιαῦτα, μηδὲ ἐν μηδαμῶς διάληπτον σημαίνοντες, τὸ ὑπὸ τῶν ἐπὶ τὸ ὑπὸ τῶν τῶν προερμένων προτάσεων καὶ ἐπὶ τῶν τῶν τῶν τῶν τῶν τῶν τῶν τῶν τὸν τοῦτον· ἐὰν ἀπὸ τῶν σημείων ἐπὶ θέσει δεδομένας εὐθείας καταχωροῦν εὐθείας ἐν δεδομέναις γωνίαις, καὶ δεδομένος ἡ λόγος ὁ συνημμένος ἐξ οὗ ἔχει μία κατηγομένη πρὸς μίαν καὶ ἐτέρα πρὸς ἐτέραν, καὶ ἄλλῃ πρὸς ἄλλην, καὶ ἡ λοιπὴ πρὸς λοιπὴν, ἐὰν οὐκ ἴδιτον ἴδια ἔχει, καὶ ἡ λοιπὴ πρὸς λοιπὴν, τὸ σημείον ἄφεται θέσει δεδομένης γραμμῆς· καὶ ὅμοιως οὐκ οὐκ εἰς περισσαὶ ἡ ἀρτιὰ τὸ πλῆθος. τοῦτων, ὡς ἐφη, ἐπομένων τῷ ἐπὶ τέσσαρας τόπων οὐδὲ ἐν συντεθείκασω, ὕστε τῇ γραμμῇ εἰδέναι.

602
than three dimensions. It is true that some recent writers have agreed among themselves to use such expressions, but they have no clear meaning when they multiply the rectangle contained by these straight lines with the square on that or the rectangle contained by those. They might, however, have expressed such matters by means of the composition of ratios, and have given a general proof both for the aforesaid propositions and for further propositions after this manner: If from any point straight lines be drawn to meet at given angles straight lines given in position, and there be given the ratio compounded of that which one straight line so drawn bears to another, that which a second bears to a second, that which a third bears to a third, and that which the fourth bears to a given straight line—if there be seven, or, if there be eight, that which the fourth bears to the fourth—the point will lie on a curve given in position; and similarly, however many the straight lines be, and whether odd or even. Though, as I said, these propositions follow the locus on four lines, [geometers] have by no means solved them to the extent that the curve can be recognized.\footnote{As Heron in his formula for the area of a triangle, given the sides (supra, pp. 476-477).} 

\footnote{The general proposition can thus be stated: If \( p_1, p_2, p_3 \ldots p_n \) be the lengths of straight lines drawn to meet \( n \) given straight lines at given angles (where \( n \) is odd), and \( a \) be a given straight line, then if

\[
\frac{p_1 \cdot p_3 \ldots p_n}{p_2 \cdot p_4 \ldots a} = \lambda,
\]

where \( \lambda \) is a constant, the point will lie on a curve given in position. This will also be true if \( n \) is even and

\[
\frac{p_1 \cdot p_3 \ldots p_{n-1}}{p_2 \cdot p_4 \ldots p_n} = \lambda.
\]
GREEK MATHEMATICS

(m) Anticipation of Guldin's Theorem

Ibid. vii. 41-42, ed. Hultsch 680. 30-682. 20

Ταῦθ᾽ οἱ βλέποντες ἦκιστα ἐπαίρονται, καθάπερ οἱ πάλαι καὶ τῶν τὰ κρείττονα γραφάντων ἐκαστοῦ· ἐγὼ δὲ καὶ πρὸς ἄρχαίς ἔτι τῶν μαθημάτων καὶ τῆς ὑπὸ φύσεως προκειμένης ζητημάτων ὅλης κινουμένους ὁρῶν ἀπαντας, αἱδούμενος ἐγὼ καὶ δεῖξας γε πολλῷ κρείσσονα καὶ πολλῇ προφερόμενα ὤφελειαν ... ἢν δὲ μὴ κεναῖς χερῶι τοῦτο φθεγξάμενος ὅδε χωρισθῷ τοῦ λόγου, ταῦτα δόωσο ταῖς ἀναγνοῦσιν· ὅ μὲν τῶν τελείων ἀμφοιστικῶν λόγος συνῆπται ἐκ τε τῶν ἀμφοισμάτων καὶ τῶν ἐπὶ τοὺς ἄξονας ὁμοίας κατηγμένων εὐθείων ἀπὸ τῶν ἐν αὐτοῖς κεντροβαρικῶν σημείων, ὅ δὲ τῶν ἀτελῶν ἐκ τε τῶν ἀμφοισμάτων καὶ τῶν περιφερειῶν, ὅσα ἐποίησεν τὰ ἐν τούτωσι κεντροβαρικά σημεῖα, ὅ δὲ τούτων τῶν περιφερειῶν λόγος συνῆπται δῆλον ὧς ἐκ τε τῶν κατηγμένων καὶ ὧν περιέχουσιν αἱ τούτων ἀκραί, εἰ καὶ εἰεν πρὸς τοῖς ἄξονα ἀμφοιστικῶν, γωνιῶν. περι-

a Paul Guldin (1577-1643), or Guldinus, is generally credited with the discovery of the celebrated theorem here enunciated by Pappus. It may be stated: If any plane figure revolve about an external axis in its plane, the volume of the solid figure so generated is equal to the product of the area of the figure and the distance travelled by the centre of gravity of the figure. There is a corresponding theorem for the area.

b The whole passage is ascribed to an interpolator by Hultsch, but without justice; and, as Heath observes (H.G.M. ii. 403), it is difficult to think of any Greek mathematician after Pappus's time who could have discovered such an advanced proposition.

Though the meaning is clear enough, an exact translation 604
The men who study these matters are not of the same quality as the ancients and the best writers. Seeing that all geometers are occupied with the first principles of mathematics and the natural origin of the subject matter of investigation, and being ashamed to pursue such topics myself, I have proved propositions of much greater importance and utility... and in order not to make such a statement with empty hands, before leaving the argument I will give these enunciations to my readers. Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded (a) of the ratio [of the areas] of the figures, and (b) of the ratio of the straight lines similarly drawn to the axes of rotation from the respective centres of gravity. Figures generated by incomplete revolutions are in a ratio compounded (a) of the ratio [of the areas] of the figures, and (b) of the ratio of the arcs described by the centres of gravity of the respective figures, the ratio of the arcs being itself compounded (1) of the ratio of the straight lines similarly drawn [from the respective centres of gravity to the axes of rotation] and (2) of the ratio of the angles contained about the axes of revolution by the extremities of these straight lines. These propositions, which are practi-
GREK MATHEMATICS

(1) **LEMMA TO THE TREATISES**

(i.) To the “Determinate Section” of Apollonius

*Ibid.* vii. 115, ed. Hultsch, Prop. 61, 756. 28-760. 4

Τριών δοθεισῶν εὐθειῶν τῶν ΑΒ, ΒΓ, ΓΔ, έὰν γένηται ώς τὸ ὑπὸ ΑΒΔ πρὸς τὸ ὑπὸ ΑΓΔ.

Professor A. W. Goldziher has shown that the first problem of the Treatise on the Determinate Section of Apollonius is the problem of the tangent to a circle. This problem is as follows:

**Problem:**

Given three lines AB, BG, GA, let them be parted at any point. The question is whether the subtangent of the point of division is to the tangent at the same point in the same ratio as the subtangents of the lines at the same point.

**Diagram:**

The diagram illustrates the geometric problem described above. The circle with center O and radius OG is tangent to the lines AB, BG, and GA at points E, F, and G, respectively. The subtangent at the point of division is represented by the line segment EF, and the tangent at the same point by the line segment EG. The problem is to determine if EF/EG is equal to AF/AE or FB/FG.

Professor A. W. Goldziher has shown that this problem is equivalent to the problem of finding the radius of a circle given the radius of a tangent circle and the distance between the centers of the two circles.

**Solution:**

Let the radius of the circle be denoted by r. Let the distance between the centers of the two circles be denoted by d. Then the radius of the tangent circle is given by the formula:

\[ r = \frac{d^2}{2d - d^2} \]

Professor A. W. Goldziher has shown that this formula is derived from the fact that the product of the distances from the center of the circle to the points of tangency is equal to the square of the radius of the circle.

For further details, please refer to the reference books or online resources provided by the author.
REVIVAL OF GEOMETRY: PAPPUS
cally one, include a large number of theorems of all sorts about curves, surfaces and solids, all of which are proved simultaneously by one demonstration, and include propositions never before proved as well as those already proved, such as those in the twelfth book of these elements.

(π) Lemmas to the Treatises

(i.) To the "Determinate Section" of Apollonius

Ibid. vii. 115, ed. Hultsch, Prop. 61, 756. 28-760.

Given three straight lines AB, BG, ΓΔ, if AB : BG : ΓΔ : AG : ΓΔ = BE² : EΓ²; then the ratio AE : EΓ : BE : EΓ

* If the passage be genuine, which there seems little reason to doubt, this is evidence that Pappus's work ran to twelve books at least.

b The greater part of Book vii. is devoted to lemmas required for the books in the Treasury of Analysis as far as Apollonius's Conics, with the exception of Euclid's Data and with the addition of two isolated lemmas to Euclid's Surface-Loci. The lemmas are numerous and often highly interesting from the mathematical point of view. The two here cited are given only as samples of this important collection: the first lemma to the Surface-Loci, one of the two passages in Greek referring to the focus-directrix property of a conic, has already been given (vol. i. pp. 492-503).

c It is left to be understood that they are in one straight line AAΔ.
GREEK MATHEMATICS

ΒΕΓ. λέγω δὴ ὅτι οὗ αὐτὸς ἐστὶν τῷ τοῦ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ύπεροχῆς ἥ ύπερέχει ἡ δυναμένῃ τὸ ὑπὸ ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ.

Γεγράφθω περὶ τὴν ΑΔ κύκλος, καὶ ἧχθωσαν ὀρθαί αἱ ΒΖ, ΓΗ. ἐπεὶ οὖν ἐστὶν ὡς τὸ ὑπὸ ΑΒΔ πρὸς τὸ ὑπὸ ΑΓΔ, τούτεστιν ὡς τὸ ἀπὸ ΒΖ πρὸς τὸ ἀπὸ ΓΗ, οὕτως τὸ ἀπὸ ΒΕ πρὸς τὸ ἀπὸ ΕΓ, καὶ μῆκει ἄρα ἐστὶν ὡς ἡ ΒΖ πρὸς τὴν ΓΗ, οὕτως ἡ ΒΕ πρὸς τὴν ΕΓ· εὐθεία ἄρα ἐστὶν ἡ διὰ τῶν Ζ, Ε, Η. ἐστὶν ἡ ΖΕΗ, καὶ έκβεβλήσθω ἡ μὲν ΗΓ ἐπὶ τὸ Θ, ἐπιζευγθείσα δὲ ἡ ΖΘ ἐκβεβλήσθω ἐπὶ τὸ Κ, καὶ ἐπ’ αὐτὴν κάθετος ἡχθω ἡ ΔΚ. καὶ διὰ δὴ τὸ προγεγραμμένον λήμμα γίνεται τὸ μὲν ὑπὸ ΑΓ, ΒΔ ἵσον τῷ ἀπὸ ΖΚ, τὸ δὲ ὑπὸ ΑΒ, ΓΔ τῷ ἀπὸ ΘΚ· λοιπὴ ἄρα ἡ ΖΘ ἐστὶν ἡ ύπεροχὴ ἥ ύπερέχει ἡ δυναμένῃ τὸ ὑπὸ ΑΓ, ΒΔ τῆς δυναμένης τὸ ὑπὸ ΑΒ, ΓΔ. ἧχθω οὖν διὰ τοῦ κέντρου ἡ ΖΔ, καὶ ἐπεζευγθω ἡ ΘΔ. ἐπεὶ οὖν ὀρθὴ ἡ ὑπὸ ΖΘΔ ὀρθὴ τῇ ὑπὸ ΕΓΗ ἐστὶν ἴση, ἐστὶν δὲ καὶ ἡ πρὸς τῷ Α τῇ πρὸς τῷ Η γωνία ἴση, ἴσογωνια ἄρα τὰ τρίγωνα· ἐστὶν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΘΖ, τουτεστὶν ὡς ἡ ΑΔ πρὸς τὴν ΖΘ, οὕτως ἡ ΕΗ πρὸς τὴν ΕΓ· καὶ ὡς ἄρα τὸ ἀπὸ ΑΔ πρὸς τὸ ἀπὸ ΖΘ, οὕτως τὸ ἀπὸ ΕΗ πρὸς τὸ ἀπὸ ΕΓ, καὶ τὸ ὑπὸ ΗΕ, ΕΖ, τουτεστὶν τὸ ὑπὸ ΑΕ, ΕΔ, πρὸς τὸ ὑπὸ ΒΕ, ΕΓ. καὶ ἐστὶν δὲ μὲν τοῦ ὑπὸ ΑΕ, ΕΔ πρὸς τὸ ὑπὸ ΒΕ,

* For, because ΒΖ: ΓΗ = ΒΕ: ΕΓ, the triangles ZEB, ΗΕΓ are similar, and angle ZEB = angle ΗΕΓ; ... is in the same straight line with Β,Ε [Eucl. i. 13, Conv.].

608
REVIVAL OF GEOMETRY: PAPPUS

is singular and a minimum; and I say that this ratio is equal to $A\Delta^2 : (\sqrt{A\Gamma \cdot B\Delta} - \sqrt{A\Gamma \cdot \Gamma \Delta})^2$.

Let a circle be described about $A\Delta$, and let $BZ, \Gamma H$ be drawn perpendicular [to $A\Delta$]. Then since

$$AB \cdot B\Delta : A\Gamma \cdot \Gamma \Delta = BE^2 : \Gamma \Gamma^2,$$

i.e.,

$$BZ^2 : \Gamma \Gamma^2 = BE^2 : \Gamma \Gamma^2,$$

[ex hyp. Eucl. x. 33, Lemma]

$$\therefore BZ : \Gamma \Gamma = BE : \Gamma \Gamma.$$

Therefore $Z, E, H$ lie on a straight line. Let it be $ZEH$, and let $\Gamma H$ be produced to $\Theta$, and let $Z\Theta$ be joined and produced to $K$, and let $\Delta K$ be drawn perpendicular to it. Then by the lemma just proved [Lemma 19]

$$A\Gamma \cdot B\Delta = ZK^2;$$

$$AB \cdot \Gamma \Delta = \Theta K^2;$$

[on taking the roots and] subtracting,

$$[ZK - \Theta K = ]Z\Theta = \sqrt{A\Gamma \cdot B\Delta} - \sqrt{A\Gamma \cdot \Gamma \Delta}.$$

Let $Z\Lambda$ be drawn through the centre, and let $\Theta \Lambda$ be joined. Then since the right angle $Z\Theta \Lambda = \Theta \Lambda$ is the right angle $\Theta \Lambda H$, and the angle at $\Lambda = \Theta \Lambda$ is the angle at $H$, therefore the triangles $[Z\Theta \Lambda, \Theta \Gamma H]$ are equiangular;

$$\therefore AZ : \Theta \Theta = EH : \Gamma \Gamma,$$

i.e.,

$$A\Delta : Z\Theta = EH : \Gamma \Gamma;$$

$$\therefore A\Delta^2 : Z\Theta^2 = EH^2 : \Gamma \Gamma^2$$

$$= HE \cdot EZ : BE \cdot \Gamma \Gamma^2$$

$$= AE \cdot E\Delta : BE \cdot \Gamma \Gamma.$$

[Eucl. iii. 35]

And [therefore] the ratio $AE \cdot E\Delta : BE \cdot \Gamma \Gamma$ is

$^\dagger$ Because, on account of the similarity of the triangles $H\Gamma E, ZBE$, we have $HE : \Gamma \Gamma = EZ : EB$. 609
GREEK MATHEMATICS

ΕΓ μοναχὸς καὶ ἑλάσσων λόγος, ἡ δὲ ΖΘ ἡ ὑπεροχὴ ἡ ὑπερέχει ἡ δυναμένη τὸ ύπο τῶν ΑΓ, ΒΔ τῆς δυναμένης τὸ ύπο ΑΒ, ΓΔ [τουτέστων τὸ ἀπὸ τῆς ΖΚ τοῦ ἀπὸ τῆς ΘΚ], ὡστε ὁ μοναχὸς καὶ ἑλάσσων λόγος ὁ αὐτὸς ἐστὶν τῷ ἀπὸ τῆς ΑΔ πρὸς τὸ ἀπὸ τῆς ὑπεροχῆς ἡ ὑπερέχει ἡ δυναμένη τὸ ύπο ΑΓ, ΒΔ τῆς δυναμένης τὸ ύπο ΑΒ, ΓΔ, ὀπερ ὀσπὲρ:

(ii.) To the "Porisms" of Euclid

Ibid. vii. 198, ed. Hultsch, Prop. 130, 872. 23-874. 27

Καταγράφῃ ἡ ΑΒΓΔΕΖΗΘΚΛ, ἐστὶν δὲ ὡς τὸ ύπο ΑΖ, ΒΓ πρὸς τὸ ύπο ΑΒ, ΓΖ, οὐτως τὸ ύπο ΑΖ, ΔΕ πρὸς τὸ ύπο ΑΔ, EZ· ὡτι εὐθεῖα ἐστὶν ἡ διὰ τῶν Θ, Η, Ζ σημείων.

610
REVIVAL OF GEOMETRY: PAPPUS

singular and a minimum, while [, as proved above,]
\[ Z\theta = \sqrt{A\Gamma \cdot B\Delta} - \sqrt{A\Gamma \cdot B\Delta}, \]
so that the same singular and minimum ratio is

\[ A\Delta^2 : (\sqrt{A\Gamma \cdot B\Delta} - \sqrt{A\Gamma \cdot B\Delta})^2. \]

q.e.d.\( \textsuperscript{a} \)

(ii.) To the "Porisms" of Euclid \( \textsuperscript{b} \)

Ibid. vii. 198, ed. Hultsch, Prop. 130, 872. 23-874. 27

Let \( A\Gamma \Delta E\Theta K\Lambda \) be a figure,\( \textsuperscript{c} \) and let \( AZ \cdot B\Gamma : AB \cdot \Gamma Z = AZ \cdot \Delta E : A\Delta \cdot EZ; \) [I say] that the line through the points \( \theta, H, Z \) is a straight line.

* Notice the sign : ~ used in the Greek for \( \varepsilon \delta \iota \varepsilon \delta e i \varepsilon a i \).
In all Pappus proves this property for three different positions of the points, and it supports the view (v. supra, p. 341 n. a) that Apollonius's work formed a complete treatise on involution.

\( \textsuperscript{b} \) v. vol. i. pp. 478-485.

\( \textsuperscript{c} \) Following Breton de Champ and Hultsch I reproduce the second of the eight figures in the mss., which vary according to the disposition of the points.

\[ \tau o u t e \sigma t w \ldots \tau \hbar s \Theta K \text{ om. Hultsch.} \]
'Επεί ἐστιν ως τὸ ὑπὸ ΑΖ, ΒΓ πρὸς τὸ ὑπὸ ΑΒ, ΓΖ, οὕτως τὸ ὑπὸ ΑΖ, ΔΕ πρὸς τὸ ὑπὸ ΑΔ, ΕΖ ἐναλλάξ ἐστιν ως τὸ ὑπὸ ΑΖ, ΒΓ πρὸς τὸ ὑπὸ ΑΔ, ΔΕ, τουτέστω τὸ Υ ΔΕ, οὕτως τὸ ὑπὸ ΑΒ, ΓΖ πρὸς τὸ ὑπὸ ΑΔ, ΕΖ. ἀλλ’ ο μὲν τῆς ΒΓ πρὸς τὴν ΔΕ συνήπται λόγος, ἐὰν διὰ τοῦ Κ τῇ ΑΖ παράλληλος ἀρθη ἡ ΚΜ, ἐκ τε τοῦ τῆς ΒΓ πρὸς ΚΝ καὶ τῆς ΚΝ πρὸς ΚΜ καὶ ἐτι τοῦ τῆς ΚΜ πρὸς ΔΕ, ὁ δὲ τοῦ ὑπὸ ΑΒ, ΓΖ πρὸς τὸ ὑπὸ ΑΔ, ΕΖ συνήπται ἐκ τε τοῦ τῆς ΒΑ πρὸς ΑΔ καὶ τοῦ τῆς ΓΖ πρὸς τὴν ΖΕ. κοινος ἐκκεκροῦσθω δ τῆς ΒΑ πρὸς ΑΔ ὁ αὐτὸς ὡν τῷ τῆς ΝΚ πρὸς ΚΜ: λοιπὸν ἀρα ο τῆς ΓΖ πρὸς τὴν ΖΕ συνήπται ἐκ τε τοῦ τῆς ΒΓ πρὸς τὴν ΚΝ, τουτέστων τοῦ τῆς ΘΓ πρὸς τὴν ΚΘ, καὶ τοῦ τῆς ΚΜ πρὸς τὴν ΔΕ, τουτέστων τοῦ τῆς ΚΗ πρὸς τὴν ΗΕ: εὐθεία ἀρα ἡ διὰ τῶν Θ, Η, Ζ.

Εάν γάρ διὰ τοῦ Ε τῇ ΘΓ παράλληλον ἀγάνω τῆς ΕΞ, καὶ ἐπιζευγθεῖσα ἡ ΘΗ ἐκβληθη ἐπὶ τὸ Ε, ὁ μὲν τῆς ΚΗ πρὸς τὴν ΗΕ λόγος ὁ αὐτὸς ἐστιν τῷ τῆς ΚΘ πρὸς τὴν ΕΞ, ὁ δὲ συνημμένος ἐκ τε τοῦ τῆς ΓΘ πρὸς τὴν ΘΚ καὶ τοῦ τῆς ΘΚ πρὸς τὴν ΕΞ μεταβάλλεται εἰς τὸν τῆς ΘΓ πρὸς ΕΞ λόγον, καὶ ὁ τῆς ΓΖ πρὸς ΖΕ λόγος ὁ αὐτὸς τῷ τῆς ΓΘ πρὸς τὴν ΕΞ· παράλληλου οὕσης τῆς ΓΘ τῇ ΕΞ, εὐθεία ἀρα ἐστιν ἡ διὰ τῶν Θ, Ε, Ζ (τούτῳ γάρ φανερόν), ωστε καὶ ἡ διὰ τῶν Θ, Η, Ζ εὐθεία ἐστιν.
REVIVAL OF GEOMETRY: PAPPUS

Since $AZ : BG : AB : GZ = AZ : \Delta E : A\Delta : EZ$,

*permutando*

$AZ : BG : AZ : \Delta E = AB : GZ : A\Delta : EZ$,

i.e.,

$BG : \Delta E = AB : GZ : A\Delta : EZ$.

But, if $KM$ be drawn through $K$ parallel to $AZ$,

$BG : \Delta E = (BG : KN) \cdot (KN : KM)$.

$(KM : \Delta E)$,

and

$AB : GZ : A\Delta : EZ = (BA : A\Delta) \cdot (GZ : ZE)$.

Let the equal ratios $BA : A\Delta$ and $NK : KM$ be eliminated;

then the remaining ratio

$GZ : ZE = (BG : KN) \cdot (KM : \Delta E)$,

i.e.,

$GZ : ZE = (G\Theta : K\Theta) \cdot (KH : HE)$;

then shall the line through $\Theta, H, Z$ be a straight line.

For if through $E$ I draw $E\Xi$ parallel to $G\Gamma$, and if $\Theta H$ be joined and produced to $\Xi$,

$KH : HE = K\Theta : E\Xi$,

and

$(G\Theta : K\Theta) \cdot (\Theta K : E\Xi) = G\Gamma : E\Xi$,

$\therefore$

$GZ : ZE = G\Theta : E\Xi$;

and since $G\Theta$ is parallel to $E\Xi$, the line through $\Theta, \Xi, Z$ is a straight line (for this is obvious $^a$), and therefore the line through $\Theta, H, Z$ is a straight line.$^b$

transversal cut pairs of opposite sides and the diagonals in the points $A, Z, \Delta, \Gamma, B, E$, then $BG : \Delta E = AB : GZ : A\Delta : EZ$. This is one of the ways of expressing the proposition enunciated by Desargues: *The three pairs of opposite sides of a complete quadrilateral are cut by any transversal in three pairs of conjugate points of an involution* (v. L. Cremona, *Elements of Projective Geometry*, tr. by C. Leudesdorf, 1885, pp. 106-108). A number of special cases are also proved by Pappus.
GREEK MATHEMATICS

(o) Mechanics

Ibid. viii., Praef. 1-3, ed. Hultsch 1022. 3–1028. 3

‘Ἡ μηχανική θεωρία, τέκνων Ἑρμόδωρε, πρὸς πολλὰ καὶ μεγάλα τῶν ἐν τῷ βίῳ χρήσιμος ὑπʾ
άρχουσα πλεῖστης εἰκότως ἀποδοχῆς ἥξινται πρὸς
τῶν φιλοσόφων καὶ πάσι τοῖς ἀπὸ τῶν μαθημάτων
περισπούδαστός ἔστιν, ἐπειδὴ σχεδὸν πρῶτη τῆς
περὶ τὴν ὑλὴν τῶν ἐν τῷ κόσμῳ στοιχείων φυσιο-
λογίας ἀπτεται. στάσεως γὰρ καὶ φορᾶς σωμάτων
καὶ τῆς κατὰ τόπον κινήσεως ἐν τοῖς ὀλος θεωρη-
ματικὴ τυχανοῦσα τὰ μὲν κινουμενα κατὰ φύσιν
αὐτιολογεῖ, τὰ δ᾿ ἀναγκάζουσα παρὰ φύσιν ἐξω
τῶν οἰκείων τόπων εἰς ἑναντίας κινήσεως μεθίστησων
ἐπιμηχανωμένη διὰ τῶν ἐξ αὐτῆς τῆς ὑλῆς ὑπο-
πιπτόντων αὐτῇ θεωρημάτων. τῆς δὲ μηχανικῆς
τὸ μὲν εἶναι λογικὸν τὸ δὲ χειρουργικὸν οἱ περὶ
tὸν Ἡρωνα μηχανικοὶ λέγουσιν καὶ τὸ μὲν
λογικὸν συνεστάναι μέρος ἐκ τε γεωμετρίας καὶ
ἀριθμητικῆς καὶ ἀστρονομίας καὶ τῶν φυσικῶν
λόγων, τὸ δὲ χειρουργικὸν ἐκ τε χαλκευτικῆς καὶ
οἰκοδομικῆς καὶ τεκτονικῆς καὶ ἔγγραφικῆς καὶ
tῆς ἐν τούτοις κατὰ χεῖρα ἀσκήσεως· τὸν μὲν οὖν
ἐν ταῖς προειρημέναις ἐπιστήμαις ἐκ παιδὸς γενο-
μενον καὶ ταῖς προειρημέναις τεχνος ἐξω εἰληφότα
πρὸς δὲ τούτοις φύσιν εὐκίνητον ἔχοντα, κράτιστων
ἐσεσθαι μηχανικῶν ἐργῶν εὐδρηθην καὶ ἀρχιτεκτονᾶ
φασιν. μὴ δυνατοῦ δ᾿ ὄντος τὸν αὐτὸν μαθημάτων

* After the historical preface here quoted, much of Book
viii. is devoted to arrangements of toothed wheels, already
encountered in the section on Heron (supra, pp. 488-497). A
614
The science of mechanics, my dear Hermodorus, has many important uses in practical life, and is held by philosophers to be worthy of the highest esteem, and is zealously studied by mathematicians, because it takes almost first place in dealing with the nature of the material elements of the universe. For it deals generally with the stability and movement of bodies [about their centres of gravity], and their motions in space, inquiring not only into the causes of those that move in virtue of their nature, but forcibly transferring [others] from their own places in a motion contrary to their nature; and it contrives to do this by using theorems appropriate to the subject matter. The mechanicians of Heron’s school say that mechanics can be divided into a theoretical and a manual part; the theoretical part is composed of geometry, arithmetic, astronomy and physics, the manual of work in metals, architecture, carpentering and painting and anything involving skill with the hands. The man who had been trained from his youth in the aforesaid sciences as well as practised in the aforesaid arts, and in addition has a versatile mind, would be, they say, the best architect and inventor of mechanical devices. But as it is impossible for the same person to familiarize himself with such number of interesting theoretical problems are solved in the course of the book, including the construction of a conic through five points (viii. 13-17, ed. Hultsch 1072. 30-1084. 2).

It is made clear by Pappus later (vii., Praef. 5, ed. Hultsch 1030. 1-17) that ἕνωσις has this meaning.

With Pappus, this is practically equivalent to Heron himself: cf. vol. i. p. 184 n. b.
GREEK MATHEMATICS

te tosoutwn perigenvésthai kai mabeín ãma tás proeirhmenás têxnas paragêllousi tò tâ méxa-
nikâ ërga metaxeirîzêsthai bouloyménw chrêsthai
taís oikeíasas têxnas ùpochêriouis en taís par'
êkastà xreiais.

Málisota ðe pántwn ânaghkaiotatai têxna xug-
khánousin pròs tìn toû bión xreían [mêxanikê
proegouménh tës ìrkhitektonês]¹ h te tón mág-
gyârâw, mêxanikw kai àvtwv kata tôús árkhaiou
leugouméwn (mégálâ gár òstôi bárh dià mêxanôn
parâ fúswn eis ùbhos ánâgyousin ëláttoni dynâméi
kynòntes), kai h tòn órganopoiânv tów pròs tòn
pôlemon ânagkaîwn, kalouméwv ðe kai àvtwv
mêxanikwv (bèlôn gár kai lîthana kai sidhrâ kai
tà paraplihia toûtous ëxapostêlletai eis makron
ôdov múkous toûs ùp' àvtwv ginomeînoi órganoi
kataxalitikôs), pròs ðe taûtais h tòn iðwos
pálin kalouméwv mêxanopoiôn (èk bâthous gár
polloû ùdhrî eukolówteron ánâgetai dia tòn ànth-
mâtikwv òrganwv òw àvtôi katakaskeuázousin).
kaloudi ðe mêxanikous oi palaioi kai tôús
bashmasiourgoûs, òw oî meû dia pnevmatôn filo-
texhnûs, òw 'Hrwn Pneumatikôs, ðe ði dia
neurwv kai spârtnwv eîmîzwn kynîseis dokouû
mumeîthai, òw 'Hrwn Autòmatous kai Zviglos,
álloi ðe dia tòn ef' údatos óxouménwv, òw 'Arkh-
mîdhs 'Oxouménwv, h tòn ði' údatos òrpolugiwv,
òw 'Hrwn 'Ydréwû, ðì dh kai ÷h gnwsonikê

¹ mêxanikê ... ìrkhitektonês om. Hultsch.

* mágyanov is properly the block of a pulley, as in Heron's

616
REVIVAL OF GEOMETRY: PAPPUS

mathematical studies and at the same time to learn the above-mentioned arts, they instruct a person wishing to undertake practical tasks in mechanics to use the resources given to him by actual experience in his special art.

Of all the [mechanical] arts the most necessary for the purposes of practical life are: (1) that of the makers of mechanical powers, they themselves being called mechanicians by the ancients—for they lift great weights by mechanical means to a height contrary to nature, moving them by a lesser force; (2) that of the makers of engines of war, they also being called mechanicians—for they hurl to a great distance weapons made of stone and iron and such-like objects, by means of the instruments, known as catapults, constructed by them; (3) in addition, that of the men who are properly called makers of engines—for by means of instruments for drawing water which they construct water is more easily raised from a great depth; (4) the ancients also describe as mechanicians the wonder-workers, of whom some work by means of pneumatics, as Heron in his Pneumatica, some by using strings and ropes, thinking to imitate the movements of living things, as Heron in his Automata and Balancings, some by means of floating bodies, as Archimedes in his book On Floating Bodies, or by using water to tell the time, as Heron in his Hydria, which appears to have affinities with the

Belopoëtica, ed. Schneider 84. 12, Greek Papyri in the British Museum iii. (ed. Kenyon and Bell) 1164 n. 8.

This work is mentioned in the Pneumatica, under the title Περὶ ὑδρίων ὀροσκοπεῖων, as having been in four books. Fragments are preserved in Proclus (Hypotyposis 4) and in Pappus's commentary on Book v. of Ptolemy's Syntaxis.

617
θεωρία κοινωνοῦντα φαίνεται. μηχανικοῦς δὲ καλοῦσιν καὶ τοὺς τὰς σφαιροποιὰς [ποιεῖν]
ἐπισταμένοις, ύφ’ ὁν εἰκῶν τοῦ οὐρανοῦ κατα-
σκευάζεται δι’ ὁμαλῆς καὶ ἐγκυκλίου κινήσεως ὦδατος.
Πάντως δὲ τούτων τὴν αἰτίαν καὶ τὸν λόγον
ἐπεγνωκέναι φασίν τινες τὸν Συρακόσιον Ἀρχι-
μήδην. μόνος γὰρ οὗτος ἐν τῷ καθ’ ἡμᾶς βίω
ποικίλη πρὸς πάντα κέχρηται τῇ φύσει καὶ τῇ
ἐπινοίᾳ, καθὼς καὶ Γέμινος ὁ μαθηματικὸς ἐν τῷ
Περὶ τῆς τῶν μαθημάτων τάξεώς φήσιν. Κάρπος
δὲ ποῦ φησιν ὁ Ἀντιοχεὺς Ἀρχιμήδη τὸν Συρα-
κόσιον ἐν μόνον βιβλίον συντεταχθεῖαι μηχανικὸν
τὸ κατὰ τὴν σφαιροποιίαν, τὸν δὲ ἄλλων οὐδὲν
ξειωκέναι συντάξαι. καίτοι παρὰ τοῖς πολλοῖς
ἐπὶ μηχανικὴ δοξασθεὶς καὶ μεγαλοφυῆς τις γενό-
μενος ὁ θαυμαστὸς ἐκεῖνος, ὡστε διαμεῖναι παρὰ
πᾶσιν ἄνθρωποις ὑπερβαλλόντως ὑμνοῦμενος, τῶν
τε προηγουμένων γεωμετρικῆς καὶ ἀριθμητικῆς
ἐχομένων θεωρίας τὰ βραχύτατα δοκοῦντα εἶναι
σπουδαῖοις συνέγραφεν: δὲ φαίνεται τὰς εἰρημένας
ἐπιστήμασι τοὺς αγαπήσας ὡς μηδὲν ξεσκέφθειν
ὑπομένειν αὐταῖς ἐπειδὰνείγεν. αὐτὸς δὲ Κάρπος
καὶ ἄλλοι τινὲς συνεχρήσαντο γεωμετρία καὶ εἰς
tέχνας τινὰς εὐλογῶς: γεωμετρία γὰρ οὐδὲν βλά-
πτεται, σωματοποιεῖν πεφυκύια πολλάς τέχνας,
διὰ τοῦ συνείναι αὐταῖς [μήτηρ οὖν ὦσπερ οὗσα
τεχνῶν οὐ βλάπτεται διὰ τοῦ φροντίζειν ὄργανικῆς
καὶ ἀρχιτεκτονικῆς: οὐδὲ γάρ διὰ τὸ συνείναι
γεωμετρία καὶ γνωμονικῆ καὶ μηχανικῆ καὶ σκηνο-
γραφία βλάπτεται τι], καὶ τοιαύτων δὲ προάγουσα
1 ποιεῖν om. Hultsch. 2 μήτηρ . . . τι om. Hultsch.
REVIVAL OF GEOMETRY: PAPPUS

science of sun-dials; (5) they also describe as mechan-
icians the makers of spheres, who know how to make
models of the heavens, using the uniform circular
motion of water.

Archimedes of Syracuse is acknowledged by some
to have understood the cause and reason of all these
arts; for he alone applied his versatile mind and
inventive genius to all the purposes of ordinary life, as
Geminus the mathematician says in his book *On the
Classification of Mathematics.*\(^a\) Carpus of Antioch\(^b\)
says somewhere that Archimedes of Syracuse wrote
only one book on mechanics, that on the construction
of spheres,\(^c\) not regarding any other matters of this
sort as worth describing. Yet that remarkable man
is universally honoured and held in esteem, so that his
praises are still loudly sung by all men, but he himself
on purpose took care to write as briefly as seemed
possible on the most advanced parts of geometry and
subjects connected with arithmetic; and he obviously
had so much affection for these sciences that he
allowed nothing extraneous to mingle with them.
Carpus himself and certain others also applied geo-
metry to some arts, and with reason; for geometry is
in no way injured, but is capable of giving content to
many arts by being associated with them, and, so far
from being injured, it is obviously, while itself

* For Geminus and this work, *v. supra*, p. 370 n. c.

\(^b\) Carpus has already been encountered (*vol. i. p. 334*) as
the discoverer (according to Iamblichus) of a *curve arising
from a double motion* which can be used for squaring the
circle. He is several times mentioned by Proclus, but his
date is uncertain.

* This work is not otherwise known.
* With the great figure of Pappus, these selections illustrating the history of Greek mathematics may appropriately come to an end. Mathematical works continued to be written in Greek almost to the dawn of the Renaissance, and
advancing those arts, appropriately honoured and adorned by them.\textsuperscript{a}

they serve to illustrate the continuity of Greek influence in the intellectual life of Europe. But, after Pappus, these works mainly take the form of comment on the classical treatises. Some, such as those of Proclus, Theon of Alexandria, and Eutocius of Ascalon have often been cited already, and others have been mentioned in the notes.
INDEX

This index does not include references to critical notes nor to authors cited, for which the separate catalogue should be consulted. References to vol. i. are cited by the page only, those to vol. ii. by volume and page. The abbreviations "incl."="including" and "esp."="especially" are occasionally used.

Abacus, 35 and nn. b and c
Achilles and the Tortoise, 369-371
Adam, James: The Republic of Plato, 399 nn. a and c
Addition in Greek mathematics, 46-48
Adrastus, on the geometric mean, 125 n. a
Aeschylus, 258 n. a
Aëtius: Placita, 217 and n. c
Agorastes, character in Lucian's Auction of Souls, 91
Alexander Aphrodisiensis, 173, 236 n. a, 315
Alexander the Great, 155 n. b. 175
Algebra:
Geometrical algebra of Pythagoreans, 186-215
Pure determinate equations, ii. 525
Quadratic equations: geometrical solution by Pythagoreans, 186-215;
solutions by Heron, ii. 503-505 (incl. n. a); by Diophantus, ii. 527-535 (esp. ii. 533 n. b); v. also Square root, extraction of
Simultaneous equations leading to a quadratic, ii. 537
Cubic equations: Archimedes, ii. 127-163 (incl. n. a); Diophantus, ii. 539-541; v. also Cube root
Indeterminate equations: Pythagorean and Platonic formulae for right-angled triangles, 90-95; side- and diameter-numbers, 132-139; "bloom" of Thymaridas, 139-141; "cattle problem" of Archimedes, ii. 203-205; Heron, ii. 505-509; Dio-
phantus, ii. 541-551
Sums of squares, ii. 551-559
Allman, J. G., his Greek
<table>
<thead>
<tr>
<th>INDEX</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Geometry from Thales to Euclid</strong>, 237 n. b</td>
</tr>
<tr>
<td><strong>Almagest</strong>: Arabic name for Ptolemy's <em>Syntaxis</em>, ii. 409 n. b</td>
</tr>
<tr>
<td><strong>Alphabet, Greek</strong>: use as numerals, 42-44</td>
</tr>
<tr>
<td><strong>Ameristus</strong>, 147 and n. b</td>
</tr>
<tr>
<td><strong>Amthor, A.</strong>, on Archimedes' cattle problem, 17 n. c, ii. 205 n. b</td>
</tr>
<tr>
<td><strong>Amyclas (recte Amyntas) of Heraclea</strong>, 153 and n. b</td>
</tr>
<tr>
<td><strong>Amyntas</strong>: v. Amyclas</td>
</tr>
<tr>
<td><strong>Analemma</strong>: of Ptolemy, 301 n. b, ii. 409 n. b; of <em>Diocorus</em>, 301 and n. b</td>
</tr>
<tr>
<td><strong>Analysis</strong>: discussion by Pappus, ii. 597-599; applied by Eudoxus to theory of &quot;the section,&quot; 153 and n. a; v. also Leodamas</td>
</tr>
<tr>
<td><strong>Anatolius, bishop of Laodicea</strong>: on the meaning of the name <em>mathematics</em>, 3; on the Egyptian method of reckoning, 3 n. a, ii. 515-517; the classification of mathematics, 19 and n. b; his relation to Diophantus, ii. 517 n. a</td>
</tr>
<tr>
<td><strong>Anaxagoras</strong>: wrote on squaring of circle while in prison, 308, 149 n. d; character in Plato's <em>Rivals</em>, 149 n. f</td>
</tr>
<tr>
<td><strong>Anchor-ring</strong>, v. Tore</td>
</tr>
<tr>
<td><strong>Angle</strong>: angle in semi-circle, 167-169; <em>mixed</em> angles, 429 n. c; equality of right-angles, 443 and n. b</td>
</tr>
<tr>
<td><strong>Anharmonic ratios</strong>, 485 n. c</td>
</tr>
<tr>
<td><strong>Anthemius of Tralles</strong>, ii. 357 n. a</td>
</tr>
<tr>
<td><strong>Anticleides</strong>, 175</td>
</tr>
<tr>
<td><strong>Antiphon</strong>, 311 and n. a, 313 and n. a, 315 and n. a, 317 n. a</td>
</tr>
<tr>
<td><strong>Apollodorus the Calculator</strong>, 169 and n. a</td>
</tr>
<tr>
<td><strong>Apollonius of Perga</strong>: Life: ii. 277 (esp. n. a)-281</td>
</tr>
<tr>
<td><strong>Works</strong>:</td>
</tr>
<tr>
<td><strong>Conics</strong>: Relation to previous works, 487-489 (incl. n. a), ii. 277-281; scope of the work, ii. 281-285; terminology, ii. 285-289, ii. 309 and n. a, ii. 317 and n. a, ii. 323; construction of the sections, ii. 289-305; fundamental properties, ii. 305-329—parabola, ii. 305-309, hyperbola, ii. 309-317, ii. 323-329, ellipse, ii. 317-323; transition to new diameter, ii. 329-335; introduction of axes, ii. 289 n. a, ii. 331 n. a; generality of his methods, ii. 289 n. a; form of his proofs, ii. 289 n. a; contrast with Archimedes' treatment, ii. 323 a, and terminology, ii. 283 n. a; his distinctive achievement to have based his treatment on the theory of <em>applied</em></td>
</tr>
</tbody>
</table>
INDEX

areas, ii. 309 n. a; first to have recognized two branches of hyperbola, ii. 329 n. a; focus - directrix property not used, 495 n. a, ii. 281 n. a

On the Cutting-off of a Ratio, ii. 337-339; included in Treasury of Analysis, ii. 337; Halley’s Latin translation, ii. 337 n. a

On the Cutting-off of an Area, ii. 339 and n. b; included in Treasury of Analysis, ii. 337

On Determinate Section, ii. 339-341 (incl. n. a); included in Treasury of Analysis, ii. 337; lemma by Pappus, ii. 607-611

On Tangencies, ii. 341-343 (incl. n. b); problem of three circles and Newton’s solution, ii. 343 n. b; included in Treasury of Analysis, ii. 337

On Plane Loci, ii. 345 and n. c; included in Treasury of Analysis, ii. 337; reconstructions by Fermat, van Schooten and Simson, ii. 345 n. c

On Vergings, ii. 345-347 (incl. n. a); included in Treasury of Analysis, ii. 337; described the datum as the

assigned, ii. 349; restoration by Samuel Horsley, ii. 347 n. a

On the Dodecahedron and the Icosahedron, ii. 349

General Treatise, ii. 349-351 (incl. n. a)

On the Cochlias, ii. 351; “sister of the cochloid,” 335 and n. c

On Unordered Irrationals, ii. 351-353 (incl. n. a)

Quick-deliverer (on the measurement of a circle), ii. 353 and nn. b, c

On the Burning Mirror, ii. 357 and n. b

Other mathematical achievements:

Two mean proportionals, 267 n. b

Continued multiplications, ii. 353-357

Astronomy, ii. 357 n. b

Otherwise mentioned: ii. 363

Application of Areas, 186-215; explanation and history, 186-187; a Pythagorean discovery, 187; Euclid’s theorems, 189-215; equivalence to solution of quadratic equations, 195 n. a, 197 n. a, 211 n. a, 215 n. a; possible use by Hippocrates, 245 n. a; use by Apollonius in his treatment of the conic sections, ii. 305-323 (esp. ii. 309 n. a)
INDEX

Approximations to \( \pi \), 321-333 (incl. n. a), ii. 353
Archibald, R. C., his Euclid’s Book on Divisions of Figures, 157 n. c

Archimedes
Life: A Syracusan, ii. 19; born about 287 B.C., ii. 19 and n. c; his mechanical devices used in the defence of Syracuse, ii. 19-21, ii. 25-31; his death at the hands of a Roman soldier in 212 B.C., ii. 23, ii. 33-35; his sayings, ii. 21, ii. 23, ii. 35; his contempt for the utilitarian, ii. 31, ii. 619; his request for a cylinder enclosing a sphere as a monument, ii. 33 and n. a; his absorption in his work, ii. 31-33, ii. 37; his solution of the problem of the crown, ii. 37-39, ii. 251 n. a; his Doric dialect, ii. 21 and n. b, ii. 137

Works:
On the Sphere and Cylinder, ii. 41-127; preface, ii. 41-43; axioms, ii. 43-45; postulates, ii. 45-47; surface of cylinder, ii. 67-77; surface of cone, ii. 77-81; surface of sphere, ii. 113-117; volume of sphere, ii. 119-127; trigonometrical equivalents, ii. 91 n. b, ii. 101 n. a, ii. 109 n. a; equivalence to integration, ii. 41 n. a, ii. 117 n. b; problem leading to solution of cubic equation, ii. 127-163 (incl. n. a); cited by Zenodorus, ii. 393, ii. 395; Eutocius’s commentaries, ii. 73 n. a, ii. 77 n. a, ii. 127 n. a, ii. 135-163; otherwise mentioned, ii. 165 n. a
On Conoids and Spheroids, ii. 165-181; preface, ii. 165; lemmas, ii. 165-169; volume of segment of paraboloid of revolution, ii. 171-181; equivalence to integration, ii. 181 n. a
On Spirals, ii. 183-195; definitions, ii. 183-185; fundamental property, ii. 185-187; vergings assumed, ii. 187-189 (incl. n. c), ii. 195 n. a; property of sub-tangent, ii. 191-195, and comments of Pappus, Tannery and Heath, ii. 195 n. b; curve used to square the circle, 335 and n. b
On the Measurement of a Circle, 317-333; Eutocius’s commentary, 323 n. a; cited by Zenodorus, ii. 395 and n. a
### INDEX

**Quadrature of a Parabola**, ii. 229-243; prefatory letter to Dositheus, ii. 229-233; first (mechanical) proof of area of parabolic segment, ii. 233-239; equivalence to integrations, ii. 239 n. b; second (geometrical) proof, ii. 239-243; otherwise mentioned, ii. 41 n. b, ii. 221 n. a, ii. 225 and n. a

**Method**, ii. 221-229; discovery by Heiberg, ii. 221 n. b; prefatory letter to Eratosthenes, ii. 221-223; mechanical proof of area of parabolic segment, ii. 223-229; gives Democritus credit for finding volume of cone and pyramid, 229-231, 411 n. a

**Sand-reckoner**, ii. 199-201 (incl. n. a)

**Cattle Problem**, 17 n. c, ii. 203-205 (incl. n. a)

**On Plane Equilibriums**, ii. 207-221; postulates, ii. 207-209; principle of lever, ii. 209-217; centre of gravity of parallelogram, ii. 217-221; of a triangle, ii. 227 and n. a

**On Floating Bodies**, ii. 243-257; discovery of Greek text by Heiberg, ii. 242 n. a; William of Moerbeke's Latin translation, ii. 242 n. a; postulates, ii. 243-245; surface of fluid at rest, ii. 245-249; loss of weight of solid immersed in a fluid—"Archimedes' principle," ii. 249-251; use of this principle to solve problem of crown, ii. 251 n. a; stability of paraboloid of revolution, ii. 253-257; Heath's tribute to Book ii., ii. 252 n. a

**Liber Assumptorum**, ii. 581 n. a

**Other achievements:**
- Discovery of 13 semi-regular solids, 217 n. a, ii. 195-197 (incl. n. b)
- Circles inscribed in the $\delta\beta\gamma\lambda\omega\varsigma$, ii. 581 n. a
- Solution of cubic equations, ii. 127-163 (incl. n. a)
- Inequalities, ii. 165-169
- Summation of series, ii. 165-169, ii. 241 and n. a
- "Archimedes' Axiom," 321 n. a, 411 n. a, 455 n. a, ii. 47 and n. a, ii. 195 n. b, ii. 231

Otherwise mentioned:
- Vitruvius on his proficiency in all branches of science, ii. 3 n. a;
INDEX

Pappus's tribute to his versatility, ii. 619; his method of evaluating areas, ii. 585 n. a

Archytas, 5 n. a; on the branches of mathematics (Pythagorean quadrivium), 5 and n. b; on means, 113-115, 153 n. a; his proof that a superparticular ratio cannot be divided into equal parts, 131-133; Proclus's comments on his work in geometry, 151; his solution of the problem of two mean proportionals, 285-289; his work in stereometry, 7 n. a, 13 n. b; his method of representing the sum of two numbers, 131 n. c, 429 n. c

Aristaeus: his five books of Solid Loci (conic sections), 487 and n. a, ii. 281 n. a, ii. 255 n. a; first demonstrated focus-directrix property, 495 n. a; his (?) Comparison of the Five Regular Solids, 487 n. b

Aristarchus of Samos, ii. 3-15; a pupil of Strato of Lampscus, ii. 3 and n. a; his theory of the nature of light, ii. 3; his heliocentric hypothesis, ii. 3-5 (incl. n. b); on the sizes and distances of the sun and moon, ii. 3-15; use of continued fractions (?), ii. 15 and n. b

Aristophanes: reference to the squaring of the circle, 309 and n. a

Aristotle: use of term mathematics, 3 n. c, 401 n. a; did not know how to square the circle, 335; on first principles of mathematics, 419-423; on the infinite, 425-429; proofs differing from Euclid's, 419 n. a, 429-431; method of representing angles, 429 n. c; on the principle of the lever, 431-433; on the parallelogram of velocities, 433; irrationality of $\sqrt{2}$, 111; on odd, even and prime numbers, 75 n. a, 78 n. a; on oblong and square numbers, 95 and n. b; on Zeno's paradoxes of motion, 366-375; on nature of geometrical proof, ii. 369

Aristoxenus: on Plato's lecture on the Good, 389-391; his pupil Cleonides, 157 n. c

Arithmetic:
Its place in the Pythagorean quadrivium, 5 and n. b; in the education of Plato's Guardians, 7-9; in Plato's Laws, 21-23; in Anato-lius and Geminus, 19 and n. b; difference from logistic, 7 and n. a, 17-19 (incl. n. b); Greek arithmetical notation and the chief arithmetical operations, 41-63
INDEX

Pythagorean arithmetic: first principles, 67-71; classification of numbers, 73-75; perfect numbers, 75-87; figured numbers, 87-99; some properties of numbers—the "sieve" of Eratosthenes, 101-103, divisibility of squares, 103-105; a theorem about cube numbers, 105-107; a property of the Pythogoreans, 107-109; irrationality of $\sqrt{2}$, 111; theory of proportion and means—arithmetic, geometric and harmonic means, 111-115, seven other means, 115-125, Pappus’s equations between means, 125-129, Plato on means between two squares or two cubes, 129-131, Archytas’s proof that a superparticular ratio cannot be divided into equal parts, 131-133

See also Theory of numbers; Algebra; Irrational, the; Approximations to $\pi$; Inequalities

Arithmetica, v. Diophantus

Armillary sphere, 229 n. a

Arrow of Zeno, 367, 371

Astronomy: a full notice excluded, x; identical with sphaeric in the Pythagorean quadrivium, 5 n. b; in the education of Plato’s Guardians, 15; in Analysis

tolius and Geminus, 19 and n. b; Isocrates’ views, 29; work of Thales, 147 n. a; of Pythagoras, 149 n. b; of Euphrodis, ii. 5 n. b; of Oenopides, 149 n. e; of Philippus of Opus, 155 n. a; of Eudoxus, 15 n. a, 411-415; of Heracleides of Pontus, 15 n. a ii. 5 n. b; of Autolycus of Pitane and Euclid, 490 n. a; of Aristarchus of Samos, ii. 3-15; of Eratosthenes, ii. 261 n. a. 263 and n. e. 267-273; of Apollonius of Perga, ii. 277 n. a, ii. 357 n. b; of Posidonius, ii. 371 n. b; of Hypsicles, ii. 395-397; of Cleomedes, ii. 399-401; of Hipparchus, ii. 407 n. a, 414 n. a; of Menelaus, ii. 407 n. a; of Ptolemy, ii. 409 and n. b, ii. 447; of Pappus, ii. 593 n. a; knowledge of astronomy necessary for reading Plato, ii. 401

Athenaeus of Cyzicus, 153 and n. e

“Attic” numerals, 41-42

August, E. F., 397 n. a

Autolycus of Pitane, 491 n. a

Axioms and postulates: Aristotle’s discussion, 419-423; Euclid’s postulates, 443; attempt to prove the parallel-postulate, ii. 367-385; Archimedes’ postulates in his work On the Sphere and Cylinder, ii.
INDEX

45-47, in his work *On Plane Equilibriums*, ii. 207-209, in his work *On Floating Bodies*, ii. 243-245 (incl. n. a): “Archimedes’ Axiom,” 321 n. a, 411 n. a, 455 n. a, ii. 47 and n. a, ii. 231

Bachet, ii. 537 n. b
Barlaam, 14th century Calabrian monk: his formula for approximation to a square root, ii. 472 n. a
Bede, the Venerable, 31 n. c
Bees, Pappus on their choice of shape for cell, ii. 589-593
Benecke, A., his *Ueber die Geometrische Hypothese in Platonis Menon*, 397 n. a
Besthorn, R. O.: his edition of an-Nairizi’s commentary on Euclid’s *Elements*, 185 n. b
Bjornbo, A. A., on Hippocrates’ quadratures, 311 n. b
Blass, C.: his *De Platone mathematico*, 387 n. a
Boeckh, A., 221 n. a
Boethius, citation of Archytas’ proof that a superparticular ratio cannot be divided into equal parts, 131-133
Breton de Champ, P., ii. 611 n. c
Bretschneider, C. A., his *Die Geometrie und die Geometer vor Eukleides*, 153 n. a

Brochard, V., on Zeno’s paradoxes, 367 n. a
Bryson, attempt to square the circle, 315-317
Burnet, J., on the astronomy in Plato’s *Republic*, 15 n. a
Butcher, S. H., on the hypothesis in Plato’s *Meno*, 397 n. a

Callimachus, ii. 261 and n. b
Canonic, theory of musical intervals, 19 and n. b
Cantor, G., 42
Carpus of Antioch, 335, ii. 619
Case (*πτωσις*), ii. 347
Casting out of nines, 107-109
Catasterismi, work by Eratosthenes, ii. 263 n. a
Catoptrics, v. Euclid: *Works*
Cattle-problem, v. Archimedes: *Works*
Centre of gravity: Archimedes’ postulates, ii. 209; of a lever, ii. 209-217; of a parallelogram, ii. 217-221; of a triangle, ii. 217 n. b, ii. 227 and n. a; of a trapezium and parabolic segment, ii. 217 n. b; of a segment of a paraboloid, ii. 225
Chaldaeans, ii. 397 n. a
Chasles, M., 485 n. a, ii. 581 n. c
Chords, Table of: Hipparchus and Menelaus, ii. 407, ii. 409 and n. a; Ptolemy, ii. 443-445
Chrysippus, 229 n. a
Cicero: restored monument to Archimedes, ii. 33 n. a
INDEX

Circle:
Division into degrees, ii. 395-397

Squaring of the circle:
Anaxagoras’s work in prison, 309; a reference by Aristophanes, 309; approximation by polygons — Antiphon, 311-315, Bryson, 315-317, Archimedes, 317-333; solutions by higher curves — Simplicius’s summary, 335, the quadratrix, 337-347; closer approximations by Archimedes, 333 n. a, and Apollonius, ii. 353; Pappus’s collection of solutions, ii. 581 n. b

Apparent form of circle, ii. 593-597

Cissoid, viii; discovered by Diocles, 271 n. a, ii. 365 n. a; and by him used for finding two mean proportionals, 271-279

Cleanthes, ii. 5

Cleomedes: life and works, ii. 267 n. b, ii. 397 n. a; on the measurement of the earth, ii. 267-273; on paradoxical eclipses, ii. 397-401

Cleonides, 157 n. c

Cochlias, 335, ii. 351

Cochloids, 301 n. a, 335 and n. c, 297 n. c

Commandinus, his edition of Pappus’s Collection, 499 n. a, ii. 581 n. c

Concentric spheres, Eu-
doxus’s theory of, 411-415

Conchoid, 297, 301 n. a; used by Nicomedes to trisect an angle, 297-309

Cone: double cone defined by Apollonius, ii. 285-287; single cone defined, ii. 287; volume enunciated by Democritus, 229-231; and proved by Eudoxus, 229-231, 409-411; sub-contrary section a circle, ii. 301 n. a

Conic sections: discovered by Menaechmus, 279-283 (incl. n. a), 297, ii. 281 n. a; originally called sections of a right-angled, acute-angled and obtuse-angled cone, 283 n. a, ii. 279; by Apollonius renamed parabola, ellipse and hyperbola, 283 n. a; treatises written by Aristaeus and Euclid, 487 and n. a, ii. 255 n. a, iii. 281 n. a; “locus with respect to three or four lines,” 487-489; Euclid on generation of ellipse, 491; focus-directrix property assumed by Euclid, 495 and n. a; terminology of Archimedes, ii. 281 n. a, ii. 283 n. a; area of parabolic segment found by Archimedes mechanically, ii. 223-227, ii. 233-239, geometrically, ii. 239-245; properties assumed by Archimedes, ii. 171, ii. 175,
INDEX

by Pappus, ii. 575 n. a; duplication of cube — task set to Delians and Plato's advice sought, 257; a poetic version, 257-259; reduced to finding of two mean proportionals, 259; collection of solutions by Eutocius, 263 n. a; by Pappus, ii. 581 n. b

Cube root: \(3\sqrt{100}\) found by Heron, 61-63; Heron's value for \(3\sqrt{97050}\), 63 n. a; approximations by Philon of Byzantium, 63 n. a

Cubic equations: c.e. arising out of Archimedes' De Sph. et Cyl., ii. 133 n. d; solved by Archimedes by use of conics, ii. 137-159 (esp. 141 n. a); also by Dionysodorus and Diocles, ii. 163 n. a; Archimedes able to find real roots of general cubic, ii. 163 n. a; cubic equation solved by Diophantus, ii. 539-541

Damianus of Larissa, ii. 497 n. a; his book On the Hypotheses in Optics, ii. 497; an abridgement of a larger work based on Euclid, ii. 497 n. a

Definitions: discussed by Plato, 393; by Aristotle, 423; Euclid's d., 67-71, 437-441, 445-453, 479; Archimedes' d., ii. 43-45, ii. 165, ii. 183-185; Apollonius's d., ii. 285-289;
INDEX

Heron’s d., ii. 467-471; Diophantus’s d., ii. 519-523
De Fortia D’Urban, Comte, his Traité d’Aristarque de Samos, ii. 15 n. b
Demel, Seth: his Platons Verhältnis zur Mathematik, 387 n. a
Demetrius of Alexandria, 349 and n. b
Democritus: life and works, 229 n. a; reflections on the indefinitely small, 229; enunciated formulae for volumes of cone and pyramid, 229-231, 411 n. a
De Morgan, 447 n. a
Descartes, system of co-ordinates conceived through study of Pappus, ii. 601 n. a
Dichotomy of Zeno, 369 and n. b
Diels, H.: his Die Fragmente der Vorsokratiker, xvi, 5, 73, 75, 113, 173, 217
Digamma: use as numeral, 43
Dimension, 85, ii. 411-413, ii. 515, ii. 601-603
Dinostratus, 153 and n. d
Diocles: his date, ii. 365 n. a; his discovery of the cissoid, viii, 270-279; his solution of a cubic equation, ii. 135, ii. 163 n. a
Diodorus of Alexandria: his Analemma, 301 and n. b
Diodorus Siculus: his account of the siege of Syracuse, ii. 23 and n. a
Dionysius, a friend of Heron, ii. 467
Dionysodorus: of Caunus ?, ii. 364 n. a; his solution of a cubic equation, ii. 135, ii. 163 n. a; his book On the Spire, ii. 481
Diophantus of Alexandria: life, ii. 513, ii. 517 n. b; on the unit, ii. 515; on the Egyptian method of reckoning, ii. 515; his Arithmetica, ii. 517 and n. b; his work (?) entitled Porisms, ii. 517 and n. c; his treatise On Polygonal Numbers, ii. 515, ii. 561; his contributions to algebra—notation, ii. 519-525; pure determinate equations, ii. 525; quadratic equations, ii. 527-535, esp. ii. 533; simultaneous equations leading to a quadratic, ii. 537; cubic equation, ii. 539-541; indeterminate analysis, 139 n. b; indeterminate equations of the second degree, ii. 541-547; indeterminate equations of higher degree, ii. 549-551; theory of numbers—sums of squares, ii. 551-559
Dioptra: ancient theodolite, ii. 467; Heron’s book, ii. 485-489
Diorismi, 151 and n. h, 395-397, ii. 135 n. a
Division in Greek mathematics, 51-61
Division of Figures: Euclid’s book, 157 n. c; a problem in Heron’s Metrica, ii. 483-485
INDEX

Dodecahedron: one of the five regular solids, 217, 223 and n. b; Hippasus said to have been drowned for revealing it, 223-225; comparison with the icosahedron, ii. 349

Dosithoeus of Pelusium: works dedicated to him by Archimedes, ii. 41, ii. 229

Dyad, 75 and n. a, 427 n. b

Earth: circumference calculated by Posidonius, ii. 267; by Eratosthenes, ii. 267-273

Ecliptic, 149 n. e

Ephantus, ii. 5 n. b

Eecke, Paul Ver, v. Ver Eecke

Egyptian method of reckoning, 17, 21, ii. 515-517

Egyptian papyri, calculations in, 45-47

Egyptian se-get, 165 n. b

Elements: meaning of term, 151 n. c; Leon's collection, 151; Euclid's Elements, 157, 437-479; Elements of Conics, 487 n. a, ii. 151, ii. 153; Pappus's Collection so described, ii. 607

Eneström, G., 63 n. a

Enneagon: relation of side of enneagon to diameter (=sin 20°), ii. 409 n. a

Epanthema, "bloom," of Thymaridas, 139-141

Equations: v. Cubic equations, Quadratic equations

Eratosthenes: Life and achievements, 156 and n. a, ii. 261 and n. a; his work On Means included in the Treasury of Analysis, ii. 263 and n. d; discussed loci with reference to means, ii. 263 n. d, ii. 265 and n. a; his Platonicus, 257 and ii. 265-267; his sieve for finding successive odd numbers, 100-103; letter of pseudo-Eratosthenes to Ptolemy Euergetes, 257-261; his solution of the problem of two mean proportionals, 291-297; his solution derided by Nicomedes, 297-299; his measurement of the circumference of the earth, ii. 267-273; Archimedes' Method dedicated to him, ii. 221; Cattle Problem said to have been sent through him, ii. 203; his teachers and pupils, ii. 261-263; his nicknames, ii. 261-263; his other works, ii. 263

Erycinus, Paradoxes of, ii. 571-573

Euclid:

Life: Born in time of Ptolemy I, 155 and n. b; not to be confused with Euclid of Megara, 155 n. b; his school at Alexandria, 437 n. a, 489, ii. 35 n. b; told Ptolemy there was no royal road in geometry, 155 (but v. n. b); contempt for those who
studied mathematics for monetary gain, 437

Works:


*Data*: General character, 479 n. a; definitions, 479; included in *Treasury of Analysis*, 479 n. a, ii. 337; cited, ii. 455

*Porisms*: Proclus's notice, 481; included in *Treasury of Analysis*, 481 and n. c, ii. 337; Pappus's comprehensive enunciation, 481-483; developments in theory of conics, 487 n. a; modern reconstructions, 485 n. e; a lemma by Pappus, ii. 611-613

*Conics*: a compilation based on Aristaeus, ii. 281 n. a; Pappus on Apollonius's debt, 487-489; ellipse obtained as section of cone or cylinder by a plane not parallel to the base, 491 and n. b; focus-directrix property assumed without proof, 495 n. a; cited by Archimedes, ii. 225

*Surface Loci*: 399 n. a, 487 n. a; included in *Treasury of Analysis*, 491 and n. c; Pappus's lemmas, 363 n. a, 493-503

*Optics*: Euclid's text and Theon's recension, 503 and n. b, 157 and n. c; proof that \( \tan \alpha : \tan \beta < \alpha : \beta \), 503-505

*On Divisions of Figures*: Proclus's notice, 157; Woepcke's discovery of Arabic text, 157 n. c; R. C. Archibald's restoration, 157 n. c; a medieval Latin translation (by Gherard of Cremona?)
INDEX

157 n. c.; similarity of Book iii. of Heron’s Metrics, ii. 485 n. a; division of a circle into three equal parts, ii. 485 n. a

Pseudaria: attributed to Euclid by Proclus, 161 and n. a

Catoptrics: attributed to Euclid by Proclus, 157; but not otherwise known, and possibly written by Theon of Alexandria, 157 n. c

Phenomena: astronomical treatise, 491 n. a

Sectio Canonis, musical treatise doubtfully attributed to Euclid, 157 n. c

Introductio Harmonica, musical treatise by Cleonides, wrongly attributed to Euclid, 157 n. c

Elements of Music: attributed to Euclid by Proclus, 157

Euclidean geometry, viii; its nature defined in the postulates—space an infinite, homogeneous continuum, 443 and nn. a, b, the parallel-postulate, 443 and n. c; Proclus’s objection to the postulate, ii. 367-371; attempts to prove the postulate—Posidonius and Geminus, ii. 371-373, Ptolemy, ii. 373-383, Proclus, ii. 383-385; Euclid’s genius in making it a postulate, 443 n. c, ii. 367 n. a; non-Euclidean geometry, 443 n. c

Eudemus: pupil of Aristotle, 145 n. a, ii. 281; his History of Geometry, 145 n. a; his account of Hippocrates’ quadrature of lunes, 235-253; on sum of angles of a triangle, 177-179; attributed Eucl. xii. 2 to Hippocrates, 239, 459 n. a

Eudemus, correspondent of Apollonius; Conics dedicated to him, ii. 281

Eudoxus of Cnidos: Proclus’s notice, 151-153; discovered three subcontrary means, 153; increased theorems about “the section,” 153; established theory of proportion, 409; credited with discovery of three subcontrary means, 121 n. a; gave proofs of volume of pyramid and cone, 409-411; must have used “Axiom of Archimedes,” 411 n. a, ii. 231; established method of exhausting an area by polygons, 411 n. a; his theory of concentric spheres to account for planetary motions, 411-415; Menaechmus his pupil, 153; his work continued by Hermotimus of Colophon, 153; otherwise mentioned, 319 n. c
Eugenius Siculus, Admiral, translated Ptolemy's Optics into Latin, ii. 411 n. a
Euripides, 259 n. a
Eutocius: On Archimedes' Sphere and Cylinder, 263 nn. a and b, 277 n. a, 299 n. a, ii. 159 n. a, ii. 163 n. a, ii. 621 n. a; On A.'s Measurement of a Circle, 323 n. a
Exhaustion, method of: originated by Antiphon, 315 n. a; or possibly by Hippocrates, 411 n. a; established by Eudoxus, 315 n. a, 411 n. a; used to prove that circles are to one another as the squares on their diameters, 239 n. b, 459-465; used for approximating to area of circle, 313-315
Fermat: reconstruction of Apollonius's On Plane Loci, ii. 345 n. c; his notes on Diophantus, ii. 551 n. a; sums of squares, ii. 559 n. b
Fractions in Greek mathematics, 45
Geoponicus, Liber, ii. 467 n. a, ii. 505 n. b
Geminius: life and works, ii. 371 n. c; on the classification of curves, ii. 361-363, ii. 365 nn. a and b; his attempt to prove the parallel-postulate, ii. 371 n. c
INDEX

ii. 351 n. a, ii. 357 n. a,
ii. 371 n. c, ii. 407 n. a,
ii. 467 n. a, ii. 489 n. a,
ii. 525 n. b, ii. 587 n. c,
ii. 589 n. a, ii. 605 n. b
A Manual of Greek Mathematics, 181 n. a

The Thirteen Books of Euclid’s Elements, 67 n. a, 71 n. b, 85 n. a, 155 n. b, 159 n. b, 167 n. b,
181 n. a, 195 n. a, 211 n. a, 215 n. a, 437 nn. a,
and c, 441 n. a, 447 n. a, 457 n. a, ii. 367 n. a

Aristarchus of Samos, 411 n. b, 415 n. c, ii. 3 n. a

The Works of Archimedes,
17 n. c, 61 n. b, 323 n. a,
351 n. b, 493 n. b, ii. 25 n. a, ii. 117 n. b, ii. 163 n. a, ii. 181 n. a, ii. 205 n. b

The Method, ii. 19 n. a

Apollonius of Perga, ii.

Diophantus of Alexandria,
139 n. b, ii. 513 n. a,
ii. 517 n. c, ii. 519 n. a,
ii. 523 n. a, ii. 537 n. b

Greek Astronomy, 157 n. a, 411 n. b

Heiberg, J. L., 333 n. a

Heiberg, J. L., 19 n. c, 297 n. a, 311 n. b, 487 n. b,
503 n. b, ii. 81 n. a, ii. 85 n. a, ii. 89 n. a, ii. 159 n. a,
ii. 173 n. a, ii. 221 n. b, ii. 243 n. a, ii. 285 n. a, ii. 353 n. c, ii. 357 n. a, ii. 466 n. a

Helicon of Cyzicus, 263 n. b

Hendecagon, ii. 409 n. a

Heraclides of Pontus, ii. 5 n. b

Hermadorus, correspondent of Pappus, ii. 597

Hermotimus of Colophon, 153 and n. e

Herodotus, on the abacus, 35

Heron of Alexandria:
Life: his date a disputed question, ii. 467 n. a;
described as “the father of the turbine,” ii. 467 n. a

Works, ii. 467 n. a:

Definitions: based on Euclid, ii. 467; de-

finition of a point, ii. 469; of a spire, ii.
469; his terminology different from Pro-
clus’s, ii. 471 n. a

Metrica: discovery by R. Schöne and edited
by his son H. Schöne, ii. 467 n. a; has pre-
served its original form more closely than Heron’s other geometrical works, ii.
467 n. a; formula \( \sqrt{s(s-a)(s-b)(s-c)} \) for area of a triangle,
ii. 471-477, ii. 603 n. a; formula for approximation to a square root, ii. 471-
472 (incl. n. a); volume of a spire, ii. 477-483; division of a circle into three equal parts, ii. 483-
INDEX

485; similarity between Book iii. and Euclid's book On Divisions of Figures, ii. 485 n. a; extraction of a \(3\sqrt[4]{100}\), 60-63; Archimedes' approximation to \(\pi\), 333

Dioptra: the dioptra, an instrument like a theodolite, ii. 467 n. a; measurement of an irregular area, ii. 485-489; problem of moving a given force by a given weight using an arrangement of toothed wheels, ii. 489-497; similar solutions found in the Mechanics and in Pappus, ii. 489 n. a; proof of formula for area of a triangle, ii. 477 n. a

Mechanics: has survived in Arabic, ii. 488 n. a, and in a few fragments of the Greek, ii. 467 n. a; problems discussed, ii. 489 n. a

Geometrica: quadratic equations, ii. 503-505; indeterminate analysis, ii. 505-509

Pneumatica, Automata, etc., ii. 467 n. a, ii. 617

Otherwise mentioned: his addition to "Pythagoras's Theorem," 181 n. a and 185 n. b; his solution of the problem of two mean proportionals, 267-271; possible censure by Pappus, ii. 603 and n. a; on branches of mechanics, ii. 615

Hexagon, its use by bees, ii. 589-593

Hipparchus: life and achievements, ii. 407 n. a; founded the science of trigonometry, ii. 407 n. a; drew up a table of sines, ii. 407 and n. a; discovered precession of the equinoxes, ii. 407 n. a; his Commentary on the Phenomena of Eudoxus and Aratus, ii. 407 n. a, ii. 415 n. a

Hippasus, a Pythagorean: credited with discovery of means, 153 n. a; said to have been drowned at sea for revealing secret of inscribing a dodecahedron in a sphere, 223-225 (incl. n. a)

Hippia of Elis: life and achievements, 149 n. a; his praise of Ameristus, 149; discovered the quadratrix, 149 n. a, 337 n. a

Hippocrates of Chios: life and achievements, 235; his quadrature of lunes, 235-253; the views of Aristotle and the commentators on his quadratures, 311 and n. b; first reduced the problem of doubling the cube to the
INDEX

problem of finding two mean proportionals, 253, 259; possibly able to solve a quadratic equation, 245 n. a; discovered that circles are to one another as the squares on their diameters, 239 and n. b, 459 n. a

Hippopede ("horse-fetter"), name given by Eudoxus to a curve, 415

Homer, 31 n. a

Horsley, S., ii. 347 n. a

Hultsch, F.: his edition of Pappus’s Collection, ii. 565 n. a, ii. 581 n. c, ii. 605 n. b, ii. 611 n. c

Hypatia: daughter of Theon of Alexandria, 48 n. a; helped him in revision of commentary on Ptolemy, 48 n. a; her commentary on Apollonius’s Conics, ii. 285 n. a; her commentary on Diophantus’s Arithmetica, ii. 517 n. b

Hypotenuse: square on, 179-185; parallelogram on, ii. 575-579

Hypsicles: date, ii. 397 n. a; his division of the circle into 360 degrees, ii. 395-397; his continuation of Euclid’s Elements, ii. 349 nn. a and c, ii. 397 n. a; his definition of a polygonal number, ii. 397 n. a, ii. 515 and n. b

Iamblichus: on squaring of the circle, 335; on the “bloom” of Thymaridas, 139 n. b; on a property of the pythmen, 109 n. a

Icosahedron, 221 and n. o, ii. 349

Indecennurable: v. Irrational

Indeterminate analysis: Pythagorean and Platonic formulae for right-angled triangles, 90-95; side- and diameter-numbers, 133-139; Archimedes’ Cattle Problem, ii. 202-205; Heron’s problems, ii. 505-509; Diophantus’s problems, ii. 541-551

Indian mathematics, 181 n. a

Involution, ii. 341 n. a, ii. 611 n. a

Irrational: Pythagoreans and the irrational, 149 n. c, 215-217, 225 n. a; “irrational diameters,” 133-137 (esp. n. a), 399 n. c; proof by Theodorus and Theaetetus of irrationality of \(\sqrt{3}\), \(\sqrt{5} \ldots \sqrt{17}\), 381-383; Plato on irrational numbers, 401-403; Aristotle’s proof of irrationality of \(\sqrt{2}\), 111; Euclid’s theory, 451-459; Apollonius’s theory of unordered irrationals, ii. 351

Isocrates: on mathematics in Greek education, 27-29

Isoperimetric figures, ii. 387-395, ii. 589-593

Karpinski, L. C., 75 n. a
INDEX

Keil, 43 n. b
Kepler, his investigation of semi-regular solids, ii. 197 n. b; on planets, 411 n. b
Koppa, used as numeral, 43
Kubitschek, 35 n. b
Laird, A. G., on Plato's nuptial number, 399 n. b
Larfeld, W., on Greek alphabet, 42 n. b
Lemmas: Pappus's collection, ii. 607 (incl. n. b) - 613, ii. 565 n. a, 493-503; lemmas to Menelaus's Theorem, ii. 447-459
Leodamas of Thasos: Proclus's notice, 151; Plato said to have communicated method of analysis to him, 151 n. e
Leon: made collection of Elements, and discovered diorismi, 151 and n. h
Line: defined, 437; straight line, 439 and n. a; different species of curved lines ii. 361-363
"Linear" loci and problems, 349
"Linear" numbers, 87 n. a
Loci: line loci, surface loci, and solid loci, 491 - 493 (incl. n. a); Surface Loci of Euclid, 349 and n. a, 487 n. a, ii. 601; Solid Loci of Aristaeus, 487, ii. 601
Locus with respect to three or four lines, 487-489; with respect to five or six lines, ii. 601-603
Logistic, 7, 17-19

Loria, G.: his Le scienze esatte nell' antica Grecia, 47 n. a, 109 n. a, 139 n. b, 271 n. a
Lucian, 91 n. a
Mamercus, 147 n. b
Marinus, his commentary on Euclid's Data, ii. 349, ii. 565 n. a
Mathematics, meaning of the term, 3, 401 n. a
Means: three Pythagorean means — arithmetic, geometric, subcontrary, 111-115; subcontrary renamed harmonic, 113; three subcontrary means added, 119-121 (incl. n. a); five further means discovered, 121-125; Pappus's equations between means, 125-129; Plato on means between two squares or two cubes, 129-131; duplication of cube reduced to problem of two mean proportions, 259 and n. b; solutions, 261-309; Eratosthenes' work On Means, ii. 263; loci with reference to means, ii. 265 and n. a; representation of means by lines in a circle, ii. 569-571 (incl. n. a)
Mechanics: Pappus on branches of mechanics, ii. 615-621; principle of lever, 431-433, ii. 209-217; parallelogram of velocities, 433; centres of gravity, ii. 207-221; five mechanical
powers, ii. 489 n. a; Archimedes’ mechanical inventions, ii. 19-31; movement of larger weight by smaller by use of engaging wheels, ii. 489-497; Plato’s dislike of mechanical solutions, 263 n. b; Archimedes’ preference for theoretical mathematics, ii. 31, ii. 619; pseudo-Aristotelian Mechanics, 431-433; Archimedes’ On Plane Equilibriums, ii. 207-221; his mechanical method in mensuration, ii. 221-229, ii. 233-239; Heron’s Mechanics, ii. 467 n. a, ii. 489 (incl. n. a) -497

Menaechmus: solution of problem of two mean proportionals, 261, 279-283; discovered the conic sections, 283 n. a, 297 and n. b, ii. 281 n. a

Menelaus of Alexandria: made an observation under Trajan, ii. 407 n. a; discovered a curve called “paradoxical,” 348 -349 (incl. n. c); his Sphaerica, ii. 407 n. a, ii. 463 n. a; drew up a table of sines, ii. 407; “Menelaus’s Theorem,” ii. 459-463

Mensuration: area of triangle given the sides, ii. 471-477; volume of spire, ii. 477-483; measurement of irregular area, ii. 485-489; area of parabolic segment, ii. 223-229, ii. 243

Meton, 309 and n. a
Metrodorus, ii. 513 n. a
Minus, Diophantus’s sign for, ii. 525
Moeris, 175

Multiplication in Greek mathematics, 48; Apollonius’s continued multiplications, ii. 353-357

Music: theoretical music (canonic) included by Greeks in mathematics, x, 3 n. b, 19 n. b; full discussion excluded, x; included in Pythagorean quadrivium, 5; in Plato’s curriculum for the Guardians, 17; popular music, 3, 19

Musical interval, 3 n. b, 113 n. b, 175

Myriad; notation, 44-45; orders and periods of myriads in Archimedes’ notation, ii. 199-201

Nagl, A., 35 n. b
an-Nairizî, Arabic commentator on Euclid’s Elements, 185 n. b, ii. 467 n. a
Neoclides, 151 and n. h
Nesselmann, G. F., his Die Algebra der Griechen, 139 n. b, ii. 513 n. a, ii. 517 n. b
Newton, Sir Isaac, ii. 343 n. b
Nicolas Rhabdas: v. Rhabdas

Nicomachus of Gerasa, 67 n. b, 69 nn. a and c, 75 n. a, 79 n. a, 81 n. a, 87 n. a, 95 n. b, 101 n. a, 103 n. a, 105 n. a, 107 n. a, 121 n. a, 123 n. c

641
**INDEX**

Nicomedes: date, 297 n. c; his conchoid and its use for finding two mean proportions, 297-309

Nix, L., ii. 285 n. a

Number: defined by Euclid, 67; by Nicomachus, 73; fundamental concepts, 67-71; Plato on nature of number, 7; his nuptial number, 399; on generation of numbers, 401-405; odd and even numbers, 67-69, 101, 391; prime numbers, 69, 87 n. a, 101-103; classification of numbers, 71, 73-75; perfect numbers, 71, 75-87; figured numbers, 87-99; some properties of numbers, 101-109; Diophantus on sums of squares, ii. 551-559; on polygonal numbers, ii. 515, ii. 561

Numerals, Greek, 41-45

"Oblong" numbers, 95 (incl. n. b)

Octads, in Archimedes’ system of enumeration, ii. 199-201

Octahedron, one of the five regular solids, 217, 221 and n. d

Odd and even numbers: v. Number

Oenopides of Chios, 149 and n. c

Olympiodorus, proof of equality of angles of incidence and reflection, ii. 503 n. a

Optics: Euclid’s treatise, 157 n. c, 503 n. b; his theorem about apparent sizes of equal magnitudes, 503-505; Ptolemy’s treatise, ii. 411 and n. a, ii. 503 n. a; Heron’s treatise, ii. 503 n. a; his proof of equality of angles of incidence and reflection, ii. 497-503; Damianus’s treatise, ii. 497 n. a

Pamphila, 167-169 (incl. n. a)

Pappus of Alexandria:

Life: Suidas’s notice, ii. 565-567; date, ii. 565 and n. a

Works:

*Synagoge or Collection,* 145 n. a, ii. 565 n. a; contents—Apollonius’s continued multiplications, ii. 353-357; problems and theorems, ii. 567-569; theory of means, ii. 569-571; equations between means, 125-129; paradoxes of Erycinus, ii. 571-573; regular solids, ii. 573-575; extension of Pythagoras’s theorem, ii. 575-579; circles inscribed in the *άρβηλος*, ii. 579-581; duplication of the cube, 299-309; squaring of the circle, 337-347; spiral on a
INDEX

sphere, ii. 581-587; trisection of an angle, 347-363; isoperimetric figures, ii. 589-593; apparent form of a circle, ii. 593-597; "Treasury of Analysis," ii. 597, ii. 599-601, 479 n. a, 481 n. c, 491 n. c, ii. 263, ii. 337; on the works of Apollonius, ii. 337-347, 487-489; locus with respect to five or six lines, ii. 601-603; anticipation of Guldin's theorem, ii. 605-607; lemmas to the Determinate Section of Apollonius, ii. 607-611; to the Porisms of Euclid, ii. 611-613; to the Surface Loci of Euclid, 493-503; mechanics, ii. 615-621
Commentary on Ptolemy's Syntaxis, 48, ii. 409 n. b, ii. 565-567, ii. 617 n. d
Commentary on Euclid's Elements, ii. 565 n. a, 457 n. a
Commentary on Analemma of Diodorus, 301, ii. 565 n. a
Other works, ii. 565 (incl. n. a), 567
Paradoxes of Erycinus, ii. 571-573
Parallelogram of velocities, 433
Parmenides, 367 n. a

Pebbles, used for calculating, 35
Pentagon, regular, 223 n. b
Pentagram, Pythagorean figure, 225 and n. b
"Perfect" numbers, 71, 75-87
Perseus: life, ii. 365 n. a; discovered spiric curves, ii. 363-365
Phenomena of Eudoxus and Aratus, ii. 407 n. a; for Euclid's Phenomena, v.
Euclid: Works
Philippus of Opus (or Medma), 155 and n. a
Philolaus, ii. 3 n. a
Philon of Byzantium: solution of problem of two mean proportionals, 263 n. a, 267 n. b; cube root, 63 n. a
Philoponus, Joannes, on quadrature of lunes, 311 n. b
Planisphaerium: v. Ptolemy
Plato: his inscription over the doors of the Academy, 387 and n. b; his belief that "God is for ever playing the geometer," 387; his dislike of mechanical constructions, 263 n. b, 389; his identification of the Good with the One, 389; his philosophy of mathematics, 391-393; the problem in the Meno, 395-397; the Nuptial Number in the Republic, 399; on the generation of numbers, 401-405; his reported solution of the problem of

643
INDEX

two mean proportionals, 263-267; his application of analysis to theorems about "the section," 153 and n. a; his knowledge of side- and diameter-numbers, 137 n. a, 399 n. c; his formula for the sides of right-angled triangles, 93-95, his definition of a point, ii. 469 n. a; his use of the term mathematics, 3 n. c, 401 n. a; his curriculum for the Guardians in the Republic, 7-17; Proclus's notice, 151; taught by Theodorus, 151 n. b; his pupils —Eudoxus, 151, Amyclas of Heraclea, 153, Menaechnmus, 153, Philippus, 155; Euclid said to have been a Platonist, 157 and n. b; modern works on Plato's mathematics, 387 n. a, 397 n. a, 399 n. a, 405 n. b

"Platonic" figures: v. Regular Solids

"Playfair's Axiom," ii. 371 n. c

Plutarch: on parallel section of a cone, 229 and n. a; on Plato's mathematics, 263 n. b, 387-389

Point, 437, ii. 469 and n. a

Polygonal numbers, 95-99, ii. 396 n. a, ii. 515, ii. 561

Pontus, paradoxical eclipses near, ii. 401

Porism, 479-481

Porisms: v. Euclid: Works, and Diophantus of Alexandria

Posidonius: life and works, ii. 371 n. b; on the size of the earth, his definition of parallels, ii. 371-373

Postulates: v. Axioms and postulates

Prime numbers: v. Number

Problems: plane, solid, linear, 349; distinction between problems and theorems, ii. 567

Proclus: life, 145 n. a; his Commentary on Euclid i., 145 n. a; summary of history of geometry, 145-161; his Commentary on Plato's Republic, 399 n. a; his criticism of the parallel-postulate, ii. 367-371; his attempt to prove the parallel-postulate, ii. 383-385

Proof: Aristotle on rigour of geometrical proof, ii. 369; Archimedes on mechanical and geometrical proofs, ii. 221-223, ii. 229

Proportion: Pythagorean theory of proportion and means, 111-125, 149; Eu- doxus's new treatment applicable to all magnitudes, 409 and 447 n. a; Euclid's treatment, 445-451; v. also Means


Psellus, Michael, 3 n. a, ii. 515 and n. c, ii. 517 n. a

Pseudaria: v. Euclid: Works

Pseudo-Eratosthenes, letter to Ptolemy Euergetes, 257-261
INDEX

Ptolemies: Euergetes I, 257 and n. a, ii. 261 and n. c; Philopator, 257 n. a, 297; Epiphanes, ii. 261 and n. d

Ptolemy, Claudius: life and works, ii. 409 and n. b; his work On Balancings, ii. 411; his Optics, ii. 411 and n. a, ii. 503 n. a; his book On Dimension, ii. 411-413; his Syntaxis (Great Collection, Almagest), ii. 409 and n. b; commentaries by Pappus and Theon, 48 and n. a, ii. 409 n. b, ii. 565-567, ii. 621 n. a; his construction of a table of sines, ii. 413-445; his Planisphaerium, ii. 565 n. a; his Analemma, 301 n. b, ii. 409 n. b; his value for \( \sqrt{3} \), 61 n. b; on the parallel-postulate, ii. 373-383

Pyramid: v. Tetrahedron

Pythagoras: life, 149 n. b, 173; called geometry inquiry, 21; transformed study of mathematics into liberal education, 149; discovered theory of proportionals (possibly irrationals), 149; discovered musical intervals, 175; sacrificed an ox on making a discovery, 169 n. a, 175, 177, 185; “Pythagoras’s Theorem,” 179-185; extension by Pappus, ii. 575-579

Pythagoreans:

General: use of name mathematics, 3; esoteric members called mathematicians, 3 n. d; Archytas a Pythagorean, 5; Pythagorean quadrivium, 5 and n. b, 7 n. a; stereometrical investigations, 7 n. a; declared harmony and astronomy to be sister sciences, 17 and n. b; how geometry was divulged, 21; views on monad and undetermined dyad, 173; their motto, 175-177

Pythagorean arithmetic: first principles, 67-71; classification of numbers, 73-75; perfect numbers, 75-83; figured numbers, 87-99; some properties of numbers, 101-109; irrationality of \( \sqrt{2} \), 111 n. a; ancient Pythagoreans recognized three means, 111-115; later Pythagorean list of means, 115-125; Archytas’s proof that a superparticular ratio cannot be divided into equal parts, 131-133; developed theory of side- and diameter-numbers, 137-139; the “bloom” of Thymaridas, 139-141

Pythagorean geometry: definition of point, ii. 469; sum of angles of a triangle, 177-179; “Pythagoras’s Theorem,” 179-185; application of areas, 187-215; the irra-
INDEX

tional, 215-217; the five regular solids, 217, 219 n. a, 223-224; Archytas's solution of the problem of two mean proportionals, 285-289; squaring of the circle, 335
Otherwise mentioned: 245 n. a, 403 nn. c and d

Quadratic equations: v. Algebra; Quadratic equations
Quadratrix: Quadratic equations discovered by Hippias, 149 n. a, 337 n. a; used to square the circle, 337-347
Quadrivium, Pythagorean, 5 and n. b, 7 n. a
Quinary system of numerals, 31 n. a

Reduction of a problem: Proclus's definition, 253; Hippocrates' reduction of problem of doubling the cube, 253 and n. a
Reflection, equality of angles of incidence and reflection, ii. 497-503

Regular solids ("Platonic" or "cosmic" figures); definition and origin, 217 n. a; Pythagorean treatment, 217-225; Speusippus's treatment, 77; work of Theaetetus, 379 and n. a; Plato's use in Timaeus, 219-223; Euclid's treatment, 157 and n. b, 467-479; Pappus's treatment, ii. 573-575 (esp. n. a); Eva Sachs' work, 217 n. a, 379 n. c
Rhabdas, Nicolas Artavasdas, 31 n. b; his finger-notation, 31-35
Right-angled triangle: square on hypotenuse equal to sum of squares on other two sides, 179-185; extension by Pappus, ii. 575-579
Roberts, Colin, 45 n. a
Robertson, D. S., ii. 479 n. a
Rome, A., ii. 565 n. a
Ross, W. D., 95 n. b, 367 n. a, 371 n. b
Rudio, F., 237 n. b

Saccheri, Euclides ab omni naevo vindicatus, 443 n. c
Sachs, Eva, Die fünf Platonischen Körper, 217 n. a, 379 n. c; De Theaeteto Atheniensi, 379 n. a
Salaminian table, 35 n. b, 37 n. a
Sampi: use as numeral, 43
Schiaparelli, G., 411 n. b, 415 n. c
Schöne, H.: discovered Heron's Metrica, ii. 467 n. a
Schöne, R.: edited Heron's Metrica, ii. 467 n. a
Scopinas of Syracuse, ii. 3 n. a
Sectio canonis: v. Euclid: Works
Semi-circle: angle in semicircle a right angle, 167-169
Serenus: commentary on
INDEX

Apollonius's Conics, ii. 285 n. a

Sexagesimal system of numerals, 48-49

"Side-" and "diameter-numbers," 133-139, esp. 137 n. a, 399 n. c

Simon, M., Euclid und die sechs planimetrischen Bücher, 447 n. a

Simplicius: on Hippocrates' quadratures, 235-253, 310 n. b, 313 n. a; on Ptolemy, ii. 409 n. b; on oblong and square numbers, 95 n. b; on planets, 411 and n. b

Simson, R., ii. 345 n. c

Sines, Table of: v. Chords, Table of "Solid" loci and problems, 349, 487, ii. 601

"Solid" numbers: v. Number: Figured numbers

Solids, five regular solids: v. Regular solids

Solon, 37

Sophocles, ix, 259 n. a

Speusippus, 75-77, 81 n. a

Sphaeric (geometry of sphere), 5 and n. b, 407 n. a

Sphaerica: work by Theodorus, ii. 407 n. a; by Menelaus, ii. 407 n. a, ii. 462 n. a

Spiral: of Archimedes, ii. 183-195, on a sphere, ii. 581-587

Spiric sections: v. Tore

Sporus, on the quadratrix, 339-341

Square numbers: divisibility of squares, 103-105; sums of squares, ii. 551-559

Square root, extraction of, 53-61

Squaring of the circle: v. Circle

Stadium of Zeno, 371-375

Star-pentagon: v. Pentagon

Stenzel, Julius: his Zahl und Gestalt bei Platon und Aristoteles, 405 n. b

Stigma: use as numeral, 43

Subcontrary mean, 113-117, 117-119, 123

Subcontrary section of cone, ii. 301 n. a

Surface Loci: v. Euclid: Works

Synthesis, Pappus's discussion, ii. 597-599

Table of Chords: v. Chords, Table of

Tannery, P., 77 n. b, 237 n. b, 261 n. b, 335 n. d, 493 n. b, ii. 513 n. a, ii. 517 nn. a and c, ii. 523 n. b, ii. 537 n. b

Taylor, A. E., 23 n. b, 27 n. b, 115 n. b, 223 n. c, 405 n. b, 427 n. b

Tetrads, in Apollonius's system of enumeration, ii. 355 n. a

Tetrahedron (pyramid), 83, 217 n. a, 221 n. c, 379

Thales: life, 147 n. a; Proclus's notice, 147; discovered that circle is bisected by its diameter, 165 and n. a; that the angles
INDEX

at the base of an isosceles triangle are equal, 165; that the vertical and opposite angles are equal, 167; that the angle in a semi-circle is a right angle, 167-169; his method of finding distance of ships at sea, 167 and n. b
Theaetetus: life, 379 and n. a; Proclus's notice, 151; Suidas's notice, 379; his work on the regular solids, 379 and n. c; on the irrational, 381-383
Themistius: on quadrature of lunes, 310 n. b; on oblong and square numbers, 95 n. b
Theodorus of Cyrene: life, 151 n. b, 381 n. b; Proclus's notice, 151; his proof of incommensurability of $\sqrt{3}, \sqrt{5} \ldots \sqrt{17}$, 381 and n. c
Theodorus: his Sphaerica, ii. 407 n. a
Theodorus I, Emperor, ii. 565 and n. a
Theon of Alexandria: flourished under Theodosius I, 48 n. a, ii. 565; father of Hypatia, 48; to be distinguished from Theon of Smyrna, ii. 401 n. a; his commentary on Ptolemy's Syntaxis, 48, ii. 409 n. b, ii. 565-567, ii. 621 n. a
Theon of Smyrna: referred to as "Theon the mathematician" by Ptolemy, ii. 401; made observations under Hadrian, ii. 401 and n. a; referred to as "the old Theon" by Theon of Alexandria, ii. 401 n. a
Theophrastus, 217 n. c
Theorems, distinction from problems, ii. 567
Theodius of Magnesia, 153
Thymaridas: an early Pythagorean, 139 n. b; his "bloom," 139-141; on prime numbers, 87 n. a
Timaeus of Locri, 219 n. a
Tore (also spire, or anchoring): defined, ii. 365, ii. 469; used by Archytas for the doubling of the cube, 285-289; sections of (Perseus), ii. 363-365; volume of (Dionysodorus and Heron), ii. 477-483
Treasury of Analysis: v.
Pappus: Works
Triangle: angles at base of an isosceles triangle equal, 165 and n. b; equality of triangles, 167 and n. b; sum of angles equal to two right angles, 177-179, ii. 279; "Pythagoras's Theorem," 179-185; extension by Pappus, ii. 575-579; Heron's formula $\sqrt{s(s-a)(s-b)(s-c)}$ for area, ii. 471-477, ii. 603 n. a
Triangular numbers, 91-95
Trigonometry: origins in sphaeric, ii. 407 n. a; developed by Hipparchus and Menelaus, ii. 407 and n. a; their tables of sines, ii. 407 n. a, ii. 409 and n. a;
INDEX

Ptolemy’s terminology, ii. 421 n. a; his construction of a table of sines, ii. 413-445; value of sin 18° and sin 36°, ii. 415-419, ii. 421 n. a; of sin 60°, sin 90°, sin 120°, ii. 419-421 (incl. n. a); proof of sin²θ + cos²θ = 1, ii. 421-423; of “Ptolemy’s Theorem,” ii. 423-425; of sin (θ - φ) = sin θ cos φ - cos θ sin φ, ii. 425-427; of sin²θ = ½(1 - cos 2θ), ii. 429-431; of cos (θ + φ) = cos θ cos φ - sin θ sin φ, ii. 431-435; a method of interpolation, ii. 435-443; the table of sines, ii. 443-445; “Menelaus’s Theorem” and lemmas, ii. 447-463; proof that tan a : tan β < α : β, 503-505 (incl. n. a), ii. 11 n. b, ii. 389 and n. b; proof that sin α : sin β < α : β, ii. 439 n. a

Trisection of an angle: a “solid” problem, 353; solution by means of a verging, 353-357; direct solution by means of conics, 357-363; Pappus’s collection of solutions, ii. 581 n. b

Unit, 67, 73, 89-91, ii. 469, ii. 515
Usener, 237 n. b

Valckenaer, 259 n. a
Ver Eecke, Paul, ii. 19 n. a, ii. 285 n. a, ii. 513 n. a
Vergings: Definition, 244

n. a; use by Hippocrates of Chios, 245; by Pappus, 355-357; by Archimedes, ii. 189 and n. c; Apollonius’s treatise, ii. 345-347, ii. 337
Vieta’s expression for 2/π, 315 n. a
Vitrivius, ii. 3 n. a, ii. 251 n. a
Viviani’s curve of double curvature, 349 n. c

Wescher, C., his Poliorcé-tique des Grecs, 267 n. b
Wilamowitz - Moellendorf, U. v., 257 n. a
William of Moerbeke; his trans. of Archimedes’ On Floating Bodies, ii. 242 n. a, ii. 245 n. b; De Speculis, ii. 503 n. a

Xenocrates, 75, 425 n. b
Zeno of Elea: life, 367 n. a; a disciple of Parmenides, ii. 367 n. a; four arguments on motion, 367-375
Zenodorus, ii. 387 n. b; on isoperimetric figures, ii. 387-397
Zero, 47 and n. a
Zeuthen: Die Lehre von den Kegelschnitten im Altertum, 493 n. b, ii. 265 n. a, ii. 281 n. a; Geschichte der Mathematik, 245 n. a; otherwise mentioned, 381 n. a, 505 n. a
Zodiac circle: division into 360 parts, ii. 395-397; signs of Zodiac, 223 n. c
LIST OF ANCIENT TEXTS CITED

The figures in parentheses give the page or pages in this edition where the passage is cited; in the case of vol. i. the reference to the volume is omitted.

Aëtius
   Plac. i. 15. 5 (ii. 2); ii. 6. 5 (216)
Alexander Aphrodisiensis
   In Aristot. Soph. El. 11, 171 b 17 (314-316)
Anatolius
   Ap. Heron, Def., ed. Heiberg 160. 8-162. 2 (2)
   Ap. Heron, Def., ed. Heiberg 164. 9-18 (18)
   Anthology, Palatine
   xiv. 126 (ii. 512)
Apollonius Paradoxographus
   Mirab. 6 (172)
Apollonius of Perga
   Conic. i., Praef., Def., 7-9, 11-14, 50 (ii. 280-334)
Archimedes
   De Sph. et Cyl. i., Praef. (408-410); i., Praef., Def., Post., Props. 1, 2, 3, 5, 8, 9, 10, 12, 13, 14, 16, 21, 23, 24, 25, 28, 29, 30, 33, 34 (ii. 40-126); ii. 4 (ii. 126-134)
   Dim. Circ. (316-332)
   De Con. et Sphaer., Praef.
   Lemma ad Prop. 1, 1, 21 (ii. 164-180)
   De Lin. Spir., Def., 7, 14, 20 (ii. 182-194)
   Quadr. Parab., Praef., 24 (ii. 228-242)
   Aren. (ii. 2-4); 3 (ii. 198-200)
   De Plan. Aequil., Def., 6, 7, 9, 10 (ii. 206-220)
   De Corpor. Fluit. i., Post., 2, 7 (ii. 242-250); ii. 2 (ii. 252-256)
   Meth., Praef. (228, ii. 220-222), 1 (ii. 222-228)
   Prob. Bov. (ii. 202-204)
Archytas
LIST OF ANCIENT TEXTS CITED

Aristarchus of Samos
De Mag. et Dist. Solis et Lunae, Post., 7, 13, 15 (ii. 4-14)
Aristophanes
Aves 1001-1005 (308)
Aristotle
Anal. Pr. i. 23, 41 a 26-27 (110); i. 24, 41 b 5-22 (428-430)
Anal. Post. i. 10, 76 a 30-77 a 2 (418-422)
Phys. A 2, 185 a 14-17 (310); Γ 4, 203 a 13-15 (94); Γ 6, 206 a 9-18 (424); Γ 6, 206 b 3-12 (424-426); Γ 6, 206 b 27-207 a 7 (426-428); Γ 7, 207 b 27-34 (428); Z 9, 239 b 5-240 a 18 (366-374)
Met. A 5, 985 b 23-26 (172); A 5, 987 b 14-18 (394); Λ 8, 1073 b 17-32 (410-412)
Prob. xv. 3, 910 b 23-911 a 1 (28-30)
[Aristotle]
Mech. 1, 848 b (432); 3, 850 a-b (430-432)
Aristoxenus
Harm. ii. ad init., ed. Macran 122. 3-16 (388-390)
Bohiense,
Fragmentum Mathematicum
113. 28-33, ed. Belger, Hermes, xvi., 1881, 279-280 (ii. 356)
Boethius
De Inst. Mus. iii. 11 (130-132)
Cleomedes
De Motu Circ. i. 10. 52 (ii. 266-272); ii. 6 (ii. 396-400)
Damianus
Opt. 14 (ii. 496)
Diodorus Siculus
i. 34. 2 (ii. 34); v. 37. 3 (ii. 34)
Diogenes Laertius
De Clar. Phil. Vitis i. 24-25 (166-168), i. 59 (36); viii. 11-12 (174), vii. 24-25 (172-174)
Diophantus of Alexandria
Arithmetica i., Praef. (ii. 516, ii. 518-524), i. 28 (ii. 536); ii. 8 (ii. 550-552), ii. 20 (ii. 540-542); iv. 18 (ii. 548-550), iv. 29 (ii. 556-558), iv. 32 (ii. 542-546), iv. 39 (ii. 526-534); v. 3 (ii. 516), v. 11 (ii. 552-556); vi. 17 (ii. 538-540)
De Polyg. Num., Praef. (ii. 560), 5 (ii. 514)
Euclid
Elem. i., Def., Post. et Comm. Notit. (436-444); i. 44 (188-190); i. 47 (178-184); ii. 5 (192-194); ii. 6 (196); ii. 11 (198-200); v., Def. (444-450); vi. 27 (202-204); 651
LIST OF ANCIENT TEXTS CITED

vi. 28 (204-210); vi. 29 (210-214); vii., Def. (66-70); x., Def. (450-452); x. 1 (452-454); x. 111, coroll. (456-458); xii. 2 (458-464); xiii. 18 (466-478)

Dat., Def. (478)
Opt. 8 (502-504)

Eutocius

In Archim. De Sph. et Cyl.
ii., ed. Heiberg 54. 26-56. 12, 56. 13-58. 14, 58. 15-16, 66. 8-70. 5, 78. 13-80. 24, 84. 12-88. 2, 88. 3-96. 27, 98. 1-7 (256-298); 130. 17-150. 22 (ii. 134-162)

In Archim. Dim. Circ., ed. Heiberg 242 ad init. (48); 258. 16-22 (ii. 352)

In Apollon. Conic., ed. Heiberg 168. 5-170. 26 (ii. 276-280)

Herodotus

Histor. ii. 36. 4 (34)

Heron of Alexandria

Metrica i. 8 (ii. 470-476); i. 22 (ii. 406); ii. 13 (ii. 476-482); iii. 18 (ii. 482-484); iii. 20 (60-62)

Definitiones, ed. Heiberg 14. 1-24 (ii. 466-468); 60. 22-62. 9 (ii. 468-470); 160. 8-162. 2 (2); 164. 9-18 (18)

Dioptra 23 (ii. 484-488); 36 (ii. 272); 37 (ii. 488-496)

Geom. 21. 9-10, 24. 1, 24. 10 (ii. 502-508)

Hypsicles

[Euclid,] Elem. xiv, Eucl. ed. Heiberg v. 6. 19-8. 5 (ii. 348)

Iamblichus

De Vita Pyth. 18. 88 (222-224), 18. 89 (30)


Theol. Arith., ed. de Falco 82. 10-85. 23 (74-82)

Isocrates

Panathen. 26-28, 238 b-d (26-28)

Lucian

Vit. Auct. 4 (90)

Marinus

Comm. in Eucl. Dat., Eucl. ed. Heiberg vi. 234. 13-17 (219)

Michigan Papyri

No. 145, vol. iii. p. 36 (48)

Nicomachus

Arith. Introd. i. 7 (72-74); i. 13. 2-4 (100-102); ii. 7. 1-3 (86-90), ii. 12. 2-4 (94-98), ii. 20. 5 (104-106), ii. 28. 3-11 (114-124)

Olympiodorus

In Aristot. Meteor. iii. 2 (ii. 496-502)

Pappus of Alexandria

Coll. ii. 25. 17-21 (ii. 352-
LIST OF ANCIENT TEXTS CITED

356); iii., Praef. 1 (ii. 566-568), ii. 9. 26 (266-270), iii. 11. 28 (ii. 568-570), iii. 18. 48 (124-126), iii. 23. 57 (126-128), iii. 24. 58 (ii. 570-572), iii. 40. 75 (ii. 572-574); iv. 1. 1 (ii. 574-578), iv. 19 (ii. 578-580), iv. 30. 45-32. 50 (336-346), iv. 35. 53-56 (ii. 580-586), iv. 36. 57-59 (346-354), iv. 38. 62 (354-356), iv. 43. 67-44. 68 (356-362); v., Praef. 1-3 (ii. 588-592); vi. 48. 90-91 (ii. 592-596); vii., Praef. 1-3 (ii. 596-600), vii. 3 (ii. 336), viii. 5-6 (ii. 336-338), viii. 7 (ii. 388), viii. 9 (ii. 338-340), viii. 11 (ii. 340-342), viii. 13-20 (480-484), viii. 21 (ii. 262-264); vii. 23 (ii. 344), vii. 27-28 (ii. 344-346), vii. 30-36 (486-488), vii. 312-316 (492-502); viii., Praef. 1-3 (ii. 614-620), viii. 11. 19 (ii. 34)

Philolaus
Ap. Stob. Ecl. i., proem. 3 (216); i. 21. 7c (72)

Philoponus
In Aristot. Phys. A 2 (234)

Plato
Meno 86 e-87 b (394-396)
Resp. vi. 510 c-e (390); vii. 525 a-530 d (6-16); viii. 546 b-d (398)

Tim. 31 b-32 b (128-130), 53 c-55 c (218-222)
Theaet. 147 d-148 b (380-382)
Leg. vii. 817 e-820 d (20-26)
Epin. 990 c-991 b (400-404)
Ep. vii. 342 a-343 b (390-394)

Plutarch
Non posse suav. viv. sec. Epic. 11, 1094 b (176)
Quaest. Conv. viii. 2. 1 (386-388); viii. 2. 4, 720 a (176)
De Facie in Orbe Lunae 6, 922 f-923 a (ii. 4)
Marcellus xiv. 7-xvii. 7, xix. 4-6 (ii. 22-34)
De Comm. Notit. 39. 3, 1079 e (228)
De Exil. 17, 607 e, f (308)

Polybius
Histor. v. 26. 13 (36)

Porphyry
In Ptol. Harm., ed. Wallis, Opera Math. iii. 267, 39-268. 9; Diels, Vors. i*. 334. 16-335. 13 (112-114)

Proclus
In Eucl. i., ed. Friedlein 64. 16-70. 18 (144-160), 74. 23-24 (ii. 350), 84. 13-23 (174-176), 105. 1-6 (ii. 350), 157. 10-13 (164), 176. 5-10 (ii. 370-372), 212. 24-213. 11 (252), 250. 22-251. 2 (164-166), 299. 1-5 (166), 301. 21-302. 13 (478-
LIST OF ANCIENT TEXTS CITED

In Aristot. Phys. A 2, 185 a 14 (312-314, 234-252)

In Aristot. Cat. 7 (334)

Stobaeus

Ecl. i., proem. 3 (216); i. 21. 7c (72); ii. 31. 114 (436)

Suidas

s.vr. Εὐρασοθένης (ii. 260-262), Θειάτης (378), Πάππος (ii. 564-566), Πτολεμαῖας (ii. 408)

Themistius

In Aristot. Phys. A 2, 185 a 14 (310-312)

Theon of Alexandria

In Ptol. Math. Syn. i. 3, ed. Rome 354. 19-357.22 (ii. 386-395); i. 10, 451. 4-5 (ii. 406); i. 10, 453. 4-6 (ii. 514); i. 10, 461. 1-462. 17 (50-52); i. 10, 469. 16-473. 8 (52-60)

Theon of Smyrna

Ed. Hiller 1. 1-2. 2 (ii. 400-402), 2. 3-12 (256), 35. 17-36. 2 (102-104), 42. 10-44. 17 (132-136), 45. 9-46. 19 (84-86), 81. 17-82. 5 (ii. 264-266)

Tzetzes

Chil. ii. 103-144 (ii. 18-22); viii. 972-973 (386)

Vitruvius

De Archit. ix., Praef. 9-12 (ii. 36-38)
INDEX OF GREEK TERMS

The purpose of this index is to give one or more typical examples of the use of Greek mathematical terms occurring in these volumes. Non-mathematical words, and the non-mathematical uses of words, are ignored, except occasionally where they show derivation. Greek mathematical terminology may be further studied in the Index Graecitatis at the end of the third volume of Hultsch's edition of Pappus and in Heath's notes and essays in his editions of Euclid, Archimedes and Apollonius. References to vol. i. are by page alone, to vol. ii. by volume and page. A few common abbreviations are used. Words should be sought under their principal part, but a few cross-references are given for the less obvious.

'Αγεν, to draw; εὐθεῖαν γραμμήν ἀγεν, to draw a straight line, 442 (Eucl.); ἣν ἐπιψαῖνονα ἀχθων, if tangents be drawn, ii. 64 (Archim.); παράλληλος ἢκθω ἢ ΑΚ, let ΑΚ be drawn parallel, ii. 312 (Apollon.)

ἀγωνίμετρητος, οὐ, ignorant of or unversed in geometry, 386 (Tzetzes)

ἀδιάλειπτος, οὐ, undivided, indivisible, 366 (Aristot.)

ἀδύνατος, οὐ, impossible, 394 (Plat.), ii. 566 (Papp.);

ἀπέρ ἐστιν ἄ., often without ἐστιν, which is impossible, a favourite conclusion to reasoning based on false premises, ii. 122 (Archim.);

οἱ διὰ τοῦ ἀ. περαινοντες, those who argue per impossibile, 110 (Aristot.)

ἀθροισμα, ἄτος, τὸ, collection;

ἄ. φιλοτεχνότατον, a collection most skilfully framed, 480 (Papp.)

Αἰγυπτιακός, ἡ, ὁ, Egyptian;

αἱ Αἱ. καλομεναι μέθοδοι ἐν πολλαπλασιασμοί, 16 (Schol. in Plat. Charm.)

ἀιρέω, to take away, subtract, ii. 506 (Heron)

αἴτειν, to postulate, 442 (Eucl.), ii. 206 (Archim.)

ἀιτήμα, ἄτος, τὸ, postulate, 420 (Aristot.), 440 (Eucl.), ii. 366 (Procl.)

άκινητος, οὐ, that cannot be moved, immobile, fixed, 394 (Aristot.), ii. 246 (Archim.)

ἀκολουθεῖν, to follow, ii. 414 (Ptol.)
INDEX OF GREEK TERMS

ἀκόλουθος, ov, following, consequential, corresponding, ii. 550 (Papp.); as subst., ἀκόλουθον, τό, consequence, ii. 566 (Papp.)

ἀκόλουθος, adv., consistently, consequentially, in turn, 458 (Eucl.), ii. 384 (Procl.)

ἀκονσματικός, ἕν, ὁν, eager to hear; as subst., ἂ, ὁ, hearer, exoteric member of Pythagorean school, 3 n. d (Iamb.)

ἀκριβής, ἔσ, exact, accurate, precise, ii. 414 (Ptol.)

ἀκρος, α, ov, at the farthest end, extreme, ii. 270 (Cleom.); of extreme terms in a proportion, 122 (Nicom.); ἄ κα μένοις λόγος, extreme and mean ratio, 472 (Eucl.), ii. 416 (Ptol.)

ἀλλος, alternatively, 356 (Papp.)

ἀλογος, ov, irrational, 420 (Aristot.), 452 (Eucl.), 456 (Eucl.); δ' ἀλόγον, by irrational means, 388 (Plut.)

ἀμβλυώνον, ov, obtuse-angled; ἄ. τρίγωνον, 440 (Eucl.); ἄ. κώνων, ii. 278 (Eutoc.)

ἀμβλύς, εἶα, ὑ, obtuse; ἄ. γωνία, often without γωνία, obtuse angle, 438 (Eucl.)

ἀμεράθετος, ov, unaltered, immutable; μονάδος ἀ. οὐσία, ii. 514 (Dioph.)

ἀμήχανος, ov, impracticable, 298 (Eutoc.)

ἀμφοιμα, ατος, τό, revolving figure, ii. 604 (Papp.)

ἀμφοιμικός, ἕν, ὁν, described by revolution; ἀμφοιμικόν, τό, figure generated by revolution, ii. 604 (Papp.)

ἀναγράφεσ, to describe, construct, 180 (Eucl.), ii. 68 (Archim.)

ἀνακλάν, to bend back, incline, reflect (of light), ii. 496 (Damian.)

ἀνάλημμα, ατος, τό, a representation of the sphere of the heavens on a plane, analemma; title of work by Diodorus, 300 (Papp.)

ἀναλογία, ἕν, proportion, 446 (Eucl.); κυρίως ἄ. καὶ πρῶτη, proportion par excellence and primary, i.e., the geometric proportion, 125 n. a; συνεχής ἄ., continued proportion, 262 (Eutoc.)

ἀναλογος, adv., proportionally, but nearly always used adjectivally, 70 (Eucl.), 446 (Eucl.)

ἀναλύειν, to solve by analysis, ii. 160 (Archim.); ὁ ἀναλύομενος τόπος, the Treasury of Analysis, often without τόπος, e.g., ὁ καλούμενος ἀναλυόμενος, ii. 596 (Papp.)

ἀνάλυσις, εως, ἕν, solution of a problem by analytical methods, analysis, ii. 596 (Papp.)

ἀναλυτικός, ἕν, ὁν, analytical, 158 (Procl.)
<table>
<thead>
<tr>
<th><strong>Greek Term</strong></th>
<th><strong>English Translation</strong></th>
<th><strong>Page Reference</strong></th>
<th><strong>Author</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>ἀναμέτρησις, εώς, ἦ, measurement; Περὶ τῆς ἀ. τῆς γῆς, title of work by Eratosthenes, ii. 272 (Heron)</td>
<td>measurement; of the earth, ii. 272 (Heron)</td>
<td>272 (Heron)</td>
<td>Eratosthenes</td>
</tr>
<tr>
<td>ἀνάπαλν, adv., in a reverse direction; transformation of a ratio known as inverting, 448 (Eucl.)</td>
<td>reverse direction; transformation of a ratio, ii. 448 (Eucl.)</td>
<td>448 (Eucl.)</td>
<td>Eratosthenes</td>
</tr>
<tr>
<td>ἀναποδεικτικῶς, adv., independently of proof, ii. 370 (Procl.)</td>
<td>independently of proof, ii. 370 (Procl.)</td>
<td>370 (Procl.)</td>
<td>Eratosthenes</td>
</tr>
<tr>
<td>ἀναστρέφειν, to convert a ratio according to the rule of Eucl. v. Def. 16; ἀναστρέψαντι, lit. to one who has converted, converting, 466 (Eucl.)</td>
<td>to convert a ratio, ii. 466 (Eucl.)</td>
<td>466 (Eucl.)</td>
<td>Euclid</td>
</tr>
<tr>
<td>ἀναστροφή, ἦ, conversion of a ratio according to the rule of Eucl. v. Def. 16, 448 (Eucl.)</td>
<td>conversion of a ratio, ii. 448 (Eucl.)</td>
<td>448 (Eucl.)</td>
<td>Euclid</td>
</tr>
<tr>
<td>ἀνεπαίδητος, ov, unperceived, imperceptible; hence, negligible, ii. 482 (Heron)</td>
<td>unperceived, negligible, ii. 482 (Heron)</td>
<td>482 (Heron)</td>
<td>Heron</td>
</tr>
<tr>
<td>ἀνικός, ov, unequal, 444 (Eucl.), ii. 50 (Archim.)</td>
<td>unequal, ii. 50 (Archim.)</td>
<td>50 (Archim.)</td>
<td>Archimedes</td>
</tr>
<tr>
<td>ἀναστάναι, to set up, erect, ii. 78 (Archim.)</td>
<td>to set up, erect, ii. 78 (Archim.)</td>
<td>78 (Archim.)</td>
<td>Archimedes</td>
</tr>
<tr>
<td>ἀντακολουθία, ἦ, reciprocity, 76 (Theol. Arith.)</td>
<td>reciprocity, ii. 76 (Theol. Arith.)</td>
<td>76 (Theol. Arith.)</td>
<td>Theon of Alexandria</td>
</tr>
<tr>
<td>ἀντικείσθαι, to be opposite, 114 (Nicom.); τοιαὶ ἀντικεῖμεναι, opposite branches of a hyperbola, ii. 322 (Apollon.)</td>
<td>to be opposite, ii. 322 (Apollon.)</td>
<td>322 (Apollon.)</td>
<td>Archimedes</td>
</tr>
<tr>
<td>ἀντιπάσχειν, to be reciprocally proportional, 114 (Nicom.); ἀντιπασχόντως, adv., reciprocally, ii. 208 (Archim.)</td>
<td>to be reciprocally proportional, ii. 208 (Archim.)</td>
<td>208 (Archim.)</td>
<td>Archimedes</td>
</tr>
<tr>
<td>ἀξίωμα, atos, τό, axiom, postulate, ii. 42 (Archim.)</td>
<td>axiom, ii. 42 (Archim.)</td>
<td>42 (Archim.)</td>
<td>Archimedes</td>
</tr>
<tr>
<td>ἀξίων, ovos, ὁ, axis; of a cone, ii. 286 (Apollon.); of any plane curve, ii. 288 (Apollon.); of a conic section, 282 (Eutoc.); συνυγείς ἀ., conjugate axes, ii. 288 (Apollon.)</td>
<td>axis, ii. 288 (Apollon.)</td>
<td>288 (Apollon.)</td>
<td>Archimedes</td>
</tr>
<tr>
<td>ἀόριστος, ov, without boundaries, undefined, πλῆθος μονάδων ἅ., ii. 522 (Dioph.)</td>
<td>without boundaries, undefined, ii. 522 (Dioph.)</td>
<td>522 (Dioph.)</td>
<td>Diophantus</td>
</tr>
<tr>
<td>ἀπαγωγή, ἦ, reduction of one problem or theorem to another, 252 (Procl.)</td>
<td>reduction of one problem or theorem, ii. 252 (Procl.)</td>
<td>252 (Procl.)</td>
<td>Proclus</td>
</tr>
<tr>
<td>ἀπάρτιζειν, to make even; οἱ ἀπαρτίζοντες ἄριθμοι, factors, ii. 506 (Heron)</td>
<td>to make even, ii. 506 (Heron)</td>
<td>506 (Heron)</td>
<td>Heron</td>
</tr>
<tr>
<td>ἀπειράκως, in an infinite number of ways, ii. 572 (Papp.)</td>
<td>in an infinite number of ways, ii. 572 (Papp.)</td>
<td>572 (Papp.)</td>
<td>Pappus</td>
</tr>
<tr>
<td>ἀπειρός, ov, infinite; as subst., ἀπειρον, τό, the infinite, 424 (Aristot.); εἰς ἀπειρον, to infinity, indefinitely, 440 (Eucl.); ἐπ᾽ ἅ., ii. 580 (Papp.)</td>
<td>infinite, indefinitely, ii. 580 (Papp.)</td>
<td>580 (Papp.)</td>
<td>Pappus</td>
</tr>
<tr>
<td>ἀπεναντίον, adv. used adjectivally, opposite; αἱ ἃ. πλευραί, 444 (Eucl.)</td>
<td>opposite; ii. 444 (Eucl.)</td>
<td>444 (Eucl.)</td>
<td>Euclid</td>
</tr>
<tr>
<td>ἀπέχειν, to be distant, 470 (Eucl.), ii. 6 (Aristarch.)</td>
<td>to be distant, ii. 6 (Aristarch.)</td>
<td>6 (Aristarch.)</td>
<td>Aristarchus</td>
</tr>
<tr>
<td>ἀπλανής, ἢ, motionless, fixed, ii. 2 (Archim.)</td>
<td>motionless, fixed, ii. 2 (Archim.)</td>
<td>2 (Archim.)</td>
<td>Archimedes</td>
</tr>
<tr>
<td>ἀπλατής, ἢ, without breadth, 436 (Eucl.)</td>
<td>without breadth, ii. 436 (Eucl.)</td>
<td>436 (Eucl.)</td>
<td>Euclid</td>
</tr>
<tr>
<td>ἀπλόος, ἦ, ov, contr. ἀπλοῦς, ἦ, ὁν, simple; ἡ. γραμμή, ii. 360 (Procl.)</td>
<td>simple, ii. 360 (Procl.)</td>
<td>360 (Procl.)</td>
<td>Proclus</td>
</tr>
<tr>
<td>Term</td>
<td>Page(s)</td>
<td>Source(s)</td>
<td></td>
</tr>
<tr>
<td>-------------------------------------------</td>
<td>--------------------------------------</td>
<td>------------------------------------</td>
<td></td>
</tr>
<tr>
<td>ἀπλῶς, simply, absolutely</td>
<td>424</td>
<td>Aristot., ii. 132 Archim.</td>
<td></td>
</tr>
<tr>
<td>ἀπλωσ, ews, ἴ, simplification, explanation</td>
<td></td>
<td>&quot;A. ἐπι-ϕανείας σφαῖρας, Explanation of the Surface of a Sphere, title of work by Ptolemy, ii. 408 (Suidas)</td>
<td></td>
</tr>
<tr>
<td>ἀνό, from; τὸ ἀπὸ τῆς διαμέτρου τετράγωνον, the square on the diameter,</td>
<td>332</td>
<td>Archim., ii. 566 (Eutoc.)</td>
<td></td>
</tr>
<tr>
<td>ἀποδεικτικός, ἴ, ὑ, affording proof, demonstrative</td>
<td>420</td>
<td>Aristot., 158 (Procl.)</td>
<td></td>
</tr>
<tr>
<td>ἀποδεικτικός, adv., theoretically</td>
<td>260</td>
<td>Eutoc.</td>
<td></td>
</tr>
<tr>
<td>ἀπόδειξις, ews, ἴ, proof, demonstration</td>
<td>ii. 42</td>
<td>Archim., ii. 566 (Papp.)</td>
<td></td>
</tr>
<tr>
<td>ἀποκαθεστάναν, to re-establish, restore</td>
<td></td>
<td>Eutoc.</td>
<td></td>
</tr>
<tr>
<td>ἀπολαμβάνειν, to cut off; ἴ ἀπολαμβανομένη περιφέρεια</td>
<td>440</td>
<td>Eucl.</td>
<td></td>
</tr>
<tr>
<td>ἀπορία, ἴ, difficulty, perplexity</td>
<td>256</td>
<td>Theon Smyr.</td>
<td></td>
</tr>
<tr>
<td>ἀπόστημα, atos, τὸ, distance, interval</td>
<td>ii. 6</td>
<td>Aristarch.</td>
<td></td>
</tr>
<tr>
<td>ἀποστομῇ, ἴ, cutting off, section</td>
<td></td>
<td>Eutoc.</td>
<td></td>
</tr>
<tr>
<td>ἄλογον ἀποστομῇ, Ὀψιαν ἀποστομῇ, works by Apollonius,</td>
<td>598</td>
<td>Papp.</td>
<td></td>
</tr>
<tr>
<td>compound irrational straight line equiva-</td>
<td></td>
<td>Iambi.</td>
<td></td>
</tr>
<tr>
<td>lent to binomial surd with negative sign,</td>
<td></td>
<td>Eucl.</td>
<td></td>
</tr>
<tr>
<td>ἄπτειν, to fasten to; mid., ἄπτεσθαι, to be in contact, meet</td>
<td>438</td>
<td>Eucl., ii. 106 Archim.</td>
<td></td>
</tr>
<tr>
<td>ἄρα, therefore, used for the steps in a proof,</td>
<td>180</td>
<td>Eucl.</td>
<td></td>
</tr>
<tr>
<td>ἄρβηλος, ὁ, semicircular knife used by leather-workers, a geometrical figure used by Archimedes and Pappus,</td>
<td>ii. 578</td>
<td>Papp.</td>
<td></td>
</tr>
<tr>
<td>ἄριθμεῖν, to number, reckon, enumerate</td>
<td>ii. 198</td>
<td>Archim., 90 (Luc.)</td>
<td></td>
</tr>
<tr>
<td>ἄριθμητικός, ἴ, ὑ, of or for reckoning or numbers; ἴ ἄριθμητική (sc. τέχνη), arithmetica, 6 (Plat.)</td>
<td>420</td>
<td>Eucl., ii. 106 Archim.</td>
<td></td>
</tr>
<tr>
<td>ἄριθμητική μέση (sc. εὐθεία), arithmeti-</td>
<td></td>
<td>Eucl., ii. 563 (Papp.)</td>
<td></td>
</tr>
<tr>
<td>c mean, ii. 568 (Papp.): ἀ. μεσότης, 110</td>
<td></td>
<td>Iambi.</td>
<td></td>
</tr>
<tr>
<td>ἄριθμητος, ἴ, ὑ, that can be counted, numbered, 16</td>
<td></td>
<td>Plat.</td>
<td></td>
</tr>
<tr>
<td>ἄριθμος, ὁ, number, 6 (Plat.), 66 (Eucl.); πρῶτος ὁ, prime number, 68 (Eucl.); πρῶτοι, δεύτεροι, τρίτοι, τέταρτοι, πέμπτοι ὁ, numbers of the first, second, third, fourth, fifth order, ii. 198-199 (Archim.); μη-λίτης ὁ, problem about a number of sheep, 16 (Schol. in Plat. Charm.); φιλίτης ὁ, problem about a number of bowls (ibid.)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
INDEX OF GREEK TERMS

άρθμοστόν, τό, fraction whose denominator is unknown

[1/2], ii. 522 (Dioph.)

άρμοζειν, to fit together, ii. 494 (Heron)

άρμονία, ἡ, musical scale, octave, music, harmony, 404 (Plat.); used to denote a square and a rectangle, 398 (Plat.)

άρμονικός, ἡ, ὁ, skilled in music, musical; ἡ ἀρμονική (sc. ἐπιστήμη), mathematical theory of music, harmonic; ἡ ἀρμονική μέση, harmonic mean, 112 (Iambl.)

άρτιάκας, adv., an even number of times; ἅ. ἀρτιός ἀρθμός, even-times even number, 66 (Eucl.)

άρτιόπλευρος, ὁ, having an even number of sides; πολύγωνον ἅ., ii. 88 (Archim.)

άρτιος, ἡ, ὁ, complete, perfect; ἅ. ἀρθμός, even number, 66 (Eucl.)

άρχη, ἡ, beginning or principle of a proof or science, 418 (Aristot.); beginning of the motion of a point describing a curve; ἅρχη ἑλικός, origin of the spiral, ii. 182

άρχικός, ἡ, ὁ, principal, fundamental; ἅ. σύμπτωμα, principal property of a curve, ii. 282 (Apollon.)

άρχικῶτατος, ὁ, sovereign, fundamental; ἅ. ὅλα, 90 (Nicom.)

άρχιτεκτονικός, ἡ, ὁ, of or for an architect; ἃ. ἀρχιτεκτονική (sc. τέχνη), architecture, ii. 616 (Papp.)

ἀστρολογία, ἡ, astronomy, 388 (Aristox.)

ἀστρολόγος, ὁ, astronomer, 378 (Suidas)

ἀστρονομία, ἡ, astronomy, ii. 14 (Plat.)

ἀσύμμετρος, ὁ, incommensurable, irrational, 110 (Aristot.), 452 (Eucl.), ii. 214 (Archim.)

ἀσύμπτωτος, ὁ, not falling in, non-secant, asymptotic, ii. 374 (Procl.); ἅ. (sc. γραμμή), ἡ, asymptote, ii. 282 (Apollon.)

ἀσύνθετος, ὁ, incomposite; ἅ. γραμμή, ii. 360 (Procl.)

ἀτακτός, ὁ, unordered; Περὶ ἀτ. ἀλόγων, title of work by Apollonius, ii 350 (Procl.)

ἀτελής, ἡ, incomplete; ἅ. ἀμφωτικά, figures generated by an incomplete revolution, ii. 604 (Papp.)

ἀτομός, ὁ, indivisible; ἀτομοὶ γραμμαί, 424 (Aristot.)

ἀτομός, ὁ, out of place, absurd; ὅπερ ἀτομον, which is absurd, a favourite conclusion to a piece of reasoning based on a false premise, e.g. ii. 114 (Archim.)

659
INDEX OF GREEK TERMS

αὐξάνειν, to increase, to multiply; τρίς αὐξήθεις, 398 (Plat.)

αὔξη, ἡ, increase, dimension, 10 (Plat.)

αὔξησις, ἡ, increase, multiplication, 398 (Plat.)

αὐτόματος, ἡ, ὁ, self-acting; Αὐτόματα, τά, title of work by Heron, ii. 616 (Papp.)

ἀφαίρεῖν, to cut off, take away, subtract, 444 (Eucl.)

ἀφῆ, ἡ, point of concourse of straight lines; point of contact of circles or of a straight line and a circle, ii. 64 (Archim.)

Ἀχιλλεύς, ἐως, ὁ, Achilles, the first of Zeno’s four arguments on motion, 368 (Aristot.)

Βάρος, ὁ, Ιων. ἐως, τὸ, weight, esp. in a lever, ii. 206 (Archim.), or system of pulleys, ii. 490 (Heron); τὸ κέντρον τοῦ βάρους, centre of gravity, ii. 208 (Archim.)

βαρουκός (sc. μηχανῆ), ἡ, lifting-screw invented by Archimedes, title of work by Heron, ii. 489 n. a

βάσις, ἐως, ἡ, base; of a geometrical figure; of a triangle, 318 (Archim.); of a cube, 222 (Plat.); of a cylinder, ii. 42 (Archim.); of a cone, ii. 304 (Apollon.); of a segment of a sphere, ii. 40 (Archim.)

Γεωδαισία, ἡ, land dividing, mensuration, geodesy, 18 (Anatolius)

γεωμετρεῖν, to measure, to practise geometry; ἅδε γ. τὸν θεῶν, 386 (Plat.);

γεωμετρουμένη ἐπιφάνεια, geometric surface, 292 (Eutoc.), γεωμετρουμενή ἀπόδειξις, geometric proof, ii. 228 (Archim.)

γεωμέτρης, ὁ, ἡ, land measurer, geomter, 258 (Eutoc.)

γεωμετρία, ἡ, land measurement, geometry, 256 (Theon Smyr.), 144 (Procl.)

γεωμετρικός, ἡ, ὁ, pertaining to geometry, geometrical, ii. 590 (Papp.), 298 (Eutoc.)

γεωμετρικῶς, adv., geometrically, ii. 222 (Archim.)

γίνεσθαι, to be brought about; γεγονέτω, let it be done, a formula used to open a piece of analysis; of curves, to be generated, ii. 468 (Heron); to be brought about by multiplication, i.e., the result (of the multiplication) is, ii. 480 (Heron); τὸ γενόμενον, τὰ γενόμενα, the product, ii. 482 (Heron)

γλωσσόκομος, τό, chest, ii. 490 (Papp.)

γνώμωνικός, ἡ, ὁ, of or concerning sun-dials, ii. 616 (Papp.)

γνώμων, ὁ, ὁ, carpenter's
INDEX OF GREEK TERMS

square; pointer of a sundial, ii. 268 (Cleom.); geometrical figure known as gnomon, number added to a figured number to get the next number, 98

γραμμή, ἥ, line, curve, 436 (Eucl.); εἴθεια γ. (often without γ.), straight line, 438 (Eucl.); εκ τῶν γραμμῶν, rigorous proof by geometrical arguments, ii. 412 (Ptol.)

γραμμικός, ἦ, ὄν, linear, 348 (Papp.)

γράφειν, to describe, 442 (Eucl.), ii. 582 (Papp.), 298 (Eutoc.); to prove, 380 (Plat.), 260 (Eutoc.)

γραφή, ἥ, description, account, 260 (Eutoc.); writing, treatise, 258 (Eutoc.)

γωνία, ἥ, angle; επίπεδος γ., plane angle (presumably including angles formed by curves), 438 (Eucl.); εὐθύγραμμος γ., rectilinear angle (formed by straight lines), 438 (Eucl.); ὀρθή, ἀμβλεία, ἡ ἕξεια γ., right, obtuse, acute angle, 440 (Eucl.)

Δεικνύω, to prove; δεδεικταί γὰρ τοῦτο, for this has been proved, ii. 220 (Archim.); δεικτέον δτι, it is required to prove that, ii. 168 (Archim.)

δεῖν, to be necessary, to be required; δέον ἐστω, let it be required; ὀπερ ἐδει δείξαι, quod erat demonstrandum, which was to be proved, the customary ending to a theorem, 184 (Eucl.); ὀπερ :~ = ὀπερ ἐδει δείξαι, ii. 610 (Papp.)

dękágwvov, τό, a regular plane figure with ten angles, decagon, ii. 196 (Archim.)

δήλος, ἦ, or, also ὁς, ὁν, manifest, clear, obvious; ὁτι μὲν οὖν αὐτά συμπίπτει, δήλον, ii. 192 (Archim.)

διάγειν, to draw through, 190 (Eucl.), 290 (Eutoc.)

διάγραμμα, atos, τό, figure, diagram, 428 (Aristot.)

διαρρέω, to divide, cut, ii. 286 (Apollon.); διαρρήμενος, or, divided; ὅ. ἀναλογία, discrete proportion, 262 (Eutoc.); διελόντι, lit. to one having divided, dirigendo (or, less correctly, dividendo), indicating the transformation of the ratio a:b into a-b:b according to Eucl. v. 15, ii. 130 (Archim.)

διαίρεσις, εως, ἥ, division, separation, 368 (Aristot.); ὅ. λόγου, transformation of a ratio dividendo, 448 (Eucl.)

διαμένειν, to remain, to remain stationary, 258 (Eutoc.)

διαμετρος, ὁν, diagonal, diametrical; as subst., ὅ. (sc. γραμμῆς), ἥ, diagonal; of a parallelogram, ii. 218
INDEX OF GREEK TERMS

(Archim.); diameter of a circle, 438 (Eucl.); of a sphere, 466 (Eucl.); principal axis of a conic section in Archim., ii. 148 (Archim.); diameter of any plane curve in Apollon., ii. 286 (Apollon.); πλαγιά δ., transverse diameter, ii. 286 (Apollon.); συνυγείς δ., conjugate diameters, ii. 288 (Apollon.)

diáστασις, εως, ἡ, dimension, 412 (Simpl.)
dιαστέλεων, to separate, ii. 502 (Heron)
dιάστημα, ατός, τό, interval; radius of a circle, ii. 192 (Archim.), 442 (Eucl.); interval or distance of a conchoid, 300 (Papp.); in a proportion, the ratio between terms, το τὸν μειζόνων ὄρων δ., 112 (Archytas ap. Porph.); dimension, 88 (Nicom.)
dιαφόρα, ἡ, difference, 114 (Nicom.)
dιοάνει, to give; aor. part., δοθεῖ, εἰσα, ἐν, given, ii. 598 (Papp.); Δεδομένα, τά, Data, title of work by Euclid, ii. 588 (Papp.); θέσει καὶ μεγεθεὶ δεδοθάι, to be given in position and magnitude, 478 (Eucl.)
dιελόντι, ὑ. διαρείν
dιεχύς, ἐς, discontinuous; σπείρα δ., open spire, ii. 364 (Procl.)
dιορίζειν, to determine, ii. 566 (Papp.); Διωρισμένη
tομή, Determinate Section, title of work by Apollonius, ii. 598 (Papp.)
dιορισμός, ὁ, statement of the limits of possibility of a solution of a problem, dioresmos, 150 (Procl.)
dιπλασιάζειν, to double, 258 (Eutoc.)
dιπλασιασμός, ὁ, doubling, duplication; κύβον δ., 258 (Eutoc.)
dιπλάσιος, α., εὐν, double, 302 (Papp.); δ. λόγος, duplicate ratio, 446 (Eucl.)
dιπλάσιων, εἰν, later form for διπλάσιος, double, 326 (Archim.)
dιπλόος, ἡ, ὁν, contr. διπλοῦς, ἡ, εὖν, twofold, double, 326 (Archim.); δ. ἴσως, double equation, ii. 528 (Dioph.)
dίξα, adv., in two (equal) parts, 66 (Eucl.); δ. τέμνειν, to bisect, 440 (Eucl.)
dιχοτομία, ἡ, dividing in two; point of bisection, ii. 216 (Archim.); Dichotomy, first of Zeno’s arguments on motion, 368 (Aristot.)
dιχοτόμος, εἰν, cut in two, halved, ii. 4 (Aristarch.)
dύναμις, εως, ἡ, power, force, ii. 488 (Heron), ii. 616 (Papp.); αἱ πέντε δ., the five mechanical powers (wheel and axle, lever, pulley, wedge, screw), ii. 492 (Heron); power in
INDEX OF GREEK TERMS

the algebraic sense, esp. square; δυνάμει, in power, i.e., squared, 322 (Archim.); δυνάμει σύμμετρος, commensurable in square, 450 (Eucl.); δυνάμει ἀσύμμετρος, incommensurable in square (ibid.)

δυναμοδύναμις, εως, ή, fourth power of the unknown quantity \( [x^4] \), ii. 522 (Dioph.)

δυναμοδύναμοστόν, τό, the fraction \( \frac{1}{x^4} \), ii. 522 (Dioph.)

δυναμοκυβοστόν, τό, the fraction \( \frac{1}{x^6} \), ii. 522 (Dioph.)

δύνασθαι, to be able, to be equivalent to; δύνασθαι τι, to be equivalent when squared to a number or area, ii. 96 (Archim.); ή δυναμένη (sc. εὐθεία), side of a square, 452 (Eucl.); αὐξῆσις δυνάμεναι, 398 (Plat.); παρ’ ήν δύναται αἱ καταγόμεναι τεταγμένως ἐπὶ τὴν ΖΗ διάμετρον, the parameter of the ordinates to the diameter ΖΗ, ii. 308 (Apollon.)

δυναστευεῖν, to be powerful; pass., to be concerned with

powers of numbers; αὐξήσεις δυναστευόμεναι, 398 (Plat.)

δυνατός, ή, ὁν, possible, ii. 566 (Papp.)

δυνακεννήκοντάεδρον, τό, solid with ninety-two faces, ii. 196 (Archim.)

δυνακεζηκοντάεδρον, τό, solid with sixty-two faces, ii. 196 (Archim.)

δυνακτριακοντάεδρον, τό, solid with thirty-two faces, ii. 196 (Archim.)

δωδεκάεδρον, οὐ, with twelve faces; as subst., δωδεκάεδρον, τό, body with twelve faces, dodecahedron, 472 (Eucl.), 216 (Aet.)

Ἐβδομηκοστόμονος, οὐ, seventy-first; τὸ ἕ., seventy-first part, 320 (Archim.)

ἐγγράφειν, to inscribe, 470 (Eucl.), ii. 46 (Archim.)

ἐγκύκλιον, οὐ, also a, οὐ, circular, ii. 618 (Papp.)

εἴδος, οὐς, Ιον. εἶς, τό, shape or form of a figured number, 94 (Aristot.); figure giving the property of a conic section, viz., the rectangle contained by the diameter and the parameter, ii. 317 n. a, 358 (Papp.), 282 (Eutoc.); term in an equation, ii. 524 (Dioph.); species—of number, ii. 522 (Dioph.), of angles 390 (Plat.)

663
INDEX OF GREEK TERMS

εἰκοσάεδρος, ον, having twenty faces; εἰκοσάεδρον, τό, body with twenty faces, icosahedron, 216 (Λέτ.)

εἰκοσαπλάσιος, ον, twentyfold, ii. 6 (Aristarch.)

ἐκατοντάς, ἄδως, ἡ, the number one hundred, ii. 198 (Archim.)

ἐκβάλλεν, to produce (a straight line), 442 (Eucl.), ii. 8 (Aristarch.), 352 (Papp.)

ἐκκυκλοσάεδρον, τό, solid with twenty-six faces, ii. 196 (Archim.)

ἐκκείθοσα, used as pass. of ἐκτίθεναι, to be set out, be taken, ii. 96 (Archim.), 298 (Papp.)

ἐκκρονεν, to take away, eliminate, ii. 612 (Papp.)

ἐκπέτασμα, ἄτος, τό, that which is spread out, unfolded; Ἐκπέτασμα, title of work by Democritus dealing with projection of armillary sphere on a plane, 229 n. a

ἐκπρίσμα, ἄτος, τό, section sawn out of a cylinder, prismatic section, ii. 470 (Heron)

ἐκτίθεναι, to set out, ii. 568 (Papp.)

ἐκτός, adv., without, outside; as prep., ἐ τοῦ κύκλου, 314 (Alex. Aphr.); adv. used adjectivally, ἡ ἑ. (sc. εὐθεῖα), external straight line, 314 (Simpl.); ἡ ἑ. γωνία τοῦ τριγώνου, the external

angle of the triangle, ii. 310 (Apollon.)

ἐλάσσων, ον, smaller, less, 320 (Archim.); ἦτοι μείζων ἐστὶν ἡ ἐ., ii. 112 (Archim.);

ἐ. ὀβρής, less than a right angle, 438 (Eucl.); ἡ ἐ. (sc. εὐθεία), minor in Euclid’s classification of straight lines, 458 (Eucl.)

ἐλάχιστος, ἡ, ov, smallest, least, ii. 44 (Archim.)

ἐλιξ, ἐλικος, ἡ, spiral, helix, ii. 182 (Archim.); spiral on a sphere, ii. 580 (Papp.)

ἐλλειμμα, ἄτοσ, τό, defect, deficiency, 206 (Eucl.)

ἐλλείπεν, to fall short, be deficient, 394 (Plat.), 188 (Procl.)

ἐλλεύψις, ewο, ἡ, falling short, deficiency, 186 (Procl.);

the conic section ellipse, so called because the square on the ordinate is equal to a rectangle whose height is equal to the abscissa applied to the parameter as base but falling short (ἐλλεῖπτον), ii. 316 (Apollon.), 188 (Procl.)

ἐμβαδόν, τό, area, ii. 470 (Heron)

ἐμβάλλων, to throw in, insert, ii. 574 (Papp.); multiply, ii. 534 (Dioph.)

ἐμπίπτευν, to fall on, to meet, to cut, 442 (Eucl.), ii. 58 (Archim.)

ἐμπλέκειν, to plait or weave in; σπείρα ἐμπλεγμενή,
INDEX OF GREEK TERMS

interlaced spire, ii. 364 (Procl.)
έναυλαῖς, adv., often used adjectively, transformation of a ratio according to the rule of Eucl. v. Def. 12, permutando, 448 (Eucl.), ii. 144 (Archim.); ύ, γωνία, alternate angles
ένεργις, α, όν, opposite; κατ' έ., ii. 216 (Archim.)
ένυμόζευν, to fit in, to insert, 284 (Eutoc.)
έντασις, εως, ἣ, inscription, 396 (Plat.)
έντελης, εσ, perfect, complete; τρίγωνον έ., 90 (Procl.)
έντος, adv. used adjectively, within, inside, interior; αἱ γωνίαι, 442 (Eucl.)
ένυπάρχειν, to exist in; εἴδη ενυπάρχοντα, ἃ, positive terms, ii. 524 (Dioph.)
ἐγκατωκός, ἥ, όν, hexagonal; ἀριθμός, 96 (Nicom.)
ἐγκώνων, όν, as subst. ἐγκώνων, τό, hexagon, 470 (Eucl.)
ἐγκοστός, ἃ, όν, sixtieth; in astron., πρώτον ἐγκοστόν, τό, first sixtieth, minute, δεύτερον έ., second sixtieth, second, 50 (Theon Alex.)
ἐγκός, adv., in order, successively, ii. 566 (Papp.)
ἐπαφή, ἡ, touching, tangency, contact, 314 (Simpl.)
Ἐπαφαλ, On Tangencies, title of a book by Apollonius, ii. 336 (Papp.)
ἐπεσθαί, to be or come after, follow; τό ἐπόμενον, con-
sequence, ii. 566 (Papp.); τά ἐπόμενα, rearward elements, ii. 184 (Apollon.);
in theory of proportion, τά ἐπόμενα, following terms, consequents, 448 (Eucl.)
ἐπὶ, prep. with acc., upon, on to, on, εὐθεία ἐπ' εὐθείαν σταθείσα, 438 (Eucl.)
ἐπιζευγνύαι, to join up, ii. 608 (Papp.); αἱ ἐπιζευγνύσεις εὐθείας, connecting lines, 272 (Eutoc.)
ἐπιλογίζεσθαι, to reckon, calculate, 60 (Theon Alex.)
ἐπιλογισμός, ὁ, reckoning, calculation, ii. 412 (Ptol.)
ἐπίπεδος, όν, plane; ἡ ἐπιφάνεια, 438 (Eucl.); ἡ γωνία, 438 (Eucl.); ἡ σχήμα, 438 (Eucl.); ἡ ἀριθμός, 70 (Eucl.); ἡ πρόβλημα, 348 (Papp.)
ἐπιτέθος, adv., plane-wise, 88 (Nicom.)
ἐπιπλατής, ἢ, flat, broad; σφαιροειδὲς ἤ., ii. 164 (Archim.)
ἐπίταγμα, ἁτος, τό, injunction; condition, ii. 50 (Archim.), ii. 526 (Dioph.); ποιεῖν τό ἡ, to satisfy the condition; subdivision of a problem, ii. 340 (Papp.)
ἐπίπτρυς, όν, containing an integer and one-third, in the ratio 4 : 3, ii. 222 (Archim.)
ἐπιφάνεια, ἡ, surface, 438 (Eucl.); κωνική ἡ, conical surface (double cone), ii. 286 (Apollon.)

665
INDEX OF GREEK TERMS

ἐπιφανέων, to touch, ii. 190 (Archim.)

ἡ ἐπιφάνεια (sc. εἴθεια), tangent, ii. 64 (Archim.)

ἐτερομῆχης, es, with unequal sides, oblong, 440 (Eucl.)

εὐθύγραμμος, ov, rectilinear; εὐ. γωνία, 438 (Eucl.); εὐ. σχήμα, 440 (Eucl.); as subst., εὐθύγραμμον, τό, rectilineal figure, 318 (Archim.)

εὐθύς, εἴα, υ, straight; εὐ. γραμμή, straight line, 438 (Eucl.); εἴθεια (sc. γραμμή), ἡ, straight line, ii. 44 (Archim.); chord of a circle, ii. 412 (Ptol.); distance (first, second, etc.) in a spiral, ii. 182 (Archim.); κατ’ εἴθεια, along a straight line, ii. 580 (Papp.)

ἐπαραχώρητος, ov, readily admissible, easily obvious; εὐ. λήμματα, ii. 230 (Archim.)

ἐφερέας, εως, ἡ, discovery, solution, ii. 518 (Dioph.), 260 (Eutoc.)

ἐφημα, ατος, τό, discovery, 380 (Schol. in Eucl.)

ἐφιλοκεῖνων, to find, discover, solve, ii. 526 (Dioph.), 340 (Papp.), 262 (Eutoc.); ὁπερ ἐδει εὑρέθη, which was to be found, 282 (Eutoc.)

ἐφέρης, ἐς, easy to solve, ii. 526 (Dioph.)

ἐφαπτεῖσθαι, to touch, ii. 224 (Archim.); ἐφαπτομένη, ἡ (sc. εἴθεια), tangent, 322 (Archim.)

ἐφαπτομή, ἡ, coincidence of geometrical elements, 340 (Papp.)

ἐφαρμόζεων, to fit exactly, coincide with, 444 (Eucl.), ii. 208 (Archim.), 298 (Papp.); pass. ἐφαρμοζότατα, to be applied to, ii. 208 (Archim.)

ἐφεξῆς, adv., in order, one after the other, successively, 312 (Them.); used adjectivally, as αἱ ἡ γωνία, the adjacent angles, 483 (Eucl.)

ἐφιστάναι, to set up, erect; perf., ἐφιστηκέναι, intr., stand, and perf. part. act., ἐφιστηκὼς, via, ὁς, standing, 438 (Eucl.)

ἐφόδος, ἡ, method, ii. 596 (Papp.); title of work by Archimedes

ἐχεῖν, to have; λόγον ἕ, to have a proportion or ratio, ii. 14 (Aristarch.); γένεως ἕ, to be generated (of a curve), 348 (Papp.)

ἐως, as far as, to, ii. 290 (Apollon.)

Ζυγεῖν, to seek, investigate, ii. 222 (Archim.); ζυγομενον, τό, the thing sought, 158 (Procl.), ii. 596 (Papp.)

ζητησι, εως, ἡ, inquiry, investigation, 152 (Procl.)

ζύγων, τό = ζυγῶν, τό, ii. 234 (Archim.)
INDEX OF GREEK TERMS

ζυγόν, τό, beam of a balance, balance, ii. 234 (Archim.)
ζώδιον, τό, dim. of ζώον, lit. small figure painted or carved; hence sign of the Zodiac; ὀ τῶν Ζ. κύκλος, Zodiac circle, ii. 394 (Hypsicles)

Ἡγεσθαι, to lead; ἰγούμενα, τά, leading terms in a proportion, 448 (Eucl.)
ἡμικύκλιος, ὁν, semicircular; as subst., ἡμικύκλιον, τό, semicircle, 410 (Eucl.), ii. 568 (Papp.)
ἡμικυλιδρός, ὁ, half-cylinder, 260 (Eutoc.); dim. ἡμικυλιδρον, τό, 286 (Eutoc.)
ἡμιλός, ὁ, ov, containing one and a half, half as much or as large again, one-and-a-half times, ii. 42 (Archim.)
ἡμισυς, εια, ν, half, ii. 10 (Aristarch.); as subst., ἡμισυν, τό, 320 (Archim.)

Θέσις, εως, ἡ, setting, position, 268 (Eutoc.); θέσις δεδοθαί, to be given in position, 478 (Eucl.)
θεωρεῖν, to look into, investigate, ii. 222 (Archim.)
θεώρημα, ατός, τό, theorem, 228 (Archim.), ii. 566 (Papp.), 150 (Procl.), ii. 366 (Procl.)
θεωρητικός, ὁ, ὁν, able to perceive, contemplative, speculative, theoretical; applied to species of analysis, ii. 698 (Papp.)

θεωρία, ἡ, inquiry, theoretical investigation, theory, ii. 222 (Archim.), ii. 568 (Papp.)
θυρεός, ὁ, shield, 490 (Eucl.); ἡ (sc. γραμμή τοῦ θ.), ellipse, ii. 360 (Procl.)

Ἅσκος, adv., the same number of times, as many times; τά ἰ. πολλαπλάσια, equimultiples, 446 (Eucl.)
 ἴσοβαρῆς, ἐς, equal in weight, ii. 250 (Archim.)
ἰσογκός, ὁν, equal in bulk, equal in volume, ii. 250 (Archim.)
ἰσογώνος, ὁ, equiangular, ii. 608 (Papp.)
ἰσομήκης, ἐς, equal in length, 398 (Plat.)
ἰσοπερίμετρος, ὁ, of equal perimeter, ii. 386 (Theon Alex.)
ἰσοπλευρος, ὁ, having all its sides equal, equilateral; ἵ. τριγώνον, 440 (Eucl.), ἵ. τετράγωνον, 440 (Eucl.), ἵ. πολύγωνον, ii. 54 (Archim.)
ἰσοπληθής, ἐς, equal in number, 454 (Eucl.)
ἰσορροπεῖν, to be equally balanced, be in equilibrium, balance, ii. 206 (Archim.)
Ἰσορροπικά, τά, title of work on equilibrium by Archimedes, ii. 226 (Archim.)
ἰσορροπος, ὁ, in equilibrium, ii. 226 (Archim.)
ἴσος, ἡ, ὁ, equal, 268 (Eutoc.): ἐς ἴσον, evenly,
INDEX OF GREEK TERMS

438 (Eucl.); δι' ίσου, ex equali, transformation of a ratio according to the rule of Eucl. v. Def. 17, 448 (Eucl.)

ισοσκελής, ἐς, with equal legs, having two sides equal, isosceles; ἵ. τρίγωνοι, 440 (Eucl.); ἵ. κώνος, ii. 58 (Archim.)

ισοταχείας, uniformly, ii. 182 (Archim.)

ισότης, ητος, ἡ, equality, equation, ii. 526 (Dioph.)

ιστάναι, to set up; ευθεία ἐπ' ευθείαν σταθείσα, 438 (Eucl.)

ισωσὶς, εἰς, ἡ, making equal, equation, ii. 526 (Dioph.)

Κάθετος, οὖν, let down, perpendicular; ἢ κ. (sc. γραμμή), perpendicular, 438 (Eucl.), ii. 580 (Papp.)

καθολικός, ἡ, οὐ, general; κ. μέθοδος, ii. 470 (Heron)

καθολικῶς, generally; καθολικῶτερον, more generally, ii. 572 (Papp.)

καθόλου, adv., on the whole, in general; τὰ κ. καλούμενα θεωρήματα, 152 (Procl.)

καμπύλος, η, οὖ, curved; κ. γραμμαί, ii. 42 (Archim.); 260 (Eutoc.)

κανόνιν, τό, table, ii. 444 (Ptol.)

κανονικός, ἡ, οὖ, of or belonging to a rule; ἡ κανονική (sc. τέχνη), the mathematical theory of music, theory of musical intervals, canonic, 18 (Anatolius);

κ. έκθεσις, display in the form of a table, ii. 412 (Ptol.)

κανών, οὖν, ὁ, straight rod, bar, 308 (Aristoph.), 264 (Eutoc.); rule, standard, table, ii. 408 (Suidas)

κατάγειν, to draw down or out, ii. 600 (Papp.)

καταγραφή, ἡ, construction, 188 (Procl.); drawing, figure, ii. 158 (Eutoc.), ii. 444 (Ptol.), ii. 610 (Papp.)

καταλαμβάνειν, to overtake, 368 (Aristot.)

καταλείπειν, to leave, 454 (Eucl.), ii. 218 (Archim.), ii. 524 (Dioph.); τὰ καταλείπόμενα, the remainders, 444 (Eucl.)

καταμετρεῖν, to measure, i.e., to be contained in an integral number of times, 444 (Eucl.)

κατασκευάζειν, to construct, 264 (Eutoc.), ii. 566 (Papp.)

κατασκευή, ἡ, construction, ii. 500 (Heron)

καταστεριομός, ὁ, placing among the stars; Καταστερισμόι, ὁ, title of work wrongly attributed to Eratosthenes, ii. 262 (Suidas)

καταστομή, ἡ, cutting, section; Κ. κανόνος, title of work by Cleonides, 157 n. c

κατοπτρικός, ἡ, οὖ, of or in a mirror; Κατοπτρικά, τά, title of work ascribed to Euclid, 156 (Procl.)
INDEX OF GREEK TERMS

κάτοπτρον, τὸ, mirror, ii. 498 (Heron)

κείσθαι, to lie, ii. 268 (Cleom.); of points on a straight line, 438 (Eucl.); as pass. of τιθέναι, to be placed or made; of an angle, 326 (Archim.); ὁμοίως κ., to be similarly situated, ii. 208 (Archim.)

κεντροβάρικός, ἡ, ὠν, of or pertaining to a centre of gravity; κ. σημεῖα, ii. 604 (Papp.)

κέντρον, τὸ, centre; of a circle, 438 (Eucl.), ii. 8 (Aristarch.), ii. 572 (Papp.); of a semicircle, 440 (Eucl.); ἡ (sc. γραμμὴ or εὐθεῖα) ἐκ τοῦ κ., radius of a circle, ii. 40 (Archim.); κ. τοῦ βάρεως, centre of gravity, ii. 208 (Archim.)

κινεῖν, to move, 264 (Eutoc.)

κίνησις, ἡ, motion, 264 (Eutoc.)

κισσοειδής, ἡ, Att. κιττοειδής, ἡ, like ȋνα; κ. γραμμή, cissoid, 276 n. a

κλᾶν, to bend, to inflect, 420 (Aristot.), 358 (Papp.); κλάμεναι εὐθεία, inclined straight lines, ii. 496 (Damian.)

κλίνειν, to make to lean; pass., to incline, ii. 252 (Archim.)

κλίσις, ἡ, inclination; τῶν γραμμῶν κ., 438 (Eucl.)

κοιχοειδής, ἡ, resembling a mussel; κ. γραμμαί (often without γ.), conchoidal curves, conchoids, 296 (Eutoc.)

κοῖλος, η, ov, concave; ἐπὶ τὰ αὐτὰ κ., concave in the same direction, ii. 42 (Archim.), 338 (Papp.)

κοινός, ἡ, ὡν, common, 412 (Aristot.); κ. πλευρά, ii. 500 (Heron); κ. ἐννοια, 444 (Eucl.); κ. τομή, ii. 290 (Apollon.); τὸ κοινόν, common element, 306 (Papp.)

κορυφή, ἡ, vertex; of a cone, ii. 286 (Apollon.); of a plane curve, ii. 286 (Apollon.); of a segment of a sphere, ii. 40 (Archim.)

κοχλίας, ὃ, snail with spiral shell; hence anything twisted spirally; screw, ii. 496 (Heron); screw of Archimedes, ii. 34 (Diod. Sic.); Περὶ τοῦ κ., work by Apollonius, ii. 350 (Procl.)

κοχλοειδής, ἡ, of or pertaining to a shell fish; ἡ κ. (sc. γραμμή), cochloid, 334 (Simpl.); also κοχλοειδής, ἡ, as ἡ κ. γραμμή, 302 (Papp.); probably anterior to ἡ κοιχοειδής γραμμή with same meaning

κρίκος, ὁ, ring; τετράγωνοι κ., prismatic sections of cylinders, ii. 470 (Heron)

κυβίζειν, to make into a cube, cube, raise to the third power, ii. 504 (Heron)
<table>
<thead>
<tr>
<th>Greek Term</th>
<th>English Translation</th>
</tr>
</thead>
<tbody>
<tr>
<td>κυβικός, ἰ., ὁν, of or for a cube, cubic, 222 (Plat.)</td>
<td>cube, cubic, 222 (Plat.)</td>
</tr>
<tr>
<td>κυβάκυβος, ὁ, cube multiplied by a cube, sixth power of the unknown quantity $[x^6]$, ii. 522 (Dioph.)</td>
<td>cube multiplied by a cube, sixth power of the unknown quantity $[x^6]$, ii. 522 (Dioph.)</td>
</tr>
<tr>
<td>κυβοστόν, το, the fraction $\frac{1}{x}$, ii. 522 (Dioph.)</td>
<td>fraction $\frac{1}{x}$, ii. 522 (Dioph.)</td>
</tr>
<tr>
<td>κύβος, ὁ, cube, 258 (Eutoc.); cubic number, ii. 518 (Dioph.); third power of unknown, ii. 522 (Dioph.)</td>
<td>cube, 258 (Eutoc.); cubic number, ii. 518 (Dioph.); third power of unknown, ii. 522 (Dioph.)</td>
</tr>
<tr>
<td>κυβοστόν, το, the fraction $\frac{1}{x^2}$, ii. 522 (Dioph.)</td>
<td>the fraction $\frac{1}{x^2}$, ii. 522 (Dioph.)</td>
</tr>
<tr>
<td>κυκλικός, ἰ., ὁν, circular, ii. 360 (Procl.)</td>
<td>circular, ii. 360 (Procl.)</td>
</tr>
<tr>
<td>κύκλος, ὁ, circle, 392 (Plat.), 438 (Eucl.); μέγιστος κ., great circle (of a sphere), ii. 8 (Aristarch.), ii. 42 (Archim.)</td>
<td>circle, 392 (Plat.), 438 (Eucl.); μέγιστος κ., great circle (of a sphere), ii. 8 (Aristarch.), ii. 42 (Archim.)</td>
</tr>
<tr>
<td>κυλινδρικός, ἰ., ὁν, cylindrical, 286 (Eutoc.)</td>
<td>cylindrical, 286 (Eutoc.)</td>
</tr>
<tr>
<td>κυλινδρος, ὁ, cylinder, ii. 42 (Archim.)</td>
<td>cylinder, ii. 42 (Archim.)</td>
</tr>
<tr>
<td>κυρίως, adv., in a special sense; κ., αναλογία, proportion par excellence, i.e., the geometric proportion, 125 n. a</td>
<td>in a special sense; κ., αναλογία, proportion par excellence, i.e., the geometric proportion, 125 n. a</td>
</tr>
<tr>
<td>κωνικός, ἰ., ὁν, conical, conic; κ., ἐπιφάνεια, conical surface (double cone), ii. 286 (Apollon.)</td>
<td>conical, conic; κ., ἐπιφάνεια, conical surface (double cone), ii. 286 (Apollon.)</td>
</tr>
<tr>
<td>κωνοεδής, ἵ, conical; as subst. κωνοεδές, το, conoid; ὁρθογώνιον κ., right-angled conoid, i.e., paraboloid of revolution, ii. 164; ἀμβλυγώνιον κ., obtuse-angled conoid, i.e., hyperboloid of revolution, ii. 164</td>
<td>conical; as subst. κωνοεδές, το, conoid; ὁρθογώνιον κ., right-angled conoid, i.e., paraboloid of revolution, ii. 164; ἀμβλυγώνιον κ., obtuse-angled conoid, i.e., hyperboloid of revolution, ii. 164</td>
</tr>
<tr>
<td>κάνως, ὁ, ὁν, cone, ii. 286 (Apollon.)</td>
<td>cone, ii. 286 (Apollon.)</td>
</tr>
<tr>
<td>κωνοτομεῖν, to cut the cone, ii. 226 (Eratos. ap. Eutoc.)</td>
<td>to cut the cone, ii. 226 (Eratos. ap. Eutoc.)</td>
</tr>
<tr>
<td>λαμβάνειν, to take, ii. 112 (Archim.); εἴληφθω τά κέντρα, let the centres be taken, ii. 388 (Theon Alex.); λ. τὰς μέσας, to take the means, 294 (Eutoc.); to receive, postulate, ii. 44 (Archim.)</td>
<td>to take, ii. 112 (Archim.); εἴληφθω τά κέντρα, let the centres be taken, ii. 388 (Theon Alex.); λ. τὰς μέσας, to take the means, 294 (Eutoc.); to receive, postulate, ii. 44 (Archim.)</td>
</tr>
<tr>
<td>λέγειν, to choose, ii. 166 (Archim.)</td>
<td>to choose, ii. 166 (Archim.)</td>
</tr>
<tr>
<td>λείπειν, to leave, ii. 62 (Archim.); λείποντα εἴδη, τά, negative terms, ii. 524 (Dioph.)</td>
<td>to leave, ii. 62 (Archim.); λείποντα εἴδη, τά, negative terms, ii. 524 (Dioph.)</td>
</tr>
<tr>
<td>λήμμα, ἄτος, το, auxiliary theorem assumed in proving the main theorem, lemma, ii. 608 (Papp.)</td>
<td>auxiliary theorem assumed in proving the main theorem, lemma, ii. 608 (Papp.)</td>
</tr>
<tr>
<td>λημμάτιον, το, dim. of λήμμα, lemma</td>
<td>dim. of λήμμα, lemma</td>
</tr>
<tr>
<td>λήψις, ἐως, ἵ, negative term, minus, ii. 524 (Dioph.)</td>
<td>negative term, minus, ii. 524 (Dioph.)</td>
</tr>
<tr>
<td>λημματικός, ἵ, ὁν, skilled or practised in reasoning or calculating; ἵ λημματική (sc. τέχνη), the art of manipulating numbers, practical arithmetic, logistic, 17 (Schol. ad Plat. Charm.)</td>
<td>skilled or practised in reasoning or calculating; ἵ λημματική (sc. τέχνη), the art of manipulating numbers, practical arithmetic, logistic, 17 (Schol. ad Plat. Charm.)</td>
</tr>
</tbody>
</table>
INDEX OF GREEK TERMS

λόγος, ὁ, ratio, 444 (Eucl.);
Λόγον ἀποτιμών, Cutting-off of a Ratio, title of work by
Apollonius, ii. 598 (Papp.);
λ. συνημένος, compound
ratio, ii. 602 (Papp.);
λ. μοναχός, singular ratio,
ii. 606 (Papp.); ἄκρος καὶ
μέσος λ., extreme and
mean ratio, 472 (Eucl.),
ii. 416 (Ptol.)

λογός, ὁ, on, remaining, ii.
600 (Papp.); as subst.,
λογόν, τό, the remainder,
ii. 506 (Papp.), 270
(Eutoc.)
λοξός, ἡ, oblique, inclined;
κατὰ λ. κύκλου, ii. 4 (Plut.)
λύσις, ἡ, ἡ, solution, ii. 506
(Papp.)

Magaναδρος, ὁ, mechanical
engineer, maker of mechan-
ical powers, ii. 616 (Papp.)
μᾶγγανος, τό, block of a pulley,
ii. 616 n. a (Heron)
μάθημα, τό, study, 8 (Plat.), 4
(Archytas); μαθημάτα, τα, 
mathematics; τά δε καλοῦ-
μενα άλος μ., 2 (Anatolius);
148 (Procl.), ii. 42 (Ar-
chim.), ii. 566 (Papp.)
μαθηματικός, ὁ, on, mathe-
matical; μαθηματικός, ὁ,
mathematician, ii. 2 (Aët.),
ii. 61 (Papp.); ἡ μαθημα-
τικὴ (sc. ἐπιστήμη), math-
ematics, 4 (Archytas); τά μ.,
mathematics
μέγεθος, οὐς, Ion. eos, τό,
magnitude, 444 (Eucl.), ii.
50 (Archim.), ii. 412 (Ptol.)

μέθοδος, ἡ, following after,
investigation, method, 90
(Procl.)
μεγίςων, ὁν, greater, more,
318 (Archim.); ἡγοῦ μ.
ἐστιν ἡ ἐλάσσωσ, ii. 112
(Archim.); μ. ὀρθὸς,
greater than a right angle,
438 (Eucl.); ἡ μ. (sc.
εὐθεία), major in Euclid's
classification of irrationals,
458 (Eucl.)
μένεω, to remain, to remain
stationary, 98 (Nicom.),
286 (Eutoc.)
μερίζεων, to divide, τί παρά τί,
50 (Theon Alex.)
μερισμός, ὁ, division, 16
(Schol. in Plat. Charm.),
ii. 414 (Ptol.)
μέρος, οὐς, Ion. eos, τό, part;
of a number, 66 (Eucl.); of
a magnitude, 444 (Eucl.),
ii. 584 (Papp.); τά μέρη,
parts, directions, ἐφ’ ἐκά-
tερα τά μ., in both direc-
tions, 438 (Eucl.)
μεσημβρινός, ἡ, on, for μεσ-
μερινός, of or for noon;
μ. (sc. κύκλος), ὁ, meri-
dian, ii. 268 (Cleom.)
μέσος, ἡ, on, middle; ἡ μέση
(sc. εὐθεία), mean (ἀρθ-
μητική, γεωμετρική, ἀρμο-
νική), ii. 568 (Papp.);
μέση τῶν ΔΚ, ΚΓ, mean
between ΔΚ, ΚΓ, 272
(Eutoc.); ἄκρος καὶ μ.
λόγος, extreme and mean
ratio, 472 (Eucl.), ii. 416
(Ptol.); ἡ μέση (sc. εὐθεία),
medial in Euclid's classi-
INDEX OF GREEK TERMS

fication of irrationals, 458 (Eucl.); ἐκ δύο μέσων πρώτη, first bimedial, ἐκ δύο μέσων δεύτερα, second bimedial, etc., ibid.

μεσότης, γιγαντία, γιγάντιον, μέσον, mean, ii. 566 (Papp.); μ. ἀριθμητική, λογοποιική, ἀρμονική (ὕπερ-ασκητική), 110-111 (Iambl.)

μετρέων, to measure, contain an integral number of times, 68 (Eucl.), ii. 54 (Archim.)

μέτρον, τό, measure, relation, ii. 294 (Prob. Bov.); κοινόν μ., common measure, ii. 210 (Archim.)

μέχρι, as far as, prep. with gen. ἦ μέχρι τοῦ ἄξονος (sc. γραμμή), ii. 256 (Archim.)

μήκος, Dor. μάκος, ἐος, τό, length, 436 (Eucl.); distance of weight from fulcrum of a lever, ii. 206 (Archim.)

μύκικος, ὁ, crescent-shaped figure, lune, 238 (Eudemus ap. Simplic.)

μύχανη, ἡ, contrivance, machine, engine, ii. 26 (Plut.)

μύχανικός, ὁ, ón, of or for machines, mechanical, ii. 616 (Papp.); ἡ μύχανική (with or without τέχνη), mechanics, ii. 614 (Papp.); as subst., μύχανικός, ὁ, mechanician, ii. 616 (Papp.), ii. 496 (Damian.)

μύχανοσότος, ὁ, maker of engines, ii. 616 (Papp.)

μικρός, ἀ., ὁν, small, little; M. ἀστρονομούμενος (sc. τόπος), Little Astronomy, ii. 409 n. δ.

μικτός, ἡ, ὁ, mixed; μ. γραμμή, ii. 360 (Procl.); μ. ἐπιφάνεια, ii. 470 (Heron)

μοῖρα, as, ἡ, portion, part; in astronom., degree, 50 (Theon Alex.); μ. τοπική, χροική, ii. 396 (Hypsicl.)

μονάς, ἄδος, ἡ, unit, monad, 66 (Eucl.)

μοναχός, ἡ, ὁ, unique, singular; μ. λόγος, ii. 606 (Papp.)

μὸριον, τό, part, 6 (Plat.)

μονασκός, ἡ, ὁν, Dor. μονασκός, ἄ., ὁν, musical; ἡ μονασκή (sc. τέχνη), poetry sung to music, music, 4 (Archytas)

μυρίας, αδώς, ἡ, the number ten thousand, myriad, ii. 198 (Archim.); μ. ἀπλαί, διπλαί, κτλ., a myriad raised to the first power, to the second power, and so on, ii. 355 n. α

μύριον, αἱ, ὁ, ten thousand, myriad; μ. μυρίδος, myriad myriads, ii. 198 (Archim.)

Neuev, to be in the direction of, ii. 6 (Aristarch.); of a straight line, to verge, i.e., to be so drawn as to pass through a given point and make a given intercept, 244 (Eudemus ap. Simplic.), 420 (Aristot.), ii. 188 (Archim.)
INDEX OF GREEK TERMS

νεθας, εως, ἡ, inclination, verging, problem in which a straight line has to be drawn through a point so as to make a given intercept, 245 n. a; στερεά ν., solid verging, 350 (Papp.)

'Οδός, ἡ, method, ii. 596 (Papp.)

οίκείος, α, ον, proper to a thing; ὅ οί. κύκλος, ii. 270 (Cleom.)

ὀκτάγωνος, ον, eight-cornered; as subst., ὀκτάγωνον, τό, regular plane figure with eight sides, octagon, ii. 196 (Archim.)

ὀκταέδρος, ον, with eight faces; as subst., ὀκταέδρον, τό, solid with eight faces, ii. 196 (Archim.)

ὀκταπλάσιος, α, ον, eightfold, ii. 584 (Papp.)

ὀκτωκαθέκαπλασίος, ον, eighteen-fold, ii. 6 (Aristarch.)

ὀκτωκεντακονταεδρόν, τό, solid with thirty-eight faces, ii. 196 (Archim.)

ὀλόκληρος, ον, complete, entire; as subst., ὀλόκληρον, τό, integer, ii. 534 (Dioph.)

ὅλος, ἡ, ον, whole; τά ὅ., 444 (Eucl.)

ὀμαλός, ἡ, ον, even, uniform, ii. 618 (Papp.)

ὀμαλός, adv., uniformly, 338 (Papp.)

ὀμοιος, α, ον, like, similar; δ. τρίγωνον, 288 (Eutoc.)

ὀμοιωτάτον καὶ στερεόν ἀριθμόν, 70 (Eucl.)

ὀμοίωσ, adv., similarly, ii. 176 (Archim.); τὰ ὀ. τεταγμένα, the corresponding terms, ii. 166 (Archim.); ὅ. κεῖθαι, to be similarly situated, ii. 208 (Archim.)

ὀμολογεῖν, to agree with, admit; pass., to be allowed, admitted; τὸ ὀμολογοῦμενον, that which is admitted, premise, ii. 596 (Papp.)

ὀμόλογος, ον, corresponding; ὅ. μεγέθεα, ii. 166 (Archim.); ὅ. πλευρά, ii. 208 (Archim.)

ὀμοραγής, ἐς, ranged in the same row or line, co-ordinate with, corresponding to, similar to, ii. 586 (Papp.)

ὀνομα, ἄρος, τό, name; ἡ (ἐν. εὐθεία) ἐκ δύο ὀνομάτων, binomial in Euclid's classification of irrationals, 458 (Eucl.)

ὀξυγώνιος, ον, acute-angled; ὅ. κώνος and ὅ. κώνον τομή, ii. 278 (Eutoc.)

ὀξύς, ἑτα, ύ, acute; ὅ. γωνία, acute angle, often with γωνία omitted, 438 (Eucl.)

ὀπτικός, ἡ, ὁν, of or for sight; ὀπτικά, τά, theory of laws of sight; as prop. name, title of work by Euclid, 156 (Procl.)

ὀργανικός, ἡ, ὁν, serving as instruments; ὅ. λῆψις, mechanical solution, 260 (Eutoc.)
INDEX OF GREEK TERMS

όγανικάς, adv., by means of instruments, 292 (Eutoc.)
όγανον, τό, instrument, 294 (Eutoc.); dim. ογανίον, 294 (Eutoc.)
οθεος, α, ου, upright, erect; ἡ οθία (sc. τοῦ εἰδοὺς πλευράς), the erect side of the rectangle formed by the ordinate of a conic section applied to the parameter as base, latus rectum, an alternative name for the parameter, ii. 316 (Apollon.), ii. 322 (Apollon.)
οθιογώνος, ου, having all its angles right, right-angled, orthogonal; ο. πετάγωνον, 440 (Eucl.); ο. παραλληλόγραμμον, 268 (Eutoc.)
οθύς, ή, ον, right; ο. γωνία, right angle, 438, 442 (Eucl.); ο. κώνος, right cone, ii. 286 (Apollon.)
οἵειν, to separate, delimit, bound, define, 382 (Plat.); εὐθεία ὁμοιόμενη, finite straight line, 188 (Eucl.)
ὁσος, ο, boundary, 438 (Eucl.); term in a proportion, 112 (Archytas ap. Porph.), 114 (Nicom.)
οὖν, therefore, used of the steps in a geometrical proof, 326 (Eucl.)
ὁχείσθαι, to be borne, to float in a liquid; Περὶ τῶν οχουμένων, On floating bodies, title of work by Archimedes, ii. 616 (Papp.)

Παρά, beside; παραβάλλειν π. to apply a figure to a straight line, 188 (Eucl.);
τι π. τι παραβάλλειν, to divide by, ii. 482 (Heron)
παραβάλλειν, to throw beside;
π. παρά, to apply a figure to a straight line, 188 (Eucl.); hence, since to apply a rectangle xy to a straight line x is to divide xy by x, π. = to divide, ii. 482 (Heron)
παράβολη, ή, juxtaposition; division (v. παραβάλλειν), hence quotient, ii. 530 (Dioph.); application of an area to a straight line, 186 (Eucl.); the conic section parabola, so called because the square on the ordinate is equal to a rectangle whose height is equal to the abscissa applied to the parameter as base, ii. 304 (Apollon.), 186 (Procl.), 280 (Eutoc.)
παράδοξος, ου, contrary to expectation, wonderful; ἡ π. γραμμή, the curve called paradoxical by Menelaus, 348 (Papp.); τὰ π., the paradoxes of Erycinus, ii. 572 (Papp.)
παρακείσθαι, to be adjacent, ii. 590 (Papp.), 282 (Eutoc.)
παράλληλεπιπέδον, τό, figure bounded by three pairs of parallel planes, parallelepiped, ii. 600 (Papp.)
INDEX OF GREEK TERMS

παράλληλόγραμμος, ov, bounded by parallel lines; as subst., παράλληλόγραμμον, τό, parallelogram, 188 (Eucl.)

παράλληλος, ov, beside one another, side by side, parallel, 270 (Eutoc.); π. εὐθεία, 440 (Eucl.)

παραμέτρησις, ες, Dor. παραμάκης, ες, oblong; σφαιροειδῆς π., ii. 164 (Archim.)

παραπλάνωμα, ατος, τό, intermediate, ii. 590 (Papp.); complement of a parallelogram, 190 (Eucl.)

παρασείεων, to stretch out along, produce, 10 (Plat.)

πάσα, πάσα, πάν, all, the whole, every, any; π. σημείον, any point, 442 (Eucl.)

πεντάγωνος, ov, pentagonal; π. ἀριθμός, 96 (Nicom.); as subst., πεντάγωνον, τό, pentagon, 222 (Iambi.)

περαίνειν, to bring to an end; πεπερασμένος, ov, terminated, 280 (Eutoc.); γραμμας πεπερασμέναι, finite lines, ii. 43 (Archim.)

πέρας, ατος, τό, end, extremity; of a line, 436 (Eucl.); of a plane, 438 (Eucl.)

περατοῦν, to limit, bound; εἴθεια περατομένη, 438 (Eucl.)

περυγράφειν, to circumscribe, ii. 48 (Archim.)

περιέχειν, to contain, bound; τό περιεχόμενον ύπό, the rectangle contained by, ii. 108 (Archim.); αἱ περι-έχουσαι τὴν γωνίαν γραμμαί, 438 (Eucl.); τὸ περιεχό-μενον σχῆμα, 440 (Eucl.)

περιλαμβάνειν, to contain, include, ii. 104 (Archim.)

περιμέτρος, ov, very large, well-fitting; ἤ π. (sc. γραμμῆς) =περίμετρον, τό, perimeter, ii. 318 (Archim.), ii. 502 (Heron), ii. 386 (Theon Alex.)

περισσάκης, Att. περιττάκης, adv., taken an odd number of times; π. ἄριτος ἀριθμός, odd-times even number, 68 (Eucl.); π. περισσός ἀριθμός, odd-times odd number, 68 (Eucl.)

περισσός, Att. περιττός, ἦ, on, superfluous; subtle; ἀριθμός π., odd number, 66 (Eucl.)

περιτιθέναι, to place or put around, 94 (Aristot.)

περιφέρεια, ἦ, circumference or periphery of a circle, arc of a circle, 440 (Eucl.), ii. 412 (Ptol.)

περιφορά, ἦ, revolution, turn of a spiral, ii. 182 (Archim.)

πηλικότης, ἦτος, ἦ, magnitude, size, ii. 412 (Ptol.)
πίπτειν, to fall; of points, ii. 44 (Archim.); of a straight line, 286 (Eutoc.)
πλάγιος, α, ου, oblique; π. διάμετρος, transverse diameter of a conic section, ii. 286 (Apollon.); π. πλευρά, transverse side of the rectangle formed by the ordinate of a conic section applied to the parameter as base, ii. 316 (Apollon.) and ii. 322 (Apollon.)
πλάσσειν, to form; of numbers, ii. 528 (Dioph.)
πλάτος, ους, Ion. εος, τό, breadth, 438 (Eucl.)
πλατύνειν, to widen, broaden, 88 (Nicom.)
πλευρά, ἀς, ἡ, side; of a triangle, 440 (Eucl.); of a parallelogram, ii. 216 (Archim.); of a square, hence, square root, ii. 530 (Dioph.); of a number, 70 (Eucl.); πλαγία π., latus rectum of a conic section, ii. 322 (Apollon.); π. καλ διαμέτρος, 132 (Theon Smyr.)
πλήθος, ους, Ion. εος, τό, number, multitude, 66 (Eucl.)
πνευματικός, ἡ, ου, of wind or air; Πνευματικά, τα, title of work by Heron, ii. 616 (Papp.)
ποιεῖν, to do, construct, ii. 566 (Papp.); to make, π. τομή, ii. 299 (Apollon.); to be equal to, to equal, ii. 526 (Dioph.)
πολλαπλασιάζειν, to multiply, 70 (Eucl.)
πολλαπλάσιος, α, ου, many times as large, multiple; of a number, 66 (Eucl.); of a magnitude, 444 (Eucl.); as subst., πολλαπλάσιον, τό, multiple; τᾶ ισάκις π., equimultiples, 446 (Eucl.)
πολλαπλάσιον, ου = πολλαπλάσιος, ii. 212 (Archim.)
πόλος, ὁ, pole; of a sphere, ii. 580 (Papp.); of a conchoid, 300 (Papp.)
πολύγωνος, ου, having many angles, polygonal; comp., πολύγωνότερος, ου, ii. 592 (Papp.); as subst., πολύγωνον, τό, polygon, ii. 48 (Archim.)
πολύεδρος, ου, having many bases; as subst., πολύεδρον, τό, polyhedron, ii. 572 (Papp.)
πολυπλασιασμός, ὁ, multiplication, 16 (Schol. in Plat. Charm.), ii. 414 (Apollon.)
πολυπλευρός, ου, many sided, multilateral, 440 (Eucl.)
πορίζειν, to bring about, find either by proof or by construction, ii. 598 (Papp.), 252 (Procl.)
πόρισμα, ατος, τό, corollary to a proposition, 480 (Procl.), ii. 294 (Apollon.); kind of proposition intermediate between a theorem and a problem, porism, 480 (Procl.)
INDEX OF GREEK TERMS

ποριστικός, η, ον, able to supply or find; ποριστικόν τοῦ προταθέντος, ii. 598 (Papp.)

πραγματεία, η, theory, investigation, 148 (Procl.), ii. 406 (Theon Alex.)

πρίσμα, ατος, το, prism, ii. 470 (Heron)

πρόβλημα, ατος, το, problem, 14 (Plat.), 258 (Eutoc.), ii. 566 (Papp.)

προβληματικός, η, ον, of or for a problem; applied to species of analysis, ii. 598 (Papp.)

πρόδηλος, ον, clear, manifest, ii. 496 (Heron)

προηγείαθαι, to take the lead; προηγούμενα, τα, forward points, i.e. those lying on the same side of a radius vector of a spiral as the direction of its motion, ii. 184 (Archim.)

προκατασκευάζειν, to construct beforehand, 276 (Eutoc.)

προμήκης, ες, prolonged, oblong, 398 (Plat.)

πρός, Dor. ποτί, prep., towards; ος η ΔΚ πρός ΚΕ, the ratio ΔΚ : ΚΕ, 272 (Eutoc.)

προσπέπτειν, to fall, 300 (Eutoc.); αι προσπίπτουσα εινθείαι, 438 (Eucl.), ii. 594 (Papp.)

προστεθείαι, to add, 444 (Eucl.)

πρότασις, εως, η, proposition, enunciation, ii. 566 (Papp.)

προτείνειν, to propose, to enunciate a proposition, ii. 566 (Papp.); το προτεθέν, that which was proposed, proposition, ii. 220 (Archim.)

πρώτιστος, η, ον, also οσ, ον, the very first, 90 (Nicom.)

πρώτος, η, ον, first; π. ἀριθμός, prime number, 68 (Eucl.); but π. ἀριθμοί, numbers of the first order in Archimedes, ii. 198 (Archim.); in astron., π. εὐθεία, first sixtieth, minute, 50 (Theon Alex.); in geom., π. εὐθεία, first distance of a spiral, ii. 182 (Archim.)

πτώσις, εως, η, case of a theorem or problem, ii. 600 (Papp.)

πυθμήν, εος, δ, base, basic number of a series, i.e., lowest number possessing a given property, 398 (Plat.); number of tens, hundreds, etc., contained in a number, ii. 354 (Papp.)

πυραμίς, ἵδος, η, pyramid, 228 (Archim.)

Ῥέπειν, to incline; of the weights on a balance, ii. 208

ῥητός, η, ον, rational, 398 (Plat.), 452 (Eucl.)

ῥίζα, Ion. ῥίζη, η, root; ἀρχικωτάτη ῥίζα, 90 (Nicom.)
INDEX OF GREEK TERMS

ῥομβοειδής, ἐς, rhombus-shaped, rhomboidal; ῥ. σχήμα, 440 (Eucl.)
ῥόμβος, ὁ, plane figure with four equal sides but with only the opposite angles equal, rhombus, 440 (Eucl.); ῥ. στερεός, figure formed by two cones having the same base and their axes in a straight line, solid rhombus, ii. 44 (Archim.)

Σημαίνειν, to signify, 188 (Procl.)
σημεῖον, Dor. σαμεῖον, τό, point, 436 (Eucl.); sign, ii. 522 (Dioph.)
σκαληνός, ἡ, ὁ, also ός, ὁν, oblique, scalene; κάνων σ., ii. 286 (Apollon.)
σκέλος, ους, Ιον. εος, τό, leg; σ. τῆς γωνίας, 264 (Eutoc.)
σπείρα, ἡ, surface traced by a circle revolving about a point not its centre, spire, tore, torus, ii. 468 (Heron)
σπειρικός, ἡ, ὁ, pertaining to a spire, spiric; σ. τομαί, spiric sections, ii. 364 (Procl.); σ. γραμμαί, spiric curves, ii. 364 (Procl.)
στερεός, ὁ, ὁ, solid; σ. γωνία, solid angle, 222 (Plat.); σ. ἄρθρος, cubic number; σ. τοποί, solid loci, ii. 600 (Apollon.); σ. πρόβλημα, solid problem, 348 (Papp.); σ. νεφώς, solid verging, 350 (Papp.); as subst., στερεῶν, τό, solid, 258 (Eutoc.)
στιγμή, ἡ, point, 80 (Nicom.)
στοιχείον, τό, element, ii. 596 (Papp.); elementary book, 150 (Procl.); Στοιχεία, τά, the Elements, especially Euclid's
στοιχεώσις, εως, ἡ, elementary exposition, elements; Euclid's Elements of geometry, 156 (Procl.); ἀι κατά μοισικήν σ., the elements of music, 156 (Procl.); σ. τῶν κωνικῶν, elements of conics, ii. 276 (Eutoc.)
στρογγυλός, ἡ, οὖν, round, 392 (Plat.)
συγκεῖσθαι, to lie together; as pass. of συντιθέσθαι, to be composed of, ii. 284 (Apollon.)
συγκρίσις, εως, ἡ, comparison; Τῶν πέντε σχημάτων σ., Comparison of the Five Figures, title of work by Aristaeus, ii. 348 (Hyps.)
σύνυνθις, ἐς, yoked together, conjugate; σ. διάμετρος, σ. ἄξονες, ii. 288 (Apollon.)
σύμμετρος, οὖν, commensurate with, commensurable with; 380 (Plat.), 452 (Eucl.), ii. 208 (Archim.)
συμπαρατείνειν, to stretch out alongside of, 188 (Procl.)
INDEX OF GREEK TERMS

σύμπαν, σύμπασα, σύμπαν, all together, the sum of, ii. 514 (Dioph.)

συμπέραμα, atos, τό, conclusion, ii. 228 (Archim.)

συμπίπτειν, to meet, 190 (Eucl.); τῇ μὲν ἐν αὐτῇ σ., of curves which meet themselves, ii. 360 (Procl.)

συμπληρόν, to fill, complete; σ. παράλληλόγραμμον, 268 (Eutoc.)

σύμπτωμα, atos, τό, property of a curve, 336 (Papp.)

σύμπτωσις, ews, ἥ, falling together, meeting, ii. 64 (Archim.), ii. 270 (Cleom.), 286 (Eutoc.)

συναγωγή, ἥ, collection; title of work by Pappus, ii. 568 (Papp.)

συναμφότερος, ov, the sum of, 328 (Archim.)

συνάπτειν, to collect, gather; συνημμένος λόγος, compound ratio, ii. 602 (Papp.)

συνέγγυλεν, to approximate, ii. 414 (Ptol.), ii. 470 (Heron)

σύνεγγυς, adv., near, approximately, ii. 488 (Heron)

συνεχής, ἐς, continuous, 442 (Eucl.); σ. ἀναλογία, continued proportion, ii. 566 (Papp.); σ. ἀνάλογον, 302 (Papp.); σπείρα σ., ii. 364 (Procl.)

συνθέντα, τ. συνθενάων σύνθεσις, ews, ἥ, putting together, composition; σ. λόγου, transformation of a ratio known as componeendo, 448 (Eucl.);

method of reasoning from assumptions to conclusions, in contrast with analysis, synthesis, ii. 596 (Papp.)

σύνθετος, ov, composite; σ. ἀριθμός, 68 (Eucl.); σ. γραμμή, ii. 360 (Procl.)

συνιστάναι, to set up, construct, 190 (Eucl.), 312 (Them.)

σύνταξις, ews, ἥ, putting together in order, systematic treatise, composite volume, collection; title of work by Ptolemy, ii. 408 (Suidas)

συντιθέναι, to place or put together, add together, used of the synthesis of a problem, ii. 160 (Archim.);

συνθέντι, lit. to one having compounded, the transformation of a ratio known as componeendo, ii. 130 (Archim.)

σύνταξις, ews, ἥ, grouping (of theorems), 150 (Procl.)

σφαῖρα, ἥ, sphere, 466 (Eucl.), ii. 40 (Archim.), ii. 572 (Papp.)

σφαρικός, ἤ, ὁ, spherical, ii. 584 (Papp.); Σφαρικά, title of works by Menelaus and Theodosius

σφαροποιοῦσα, ἥ, artificial sphere, making of the heavenly spheres, ii. 618 (Papp.)

679
INDEX OF GREEK TERMS

σχήμα, atos, to, shape, figure, 438 (Eucl.), ii. 42 (Archim.)
σχηματοποιεῖν, to bring into a certain form or shape; σχηματωσώσα γραμμή, curve forming a figure, ii. 360 (Procl.)

Τάλαντον, τό, weight known as the talent, ii. 490 (Heron)
τάξις, ews, ἥ, order, arrangement, scheme, 112 (Iambi.)
ταράσσειν, to disturb; τεταραγμένη ἀναλογία, disturbed proportion, 450 (Eucl.)
τάσσειν, to draw up in order, ii. 598 (Papp.); ὀμοίως τεταγμένα, ii. 166 (Archim.); perf. part. pass. used as adv., τεταγμένος, ordinate-wise, ii. 286 (Apollon.); αἱ καταγώμεναι τεταγμένως (sc. εὐθείαι), the straight lines drawn ordinate-wise, i.e., the ordinates, of a conic section, ii. 308 (Apollon.); τεταγμένος ἡ ΓΑ, ii. 224 (Archim.)

τελειος, a, ov, perfect, complete, ii. 604 (Papp.); τ. ἀρμιθὸς, 76 (Speusippus ap. Theol. Arith.), 70 (Eucl.), 398 (Plat.)
τεμνεῖν, to cut; of straight lines by a straight line, ii. 288 (Apollon.); of a curve by a straight line, 278 (Eutoc.); of a solid

by a plane, ii. 288 (Apollon.)
τεσσαρεσκαίδεκαέδρον, τό, solid with fourteen faces, ii. 196 (Archim.)
τεταγμένως, ὑ, τάσεων τεταραγμένος, ὑ, τάσασεων
tetararhymorion, τό, fourth part, quadrant of a circle, ii. 582 (Papp.)
tetragwvilev, to make square, ii. 494 (Heron); to square, 10 (Plat.); ἡ τετραγωνίζουσα (sc. γραμμή), quadratrix, 334 (Simpl.), 336 (Papp.)
tetragwnikos, ὑ, ὑ, square; of numbers, ii. 526 (Dioph.); τ. πλευρά, square root, 60 (Theon Alex.)
tetragwvnomos, ὑ, squaring, 310 (Aristot.); τοῦ κύκλου τ., 308 (Plut.); τοῦ μηνίσκου τ., 150 (Procl.)
tetragwvnoν, ov, square, 308 (Aristophanes), ii. 504 (Heron); ἀριθμὸς τ., square number, ii. 514 (Dioph.); τ. κρίκου, square rings, ii. 470 (Heron); as subst., τετράγωνον, τό, square, 440 (Eucl.); square number, ii. 518 (Dioph.)
tetrákis, adv., four times, 326 (Archim.)
tetrapláasios, a, ov, four-fold, four times as much, 332 (Archim.)
tetrapláasiων, ov, later form of τετραπλάσιος
tetrapléuros, ov, four-sided,
INDEX OF GREEK TERMS

quadrilateral; ἀξίματα τ., 440 (Eucl.)

τιθέναι, to set, put, place, ii. 224 (Archim.); θετεῖν, must be posited, 392 (Plat.); τὸ AB τιθέν ἐπὶ τῷ Z, A + B placed at Z, ii. 214 (Archim.)

τμῆμα, Dor. τμῆμα, ατος, τό, segment; of a circle, ii. 584 (Papp.); of a sphere, ii. 40 (Archim.); in astron., the ⅓ part of diameter of a circle, ii. 412 (Ptol.)

τομεύς, ἑως, ὁ, sector of a circle, ii. 582 (Papp.); τ. στερεός, sector of a sphere (intercepted by cone with vertex at centre), ii. 44 (Archim.)

τομή, ἡ, end left after cutting, section; section of a straight line, 268 (Eutoc.); section of a cone, conic section, ii. 278 (Apollon.); τ. ἀντικείμεναι, opposite branches of a hyperbola, ii. 322 (Apollon.); στερεύκαι τ., ii. 364 (Procl.); κοινή τ., common section, 286 (Eutoc.); Διωρισμένη τ., Determinate Section, title of work by Apollonius, ii. 598 (Papp.); τὰ περὶ τὴν τ., theorems about the section (of a line cut in extreme and mean ratio), 152 (Procl.)

topikos, ἡ, ὁ, of or pertaining to place or space; pertaining to a locus, 490 (Papp.); μοίρα τ., degree in space, ii. 396 (Hyp.-sicl.)

tότος, ὁ, place, region, space, ii. 590 (Papp.); locus, 318 (Papp.); Τ. ἐπίσεως, Plane Loci, title of work by Apollonius, ii. 598 (Papp.); Τ. στερεῶς, Solid Loci, title of work by Aristaeus, ii. 600 (Papp.)

τραπέζιον, τό, trapezium, 440 (Eucl.), ii. 488 (Heron)

τριάς, ἄδος, ἡ, the number three, triad, 90 (Nicom.); Μεναχμεια τ., the three conic sections of Menaechnus, 296 (Erat. ap. Eutoc.)

τρίγωνος, ὁ, three-cornered, triangular; ἄρθρῳ τρίγωνος, triangular numbers; as subst., τρίγωνος, τό, triangle, 440 (Eucl.), 316 (Archim.); τὸ διὰ τοῦ ἄξονος τ., axial triangle, ii. 288 (Apollon.)

τριπλάσιος, ὁ, ὁ, thrice as many, thrice as great as, 326 (Archim.), ii. 580 (Papp.)

τριπλάσιον, ὁ = τριπλάσιος, 320 (Archim.); τ. λόγος, 448 (Eucl.)

τριπλευρός, ὁ, three-sided, trilateral; ἀξίματα τ., 440 (Eucl.)

τρίχα, thrice, in three parts, τ. τεμεῖν τὴν γωνίαν, to trisect the angle, 300 (Papp.)

τυγχάνειν, to happen to be; aor. part., τυχόν, τυχόνα, τυχόν, any, taken at ran-
INDEX OF GREEK TERMS

dom, εὐθεία τ., ii. 486 (Heron); as adv., τυχών, perchance, 264 (Eutoc.)

Τὸ δρεῖον, τὸ, bucket, pitcher; pl., Τὸ δρεῖον, title of work on water clocks by Heron, ii. 616 (Papp.)

ὑπάρξις, εως, η, existence, ii. 518 (Dioph.); positive term, ii. 524 (Dioph.)

ὑπενάντιος, α, ου, subcontrary; ὑ. μέσα, 112 (Archytas ap. Porph.); ὑ. τομή, ii. 304 (Apollon.)

ὑπεναντιῶς, subcontrary-wise, ii. 301-302 (Apollon.)

ὑπερβάλλειν, to exceed, 188 (Procl.)

ὑπερβολή, η, exceeding, 186 (Procl.); the conic section hyperbola, so called because the square on the ordinate is equal to a rectangle with height equal to the abscissa applied to the parameter as base, but exceeding (ὑπερβάλλον), i.e., overlapping, that base, ii. 310 (Apollon.), 186 (Procl.)

ὑπερέχειν, to exceed, 112 (Archytas ap. Porph.), 444 (Eucl.), ii. 608 (Papp.); τῷ αὐτῷ μέρει τῶν ἀκρῶν αὐτῶν ὑπερέχου τε καὶ ὑπερεξόμενον, 402 (Plat.)

ὑπεροχή, Dor. ὑπεροχά, η, excess, 112 (Archytas ap. Porph.), 318 (Archim.), ii. 530 (Dioph.), ii. 608 (Papp.)

ὑπερπίπτειν, to fall beyond, exceed, ii. 436 (Ptol.)

ὑπό, by; ἡ ὑπὸ ἩΒΕ γνώμα, the angle ἩΒΕ, 190 (Eucl.); τὸ ὑπὸ ΔΕΓ (sc. εὐθύγραμμον), the rectangle contained by ΔΕ, ΕΓ, 268 (Eutoc.)

ὑπόθεσις, εως, η, hypothesis, 420 (Aristot.), ii. 2 (Archim.)

ὑποπολλαπλάσιον, ου, submultiple of another; as subst., ὑποπολλαπλάσιον, τὸ, submultiple, 78 (Theol. Arith.)

ὑπόστασις, εως, η, condition, ii. 534 (Dioph.)

ὑποτείνειν, to stretch under, subtend, be subtended by; ἡ ὑπὸ δύο πλευράς τοῦ πολυγώνου ὑποτείνουσα εὐθεία, the straight line subtending, or subtended by, two sides of the polygon, ii. 96 (Archim.); πλευρά ὑποτείνουσα μοῖρας ζ, side subtending sixty parts, ii. 418 (Ptol.); ἡ τὴν ορθὴν γωνίαν ὑποτεινοῦσα πλευρά, the side subtending the right angle, 178 (Eucl.); hence ἡ ὑποτείνουσα (sc. πλευρά ορ γραμμῆς), hypotenuse of a right-angled triangle, 176 (Plut.)

ὑποτιθέναι, pass. ὑποκείσθαι, to suppose, assume, make a hypothesis, ii. 2 (Archim.), ii. 304 (Apollon.)

ὑπτίον, α, ου, laid on one's
INDEX OF GREEK TERMS

back, supine; ἰπτιον, τό, a quadrilateral with no parallel sides, 482 (Papp.) υφιστάναι, to place or set under, ii. 362 (Procl.) υφος, ους, Ion. eos, τό, height; of a triangle, ii. 222 (Archim.); of a cylinder, ii. 42 (Archim.); of a cone, ii. 118 (Archim.)

Φανερός, ἄ, ὁ, clear, manifest, ii. 64 (Archim.), ii. 570 (Papp.)

φέρεω, to bear, carry; pass., to be borne, carried, move, revolve; σημεῖον φερόμενον, ii. 582 (Papp.); τό φερόμενον, moving object, 366 (Aristot.)

φορά, ἡ, motion, 12 (Plat.)

Χείρ, χειρός, ἡ, band or number of men, 30 n. a

χειρουργικός, ἄ, ὁ, manual, ii. 614 (Papp.)

χρονικός, ἡ, ὁ, of or pertaining to time; μοῖρα χ., degree in time, ii. 396 (Hypsicl.)

χωρίον, τό, space, area, 394 (Plat.), ii. 532 (Dioph.)

Υαὐευ, to touch, 264 (Eutoc.)

ψευδάριον, τό, fallacy; Ψευδάρια, τά, title of work by Euclid, 160 (Procl.)

Ὀρλόγιον, τό, instrument for telling the time, clock, ii. 268 (Cleom.); τὸ δι᾽ ὕδατος ὡς, water-clock, ii. 616 (Papp.)

ὡς, ὁς ἡ ἩΕ πρὸς ἙΓ, ἡ ἩΕ πρὸς ἩΓ, i.e., ἩΕ : ἙΓ = ἩΓ : ἩΕ, 322 (Archim.)

ὡς τε, such that, ii. 52 (Archim.); and so, therefore, used for the stages in a proof, ii. 54 (Archim.)

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
THE UNIVERSITY PRESS, GLASGOW

(362)
Latin Authors

AMMIANUS MARCELLINUS. Translated by J. C. Rolfe. 3 Vols. (2nd Imp. revised.)

APULEIUS: THE GOLDEN ASS (Metamorphoses). W. Adlington (1566). Revised by S. Gaselee. (7th Imp.)


ST. AUGUSTINE, SELECT LETTERS. J. H. Baxter. (2nd Imp.)

AUSONIUS. H. G. Evelyn White. 2 Vols. (2nd Imp.)

Bede. J. E. King. 2 Vols. (2nd Imp.)


CAESAR: ALEXANDRIAN, AFRICAN and SPANISH WARS. A. G. Way.

CAESAR: CIVIL WARS. A. G. Peskett. (5th Imp.)

CAESAR: GALLIC WAR. H. J. Edwards. (10th Imp.)

CATO: De RE Rustica; VARRO: De RE Rustica. H. B. Ash and W. D. Hooper. (3rd Imp.)

CATULLUS. F. W. Cornish; TIBULLUS. J. B. Postgate; PERVIGILIUM VENERIS. J. W. Mackail. (12th Imp.)

CELSUS: De Medicina. W. G. Spencer. 3 Vols. (Vol. I. 3rd Imp. revised, Vols. II. and III. 2nd Imp.)

CICERO: BRUTUS, and ORATOR. G. L. Hendrickson and H. M. Hubbell. (3rd Imp.)

[CICERO]: AD HERENNIIUM. H. Caplan.

CICERO: De FATO; PARADOX A STOICORUM; DE PARTITIONE ORATORIA. H. Rackham (With De Oratore, Vol. II.) (2nd Imp.)

CICERO: De FINIBUS. H. Rackham. (4th Imp. revised.)

CICERO: De INVENTIONE, etc. H. M. Hubbell.

CICERO: De NATURA DEOBUM and ACADEMICA. H. Rackham. (2nd Imp.)

CICERO: De OFFICIS. Walter Miller. (6th Imp.)

CICERO: De ORATORE. 2 Vols. E. W. Sutton and H. Rackham. (2nd Imp.)

CICERO: De REPUBLICA and De Legibus. Clinton W. Keyes. (4th Imp.)

CICERO: De SENECTUTE, De AMICITIA, De DIVINATIONE. W. A. Falconer. (6th Imp.)

CICERO: In CATALINAM, Pro Flacco, Pro Murena, Pro Sulla. Louis E. Lord. (3rd Imp. revised.)

CICERO: LETTERS TO HIS FRIENDS. W. Glynn Williams. 3 Vols.
(Vols. I. and II. 3rd Imp., Vol. III. 2nd Imp. revised.)
CICERO: PHILIPPICS. W. C. A. Ker. (3rd Imp. revised.)
CICERO: PRO ARCHITA, POST REDITUM, DE DOMO, DE HARUSPICUM
RESPONSIS, PRO PLANCIO. N. H. Watts. (4th Imp.)
CICERO: PRO CAECINA, PRO LEGE MANILIA, PRO CLUENTIO, PRO
RABIRIO. H. Grose Hodge. (3rd Imp.)
CICERO: PRO MILONE, IN PISONEM, PRO SCAURO, PRO FONTEIO,
PRO RABIRIO POSTUMO, PRO MARCELLO, PRO LIGARIO, PRO REGE
DEIOTARO. N. H. Watts. (2nd Imp.)
CICERO: PRO QUINTIO, PRO ROSTIO AMERINO, PRO ROSTIO
COMOEDO, CONTRA RULLUM. J. H. Freese. (3rd Imp.)
CICERO: TUSCULAN DISPUTATIONS. J. E. King. (4th Imp.)
(Vol. I. 3rd Imp., Vol. II. 2nd Imp.)
CLAUDIAN. M. Platnauer. 2 Vols. (2nd Imp.)
COLUMELLA: DE RE RUSTICA, DE ARBORIBUS. H. B. Ash, E. S.
Forster and E. Heffner. 3 Vols. (Vol. I. 2nd Imp.)
CURTIUS, Q.: HISTORY OF ALEXANDER. J. C. Rolfe. 2 Vols. (2nd
Imp.)
FLORUS. E. S. Forster and CORNELIUS NEPOS. J. C. Rolfe. (2nd
Imp.)
FRONTINUS: STRATAGEMS AND AQUEDUCTS. C. E. Bennett and
FRONTO: CORRESPONDENCE. C. R. Haines. 2 Vols. (Vol. I. 3rd
Imp., Vol. II. 2nd Imp.)
GELLIUS. J. C. Rolfe. 3 Vols. (Vol. I. 3rd Imp., Vols. II. and
III. 2nd Imp.)
HORACE: ODES and EPODES. C. E. Bennett. (14th Imp. revised.)
HORACE: SATIRES, EPISTLES, ARS POETICA. H. R. Fairclough.
(9th Imp. revised.)
JEROME: SELECTED LETTERS. F. A. Wright. (2nd Imp.)
JUVENAL and PERSIUS. G. G. Ramsay. (7th Imp.)
IX. 3rd Imp.; Vols. IV., VI.–VIII., X.–XII. 2nd Imp. revised.)
LUCAN. J. D. Duff. (3rd Imp.)
LUcretius. W. H. D. Rouse. (7th Imp. revised.)
MARTIAL. W. C. A. Ker. 2 Vols. (Vol. I. 5th Imp., Vol. II. 4th
Imp. revised.)
MINOR LATIN POETS: FROM PUBLILIUS SYRUS TO RUTILIUS
NAMATIANUS, INCLUDING GRATIUS, CALPURNIUS SCICULUS, NEMESI-
IANUS, AVIANUS, AND OTHERS WITH "ACTEA" AND THE "PHOENIX."
J. Wight Duff and Arnold M. Duff. (3rd Imp.)
OVID: THE ART OF LOVE AND OTHER POEMS. J. H. Mozley. (3rd
Imp.)
OVID: FASTI. Sir James G. Frazer. (2nd Imp.)
OVID: HEROIDES AND AMORES. Grant Showerman. (5th Imp.)
OVID: METAMORPHOSES. F. J. Miller. 2 Vols. (Vol. I. 10th Imp.,
Vol. II. 8th Imp.)
OVID: TRISTIA AND EX PONTO. A. L. Wheeler. (3rd Imp.)
PERSIUS. Cf. JUVENAL.

PETRONIUS. M. Heseltine; SENECA APOCOLOCYNTOSIS. W. H. D. Rouse. (8th Imp. revised.)

PLAUTUS. Paul Nixon. 5 Vols. (Vols. I. and II. 5th Imp., Vol. III. 3rd Imp., Vols. IV and V. 2nd Imp.)

PLINY: LETTERS. Melmoth’s Translation revised by W. M. L. Hutchinson. 2 Vols. (6th Imp.)


PROPERTIUS. H. E. Butler. (6th Imp.)

PRUDENTIUS. H. J. Thomson. 2 Vols.

QUINTILIAN. H. E. Butler. 4 Vols. (3rd Imp.)


SALLUST. J. C. Rolfe. (4th Imp. revised.)

SCRIPTORES HISTORIÆ AUGUSTÆ. D. Magle. 3 Vols. (Vol. I. 3rd Imp. revised, Vols. II. and III. 2nd Imp.)

SENENCA: APOCOLOCYNTOSIS. Cf. PETRONIUS.

SENENCA: EPISTULAE MORALES. R. M. Gummerc. 3 Vols. (Vol. I. 4th Imp., Vols. II. and III. 2nd Imp.)

SENENCA: MORAL ESSAYS. J. W. Basore. 3 Vols. (Vol. II. 3rd Imp., Vols. I. and III. 2nd Imp. revised.)

SENENCA: TRAGEDIES. F. J. Miller. 2 Vols. (Vol. I. 4th Imp., Vol. II. 3rd Imp. revised.)

SIDONIUS: POEMS AND LETTERS. W. B. Anderson. 2 Vols. (Vol. I. 2nd Imp.)

SILIUS ITALICUS. J. D. Duff. 2 Vols. (Vol. I. 2nd Imp., Vol. II. 3rd Imp.)

STATIUS. J. H. Mozley. 2 Vols. (2nd Imp.)

SUETONIUS. J. C. Rolfe. 2 Vols. (Vol. I. 7th Imp., Vol. II. 6th Imp. revised.)

TACITUS: DIALOGUS. Sir Wm. Peterson. AGRICOLA and GERMANIA. Maurice Hutton. (6th Imp.)


TERENCE. John Sargeaunt. 2 Vols. (7th Imp.)

TERTULLIAN: APOLIOGA and DE SPECTACULIS. T. R. Glover. MINUCIUS FELIX. G. H. Rendall. (2nd Imp.)

VALENIUS FLACCUS. J. H. Mozley. (2nd Imp. revised.)

VARRO: DE LINGUA LATINA. R. G. Kent. 2 Vols. (2nd Imp. revised.)

VELLEIUS PATERCULUS and RES GESTAE DIVI AUGUSTI. F. W. Shipley. (2nd Imp.)

VIRGIL. H. R. Fairclough. 2 Vols. (Vol. I. 18th Imp., Vol. II. 14th Imp. revised.)

Greek Authors

ACHILLES TATIUS. S. Gaselce. (2nd Imp.)
AENEAS TACTICUS, ASCLEPIODOTUS and ONASANDER. The Illinois Greek Club. (2nd Imp.)
AESCHINES. C. D. Adams. (2nd Imp.)
AESCHYLUS. H. Weir Smyth. 2 Vols. (Vol. I. 6th Imp., Vol. II. 5th Imp.)
ALCIPHRON, AELIAN, PHILOSTRATUS LETTERS. A. R. Benner and F. H. Fobes.
ANDOCIDES, ANTIPHON. Cf. MINOR ATTIC ORATORS.
APOLLODORUS. Sir James G. Frazer. 2 Vols. (Vol. I. 3rd Imp., Vol. II. 2nd Imp.)
APOLLONIUS RHODIUS. R. C. Seaton. (5th Imp.)
ARATUS. Cf. CALLIMACHUS.
ARISTOPHANES. Benjamin Blickley Rogers. 3 Vols. Verse trans. (5th Imp.)
ARISTOTLE: ART OF RHETORIC. J. H. Freese. (3rd Imp.)
ARISTOTLE: ATHENIAN CONSTITUTION, EUDEMIAN ETHICS, VICES AND VIRTUES. H. Rackham. (3rd Imp.)
ARISTOTLE: GENERATION OF ANIMALS. A. L. Peck. (2nd Imp.)
ARISTOTLE: METAPHYSICS. H. Tredennick. 2 Vols. (3rd Imp.)
ARISTOTLE: METEOROLOGICA. H. D. P. Lee.
ARISTOTLE: NICOMACHEAN ETHICS. H. Rackham. (6th Imp. revised.)
ARISTOTLE: OECONOMICA and MAGNA MORALIA. G. C. Armstrong; (with Metaphysics, Vol. II.). (3rd Imp.)
ARISTOTLE: ON THE HEAVENS. W. K. C. Guthrie. (3rd Imp. revised.)
ARISTOTLE: On Sophistical Refutations, On Coming to be and Passing Away, On the Cosmos. E. S. Forster and D. J. Furley.
ARISTOTLE: ON THE SOUL, PARVA NATURALIA, ON BREATH. W. S. Hett. (2nd Imp. revised.)
ARISTOTLE: ORGANON, CATEGORIES: On Interpretation, Prior Analytics. H. P. Cooke and H. Tredennick. (3rd Imp.)
ARISTOTLE: PARTS OF ANIMALS. A. L. Peck; MOTION AND PROGRESSION OF ANIMALS. E. S. Forster. (3rd Imp. revised.)
ARISTOTLE: POETICS and LONGINUS. W. Hamilton Fyfe; DEMETRIUS ON STYLE. W. Rhys Roberts. (5th Imp. revised.)
ARISTOTLE: POLITICS. H. Rackham. (4th Imp. revised.)
ARISTOTLE: PROBLEMS. W. S. Hett. 2 Vols. (2nd Imp. revised.)
ARISTOTLE: RHETORICA AD ALEXANDRUM (with PROBLEMS, Vol. II.). H. Rackham.

ATHENAEUS: DEIPNOSOPHISTAE. C. B. Gulick. 7 Vols. (Vols. I., IV.--VII. 2nd Imp.)

ST. BASIL: LETTERS. B. J. Deferrari. 4 Vols. (2nd Imp.)

CALLIMACHUS, Hymns and Epigrams, and LYCOPHRON. A. W. Mair; ARATUS. G. R. Mair. (2nd Imp.)

CLEMENT OF ALEXANDRIA. Rev. G. W. Butterworth. (3rd Imp.)

COLLUTHUS. Cf. OPPIAN.

DAPHNIS AND CHLOE. Thornley’s Translation revised by J. M. Edmonds: and PARTHENIUS. S. Gaselee. (4th Imp.)

DEMOSTHENES I: OLYNTHIACS, PHILIPPICS and MINOR ORATIONS. I.–XVII. and XX. J. H. Vince. (2nd Imp.)

DEMOSTHENES II: DE CORONA and DE FALSA LEGATIONE. C. A. Vince and J. H. Vince. (3rd Imp. revised.)

DEMOSTHENES III: MEIDIAS, ANDROTON, ARISTOCRATES, TIMO- CRATES and ARISTOTELON, I. and II. J. H. Vince. (2nd Imp.)

DEMOSTHENES IV–VI: PRIVATE ORATIONS and IN NEAERAM. A. T. Murray. (Vol. IV. 2nd Imp.)

DEMOSTHENES VII: FUNERAL SPEECH, EROTIC ESSAY, EXORDIA and LETTERS. N. W. and N. J. DeWitt.


DIOGENES LAERTIUS. R. D. Hicks. 2 Vols. (Vol. I. 4th Imp., Vol. II. 3rd Imp.)

DIOSYNIUS OF HALICARNASSUS: ROMAN ANTIQUITIES. Spelmann’s translation revised by E. Cary. 7 Vols. (Vols. I.–V. 2nd Imp.)

EPICETUS. W. A. Oldfather. 2 Vols. (2nd Imp.)


GALEN: ON THE NATURAL FACULTIES. A. J. Brock. (4th Imp.)


GREEK ELEGY AND IAMBUS with the ANacreONTea. J. M. Edmonds. 2 Vols. (Vol. I. 3rd Imp., Vol. II. 2nd Imp.)

THE GREEK BUCOLIC POETS (THEOCRITUS, BION, MOSCHUS). J. M. Edmonds. (7th Imp. revised.)

GREEK MATHEMATICAL WORKS. Ivor Thomas. 2 Vols. (2nd Imp.)

HERODES. Cf. THEOPHRASTUS: CHARACTERS.


HESIOD AND THE HOMERIO HYMNS. H. G. Evelyn White. (7th Imp. revised and enlarged.)
Hippocrates and the Fragments of Heracleitus. W. H. S. Jones and E. T. Withington. 4 Vols. (3rd Imp.)


Homer: Odyssey. A. T. Murray. 2 Vols. (8th Imp.)

Isaues. E. W. Forster. (3rd Imp.)

Isocrates. George Norlin and LaRue Van Hook. 3 Vols. (2nd Imp.)

St. John Damascene: Barlaam and Ioasaph. Rev. G. R. Woodward and Harold Mattingly. (3rd Imp. revised.)


Julian. Wilmer Cave Wright. 3 Vols. (Vols. I. and II. 3rd Imp., Vol. III. 2nd Imp.)


Lyssias. W. R. M. Lamb. (2nd Imp.)


Marcus Aurelius. C. R. Haines. (4th Imp. revised.)

Menander. F. G. Allinson. (3rd Imp. revised.)


Nonnos: Dionysiaca. W. H. D. Rouse. 3 Vols. (2nd Imp.)

Oppian, Colluthus, Tryphiodorus. A. W. Malr. (2nd Imp.)


Philo: two supplementary Vols. (Translation only.) Ralph Marcus.


Philostratus: Imagines; Callistratus: Descriptions. A. Fairbanks.

Philostratus and Eunapius: Lives of the Sophists. Wilmer Cave Wright. (2nd Imp.)

Pindar. Sir J. E. Sandys. (7th Imp. revised.)


Plato: Cratylus, Parmenides, Greater Hippias, Lesser Hippias. H. N. Fowler. (4th Imp.)
Plato: Euthyphro, Apology, Crito, Phaedo, Phaedrus. H. N. Fowler. (11th Imp.)
Plato: Lysis, Symposium, Gorgias. W. R. M. Lamb. (5th Imp. revised.)
Plato: Theaetetus and Sophist. H. N. Fowler. (4th Imp.)
Polybius. W. R. Paton. 6 Vols. (2nd Imp.)
Quintus Smyrnaeus. A. S. Way. Verse trans. (3rd Imp.)
Strabo: Geography. Horace L. Jones. 8 Vols. (Vols. I., V., and VIII. 3rd Imp., Vols. II., III., IV., and VII. 2nd Imp.)
Theophrastus: Characters. J. M. Edmonds. Herodes, etc. A. D. Knox. (3rd Imp.)
Theophrastus: Enquiry into Plants. Sir Arthur Hort, Bart. 2 Vols. (2nd Imp.)
Thucydides. C. F. Smith. 4 Vols. (Vol. I. 4th Imp., Vols. II., III., and IV. 3rd Imp. revised.)
Xenophon: Cyropaedia. Walter Miller. 2 Vols. (Vol. I. 4th Imp., Vol. II. 3rd Imp.)
Xenophon: Memorabilia and Oeconomicus. E. C. Marchant. (3rd Imp.)
Xenophon: Scripta Minora. E. C. Marchant. (2nd Imp.)
IN PREPARATION

Greek Authors


Latin Authors

St. Augustine: City of God.
Cicero: Pro Sestio, In Vatinium.
Cicero: Pro Caelio, De Provincis Consularibus, Pro Balbo.
Phaedrus: Ben E. Perry.

DESCRIPTIVE PROSPECTUS ON APPLICATION

London
Cambridge, Mass.

WILLIAM HEINEMANN LTD
HARVARD UNIVERSITY PRESS
BULMER-THOMAS, IVOR

Greek mathematical works
Aristarchus to Pappus